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Singular Points of Polynomial Equations

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Abstract:
In this paper we study an important class of polynomial equations known as Schubert cycles. They are the result of equating to zero the minors of certain matrices. We provide an algorithm to compute these equations and identify their singular points.

Introduction

The set all 2-dimensional subspaces of R^n is known by the Grassmanian and denoted by $G(2,n)$. Given a 2-dimensional subspace V of R^n, one can identify V with 2x4 matrix. For instance, if V is generated by the vectors $v_1 = (a_1, b_1, c_1, d_1)$ and $v_2 = (a_2, b_2, c_2, d_2)$, then we identify V with the matrix $\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{bmatrix}$. Moreover, since v_1 and v_2 are linearly independent, the plane V can be identified with a matrix whose ith column is the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and jth column is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. In other words, the set $G(2,n)$ can be covered by the following subsets (charts):

$$C(1,1) = \begin{bmatrix} 1 & 0 & x_1 & x_2 & \cdots & x_{n-2} \\ 0 & 1 & y_1 & y_2 & \cdots & y_{n-2} \end{bmatrix}$$

$$C(1,3) = \begin{bmatrix} 1 & x_1 & 0 & x_2 & \cdots & x_{n-2} \\ 0 & y_1 & 1 & y_2 & \cdots & y_{n-2} \end{bmatrix}$$

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$$C(i,j) = \begin{bmatrix} x_* & \cdots & 1 & x_* & \cdots & 0 & \cdots \\ y_* & \cdots & 0 & x_* & \cdots & 1 & \cdots \end{bmatrix} \text{ for } 1 \le i < j \le n.$$

ith column *jth column*

The set $G(2,n)$ is a well-known object in differential geometry and several of its properties are derived from the study of the so-called Schubert cycles. See [2]

1.1 Schubert cycles

We denote by $C(n)$ the matrix whose first two columns form the identity 2x2 matrix and the remaining entries are variables; that is

$$C(n)=\begin{bmatrix}1 & 0 & x_3 & x_4 & \dots & x_n\\ 0 & 1 & y_3 & y_4 & \dots & y_n\end{bmatrix}$$

Definition
For each positive integer m and k with $1 \le m < k \le n$, we define the Schubert cycle $S_{m,k}$ to be the solution set of the following equations:

1. $x_{k+1}=0, x_{k+2}=0,\dots,x_n=0$ and $y_{k+1}=0, y_{k+2}=0,\dots,y_n=0$
2. delete the first m columns of $C(n)$ and equate to zero all the 2x2 determinants.

Example
Let $n=4$. Then there are 6 Schubert cycles:
First note that $C(4)=\begin{bmatrix}1 & 0 & x_3 & x_4\\ 0 & 1 & y_3 & y_4\end{bmatrix}$.

$S_{1,2}=\{(x_3,x_4,y_3,y_4): x_3=0, x_4=0, y_3=0, y_4=0\}=<x_3,x_4,y_3,y_4>$

$S_{1,3}=<x_4,y_4,x_3>$

$S_{1,4}=<x_3,x_4>$

$S_{2,3}=<x_4,y_4>$

$S_{2,4}=<x_3y_4-x_4y_3>$

$S_{3,4}=<\quad>$

We usually ignore the last Schubert cycle since it is defined by the empty set of equations.

Remark
Let K_s be the subspace of R^n generated by the standard vectors $\{e_1,e_2,\dots,e_s\}$. Then the formal definition of a Schubert cycle $S_{m,k}$:

$$S_{m,k}=\{v\in G(2,n): \dim(v\cap K_m)\ge 1 \text{ and } \dim(v\cap K_k)\ge 2\}.$$

Our definition gives an easy to use characterization of Schubert cycles.

Example
Consider the Schubert cycle $S_{1,3}\subset G(2,4)$. Our definition describes the intersection of this Schubert cycle with the chart $C(4)$. Let us use the formal definition of a Schubert cycle to calculate the equations that determine the intersection $S_{1,3}\bigcap C(4)$.

If $v=\begin{bmatrix}1 & 0 & x_3 & x_4\\ 0 & 1 & y_3 & y_4\end{bmatrix}$ is in $S_{1,3}$, then $\dim(v\bigcap K_1)\ge 1$ and $\dim(v\bigcap K_3)\ge 2$.

