Radially Open sets

by

Perera, Dharmakeerthi S. ,Moreno, Andres C., and Yadiberet , Cybele D.

Supervised

by

Dr. Z. Boudhraa

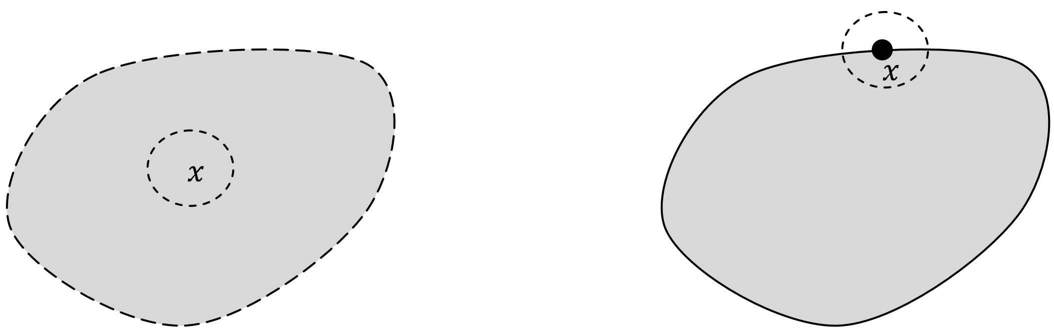
Abstract

We denote by S the unit circle $x^2 + y^2 = 1$. If P is a point on S, we can choose open line segments in each direction that do not intersect the circle at any other point. This elementary result leads to the proof that radially open sets are not always open sets in the usual sense.

Definition

Call a subset of the plane R^2 open if and only if it contains an open disk about each of its points.

Examples

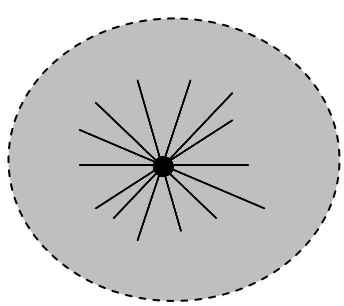


Note that the second set is not open; in fact if x is a point on the boundary then any disk surrounding x contains points not in the set.

Definition

Call a subset of the plane R^2 radially open if it contains an open line segment in each direction about each of its points.

Example



If we take a point in an open disk, then we can draw open line segments in each direction. Therefore if a subset of R^2 is open, then it is radially open.

Conjecture

Is the converse true? Are the two types of sets the same? We shall give a negative answer by showing the following result.

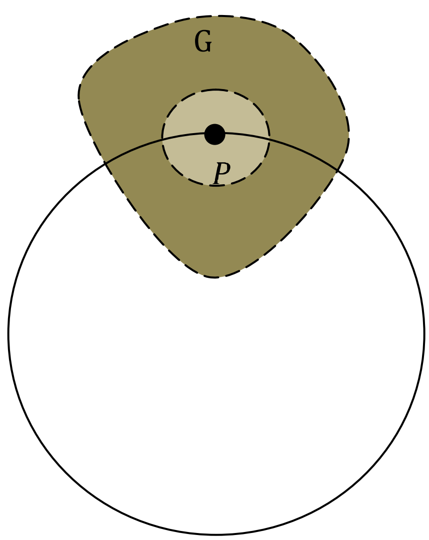
Result 1

Let S be the set of all points in R^2 satisfying the equation $x^2 + y^2 = 1$. That is S is the unit circle in R^2.

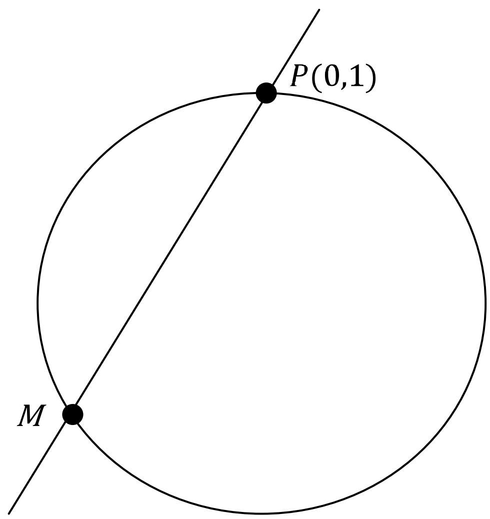
1. If $P(x, y)$ is a point on the circle S, then any open set of R^2 containing P intersects the circle at another point.
2. Given a point $P(x, y)$on the circle S, we can always construct a radially open set that contains P and does not intersect S at a different point.

Remark

The first statement is obvious; it was included so that the reader can see the difference between an open set and a radially open set.



Consider the point $P(0,1)$ on the unit circle $x^2 + y^2 = 1$. If $\vec{V}$ is an arbitrary direction, then the line $y = mx + 1$ passing through the point $(0,1)$ and parallel to $\vec{V}$ intersects the circle at a unique point M.



