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Abstract

Let (X, τ) be a topological space. The space X is a D-Space if for every function $f : X \rightarrow \tau_x$ with $x \in X$, there is a closed discrete subset D such that $X = \bigcup f(d)$. The main result of this article shows that a finite T_0 topological space is a D-Space.

1. Introduction

We begin our discussion by defining topological spaces. We are mainly interested in finite topological spaces, that is, a finite set together with a topology defined on it. If X is a set, we denote the set of all subsets of X by $P(X)$ and the empty set by \emptyset .

Definition 1.1

Let X be a non-empty set. A topology on X is a subset τ of $P(X)$ satisfying the following axioms:

1. X and \emptyset belong to τ where \emptyset is the empty set.
2. The intersection of two members of τ is also a member of τ .
3. Arbitrary union of members of τ is also in τ .

If τ is a topology on a set X , then we say that (X, τ) is a topological space. The elements of τ are called open sets and the complement of an open set is a closed set. Moreover, if Y is a subset of X then $\{G \cap Y : G \in \tau\}$ defines a topology on Y . That is, an open set in the subspace Y has the form $G \cap Y$ for some open set in the space

X. Consequently the closed sets in the subspace Y are of the form $F \cap Y$ for some closed set in X.

Example 1.2

Let $X=\{a,b,c\}$ and $\tau=\{X, \mathbf{f}, \{a\}, \{b\}, \{a,b\}\}$. Then the closed sets in the space X are

- $X-\mathbf{f}=\mathbf{f}$
- $X-\mathbf{f}=X$
- $X-\{a\}=\{b,c\}$
- $X-\{b\}=\{a,c\}$
- $X-\{a,b\}=\{c\}$

If we let $Y=\{a,c\}$ then the topology on Y induced by τ is $\tau_y =\{ Y, \mathbf{f}, \{a\}\}$. Observe that $Y-\{a\}=\{c\}$ is closed in Y.

2. Existence of closed points in finite T_0 spaces

Definition 2.1

Let X be a topological space. Then X is called a T_0 – space if and only if for each pair of distinct points $x,y \in X$, there is either an open set containing x but not y or an open set containing y but not x.

Example 2.2

Let $X=\{a,b,c\}$ and $\tau=\{X, \mathbf{f}, \{a\}, \{b\}, \{a,b\}\}$. Then X is a T_0 space.

Definition 2.3

A subset D of X is said to be discrete if every point d in D is of the form $\{d\}=G \cap D$ for some open set G.

Example 2.4

Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$. The set $D = \{b, c\}$ is a discrete subset of X

since $\{b\} = D \cap \{a, b\}$ and $\{c\} = D \cap \{a, c\}$. Note that D is closed as well. Hence the topological space X has a closed discrete subset D .

Now we have the necessary terminology to state and prove our final result.

Theorem 2.5

If X is a finite T_0 space, then there is a point x in X such that $\{x\}$ is a closed set.

Proof

If $|X| = 1$, that is $X = \{y\}$, then the result is true.

Assume that the result is true for $|X| \leq n-1$. Let $X = \{x_1, x_2, \dots, x_n\}$. Let $Y = \{x_1, x_2, \dots, x_{n-1}\}$.

Since Y is a subspace of X and X is T_0 , Y is also T_0 . By induction, there is y in Y such that $\{y\}$ is a closed set in Y . Since $\{y\}$ is closed in Y , $\{y\} = Y \cap F$ for some closed set F in X .

Either $F = \{y\}$ or $F = \{y, x_n\}$. If $F = \{y\}$ then we are done. Hence we may assume that $F = \{y, x_n\}$.

Since the space is T_0 either there is a G_y open such that x_n does not belong to G_y , hence

$(X - G_y) \cap F = \{x_n\}$ is closed in X , or there is a G_{x_n} open such that y does not belong

to it, hence $(X - G_{x_n}) \cap F = \{y\}$ is closed in X .

3. The D-space property for finite topological spaces

In this section we let (X, τ) be a topological space with $X = \{x_1, x_2, \dots, x_n\}$; that is X is a finite topological space.

Definition 3.1

The space X is a D-space if and only if for every function $f: X \rightarrow \tau$, with $x \in f(x)$, there is a closed discrete subset $D = \{d_1, d_2, \dots, d_k\}$ of X such that $X = \bigcup f(d_i)$. The two basic properties, which we shall use in the proof of our main result, are summarized in the following proposition.

Proposition 3.2

1. A closed subset of a D-space is a D-space.
2. If $X=X_1 \cup X_2$ where X_1 is a closed D-space and X_2 is a D-space, the X is a D-space.

Proof: See [2]

Theorem 3.3

Every finite T_0 -space is a D-space.

Proof:

By induction on the number of elements in X denoted by $|X|$.

The result is certainly true for $|X|=1$.

Assume the result is true for any space X with $|X| \leq n-1$.

Let $X=\{x_1, x_2, \dots, x_n\}$. By theorem 2.5 there is a point x such that $\{x\}$ is a closed subset of X . Without any loss of generality we may assume that $x = x_n$.

Now $X=\{x_1, x_2, \dots, x_{n-1}\} \cup \{X=\{x_n\}$. By induction $\{x_1, x_2, \dots, x_{n-1}\}$ is a D-space. The space X is a D-space since it is the union of a D-space and a closed D-space.

Bibliography

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