- Recall that $K_1 = Span\{1,0,...,0\}$. Hence $\dim(v \cap K_1) \geq 1$ if and only if the rank of
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & x_3 & x_4 \\ 0 & 1 & y_3 & y_4 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & x_3 & x_4 \\ 0 & 1 & y_3 & y_4 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & y_3 & y_4 \\ 0 & 0 & x_3 & x_4 \end{bmatrix}$$ is 2. Equivalently all 3x3 determinants must be equal to zero. That is $x_3 = 0$ and $x_4 = 0$.
- Similarly, $\dim(v \cap K_3) \geq 2$ if and only if the rank of the matrix
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & x_3 & x_4 \\ 0 & 1 & y_3 & y_4 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & x_4 \\ 0 & 0 & 0 & y_4 \end{bmatrix}$$ is 3. Equivalently all 4x4 determinants must be zero.

Hence we obtain the equations $x_4 = 0$ and $y_4 = 0$. That is the Schubert cycle $S_{1,3}$ is given by the equations $x_3 = 0$, $x_4 = 0$, and $y_4 = 0$.

For a full discussion of these ideas, see [1].

Definition
The dimension of $S_{m,k}$ is $(m-1)+(k-2)$.

Example

$\dim S_{1,2} = (1-1)+(2-2) = 0$

$\dim S_{1,3} = (1-1)+(3-2) = 1$

$\dim S_{1,4} = (1-1)+(4-2) = 2$

$\dim S_{2,3} = (2-1)+(3-2) = 2$

$\dim S_{2,4} = (2-1)+(4-2) = 3$

Definition $co\dim$
The co-dimension of $S_{m,k}$, $codS_{m,k}$ is $2(n-2) - \dim S_{m,k}$.

For each of the Schubert cycles described above, we can compute the matrix of partial derivatives of its polynomials:

$S_{1,2}$
$$\begin{bmatrix} \frac{dx_3}{dx_3} & \frac{dx_3}{dx_4} & \frac{dx_3}{dy_3} & \frac{dx_3}{dy_4} \\ \frac{dx_4}{dx_3} & \frac{dx_4}{dx_4} & \frac{dx_4}{dy_3} & \frac{dx_4}{dy_4} \\ \frac{dy_3}{dx_3} & \frac{dy_3}{dx_4} & \frac{dy_3}{dy_3} & \frac{dy_3}{dy_4} \\ \frac{dy_4}{dx_3} & \frac{dy_4}{dx_4} & \frac{dy_4}{dy_3} & \frac{dy_4}{dy_4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$S_{1,3}$

$$\begin{bmatrix} \frac{dx_3}{dx_3} & \frac{dx_3}{dx_4} & \frac{dx_3}{dy_3} & \frac{dx_3}{dy_4} \\ \frac{dx_4}{dx_3} & \frac{dx_4}{dx_4} & \frac{dx_4}{dy_3} & \frac{dx_4}{dy_4} \\ \frac{dy_4}{dx_3} & \frac{dy_4}{dx_4} & \frac{dy_4}{dy_3} & \frac{dy_4}{dy_4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$S_{1,4}$

$$\begin{bmatrix} \frac{dx_3}{dx_3} & \frac{dx_3}{dx_4} & \frac{dx_3}{dy_3} & \frac{dx_3}{dy_4} \\ \frac{dx_4}{dx_3} & \frac{dx_4}{dx_4} & \frac{dx_4}{dy_3} & \frac{dx_4}{dy_4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$S_{2,3}$

$$\begin{bmatrix} \frac{dx_4}{dx_3} & \frac{dx_4}{dx_4} & \frac{dx_4}{dy_3} & \frac{dx_4}{dy_4} \\ \frac{dy_4}{dx_3} & \frac{dy_4}{dx_4} & \frac{dy_4}{dy_3} & \frac{dy_4}{dy_4} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$S_{2,4}$

$$\begin{bmatrix} \frac{d(x_3y_4 - x_4y_3)}{dx_3} & \frac{d(x_3y_4 - x_4y_3)}{dx_4} & \frac{d(x_3y_4 - x_4y_3)}{dy_3} & \frac{d(x_3y_4 - x_4y_3)}{dy_4} \end{bmatrix}$$
$$= \begin{bmatrix} y_4 & -y_3 & -x_4 & x_3 \end{bmatrix}$$

We observe that the rank of each matrix is equal to the co-dimension, except for the Schubert cycle $S_{2,4}$ where the rank is zero whenever $(x_3, x_4, y_3, y_4) = (0,0,0,0)$. For this reason we say that the *origin is a singular point*.