If (x, y) are the coordinates of the point M, then

1. $y = mx + 1$ since M is a point on the line
2. $x^2 + y^2 = 1$ since M is a point on the circle

Therefore,

$$x^2 + (mx + 1)^2 = 1$$

$$(1 + m^2)x^2 + 2mx = 0$$

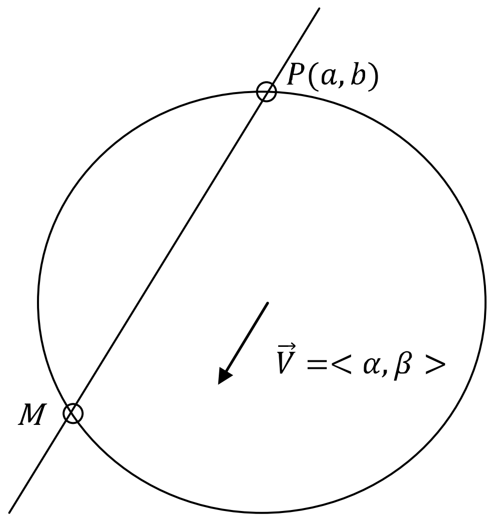
The solutions are $(0,1)$ *and* $\left(\frac{-2m}{1+m^2}, \frac{1-m^2}{1+m^2}\right)$

The distance between the points P and M.

$$d = \sqrt{\left(\frac{2m}{1+m^2}\right)^2 + \left(1 - \frac{1-m^2}{1+m^2}\right)^2} = \sqrt{\frac{4m^2}{(1+m^2)^2} + \frac{4m^2}{(1+m^2)^2}} = \frac{2\sqrt{2}m}{1+m^2}$$

Note that the line segment that contains the point P and of length d/2 does not intersect the unit circle at any other point.

Now we tackle the general case. Given a point $P(a, b)$ on the unit circle and let $\vec{V} =< \alpha, \beta >$ be an arbitrary direction. Let L be the line through the point P and parallel to the direction $\vec{V}$. Thus the slope of L is $m = \frac{\beta}{\alpha}$.



The equation of the line L is $y = mx + k$ and since $P(a, b)$ is a point of the line:

$b = ma + k$, or $k = b - ma$ and $y = mx + (b - ma)$.

If (x, y) are the coordinates of the point M, then

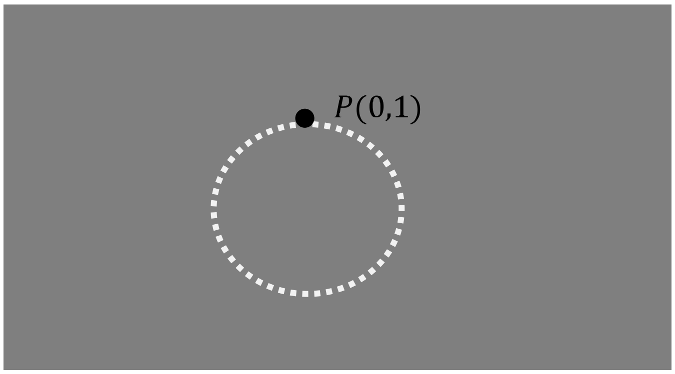
1. $y = mx + (b - ma), \quad m = \frac{\beta}{\alpha}$
2. $x^2 + y^2 = 1$

Therefore, $x^2 + [mx + (b - ma)]^2 = 1$

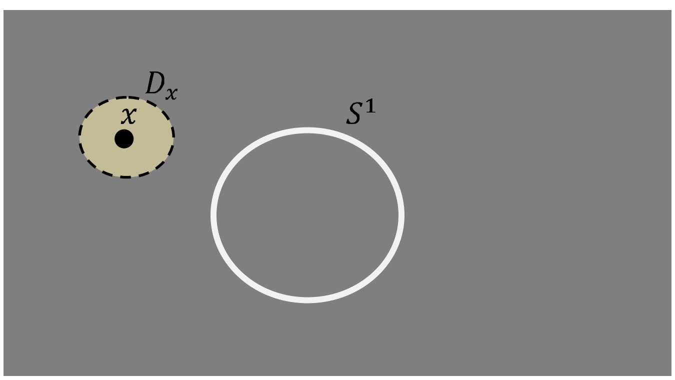
Solving for x and substituting in (1), we obtain the coordinates of the point M. Hence we can calculate the distance d_m between P and M. Thus for each point M, we take the ray of length $d_m/2$

Example

Consider the following subset of R^2. $W = (R^2 - S^1) \cup \{(0,1)\}$, that is we remove from the plane R^2 all points, but (0, 1), of the circleS^1. The set W is radially open but not open with respect to the usual topology.



The complement of an open set is called a closed set. The circle S^1 is a closed subset of the plane R^2. Thus Result 1 shows that given any point P in S^1 we can find a radially open subset of R^2 that contains P and does not intersect the circle at any other point. A subset of R^2 that satisfies the stated property is called a discrete set.



Result 2

If R^2 is endowed with the radially open sets, then S^1 is a closed discrete subset.

The following result tells us how to construct radially open subsets that are not open in the usual sense.

Result 3

If R^2 is endowed with the radially open sets, then any subset of S^1 is a closed subset of R^2.

Proof

Let $W \subseteq S^1$; we show that $R^2 - W$ is an open set. Let x be an arbitrary point in $R^2 - W$.

If $x \notin S^1$, then we select a small disk D_x (hence a radially open set) that contains x and does not intersect S^1.

That is $x \in D_x \subset R^2 - S^1 \subset R^2 - W$.

If $x \in S^1$, then we construct a radially open set G_x that contains x and does not contain any other point of S^1. Note that the intersection of G_x and W is empty since x is in $R^2 - W$; hence $x \in G_x \subset R^2 - W$.

Remark

To construct a radially open set that is not open with the usual topology, it suffices to select a subset W of S^1 that is not closed in R^2 then consider $R^2 - W$.