If $S_{m,k}$ is a Schubert cycle then it is defined by two types of equations; unless $m = 1$. Let us assume that $m \neq 1$. Then $S_{m,k}$ is defined by:

- Equations of Type I:

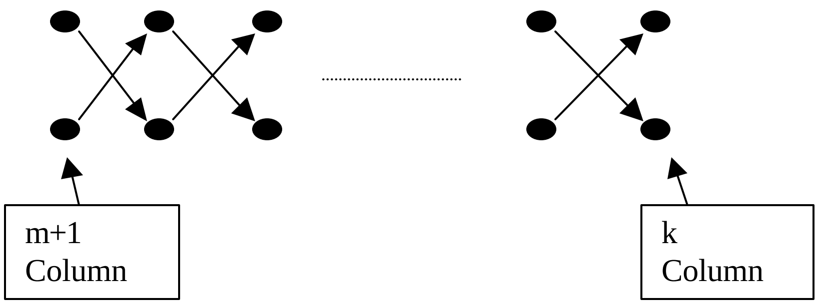
These are the result of deleting the first k columns of $C(n)$ and taking 1x1 determinants:

$$\begin{bmatrix} 1 & 0 & x_3 & x_4 & \cdots & x_m & | & x_{m+1} & \cdots & x_k & | & x_{k+1} & \cdots & x_n \\ 0 & 1 & y_3 & y_4 & \cdots & y_m & | & y_{m+1} & \cdots & y_k & | & y_{k+1} & \cdots & y_n \end{bmatrix}$$

which gives us (2n-2k) independent equations: $x_{k+1}, \ldots, x_n, y_{k+1}, \ldots, y_n$.

- Equations of Type II:

These are the result of deleting the first m columns of $C(n)$ and taking 2x2 determinants. One should observe that there are (k-m-1) independent 2x2 determinants.



Recall that a Schubert cycle is smooth iff the rank of the matrix of partial derivatives is exactly 2(n-2)-dim $S_{m,k}$ = 2n-4-(m+k-3)=2n-m-k-1. Since all the above equations are independent and the derivative of x_i or y_i is either 0 or 1, we see that a Schubert cycle $S_{m,k}$ is smooth iff the rank of the sub-matrix (formed by considering equations of type II) of partial derivatives is (2n-m-k-1)-(2n-2k)= k-m-1 (in other words, we ignore all equations of type I).
Consequently, for $S_{m,k}$ to be smooth k-m-1=0, since otherwise there is an equation of type II and hence a singular point. But k-m-1=0 iff k=m+1.

Result
If $m \neq 1$, $S_{m,k}$ is smooth iff k=m+1.
If m=1, then it is easy to see that all the equations of $S_{1,k}$ are of type I. Hence $S_{1,k}$ is smooth.

Result
$S_{1,k}$ and $S_{m,m+1}$ are the only smooth Schubert cycles.

Example
Consider $G(2,5)$ that is n=5.
The possible Schubert Cycles are as follows:

$S_{1,2} = < x_3, x_4, x_5, y_3, y_4, y_5 >$

$S_{1,3} = < x_3, x_4, x_5, y_4, y_5 >$

$S_{1,4} = < x_3, x_4, x_5, y_5 >$

$S_{1,5} = < x_3, x_4, x_5 >$

$S_{2,3} = < x_4, x_5, y_4, y_5 >$

$S_{2,4} = < x_5, y_5, x_3 y_4 - x_4 y_3 >$

$S_{2,5} = < x_3 y_4 - x_4 y_3, x_3 y_5 - x_5 y_3, x_4 y_5 - x_5 y_4 >$

$S_{3,4} = < x_5, y_5 >$

$S_{3,5} = < x_4 y_5 - x_5 y_4 >$

For each of the Schubert cycles described above, we can compute the matrix of partial derivatives of its polynomials:

$$S_{1,2} : \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S_{1,3} : \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S_{1,4} : \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S_{1,5} : \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$S_{2,3}:\begin{bmatrix}0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}$$

$$S_{2,4}:\begin{bmatrix}0&0&1&0&0&0\\0&0&0&0&0&1\\y_4&-y_3&0&-x_4&x_3&0\end{bmatrix}$$

$$S_{2,5}:\begin{bmatrix}y_4&-y_3&0&-x_4&x_3&0\\y_5&0&-y_3&-x_5&0&x_3\\0&y_5&-y_4&0&-x_5&x_4\end{bmatrix}$$

$$S_{3,4}:\begin{bmatrix}0&0&1&0&0&0\\0&0&0&0&0&1\end{bmatrix}$$

$$S_{3,5}:\begin{bmatrix}0&y_5&-y_4&0&x_5&x_4\end{bmatrix}$$

$S_{4,5}$: *No Solution*

Result

As $S_{1,k}$ and $S_{m,m+1}$ are the only smooth Schubert Cycles, the following are all smooth.

$S_{1,2}$

$S_{1,3}$

$S_{1,4}$

$S_{1,5}$

$S_{2,3}$

$S_{3,4}$

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