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Existence of Distance Functions on Nagata Spaces

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Abstract

A distance function on a set X is a function $d : X \times X \rightarrow R$ such that $d(x, y) = d(y, x)$ for all x, y in X, and $d(x, y) = 0$ if and only if $x = y$. We show if X is a Nagata space, then X has a distance function.

Definition 1

Let X be a set. A topology on X is a collection of subsets of X, denoted by $\Im$, that satisfies the following conditions.

1. X and f are in $\Im$.
2. If A and B are in $\Im$ then the intersection of A and B is also in $\Im$.
3. An arbitrary union of members of $\Im$ is also a member of $\Im$.

Example 1

Let $X = \{a, b, c, d\}$ and $\Im = \{X, f, \{b, c, d\}, \{b, d\}, \{c\}\}$.

Let us verify that $\Im$ is a topology on X.

1. X and the empty set f are in $\Im$.
2. $X \cap \{b, c, d\} = \{b, c, d\}$ is in $\Im$, $f \cap \{b, c, d\} = f$ is in $\Im$. $\{b, c, d\} \cap \{b, d\} = \{b, d\}$ is in $\Im$ and etc.
3. $X \cup f = X$ is in $\Im$, $\{b, c, d\} \cup \{b, d\} = \{b, c, d\}$ is in $\Im$, $\{b, d\} \cup \{c\} = \{b, c, d\}$ is in $\Im$ and any other arbitrary union of members of $\Im$ is a members of $\Im$ as well.

Definition 2.1

If $\Im$ is a topology on *X*, we say $(X,\Im)$ is a topological space. Each member of $\Im$ is called an open set.

Definition 2.2

A set is closed if its complement on *X* is open.

Example 2.2

Let $X=\{a,b,c,d\}$and $\Im=\{X,f,\{b,c,d\},\{b,d\},\{c\}\}$

The set $\{a,c\}$is closed since its complement $X-\{a,c\}=\{b,d\}$is open. Note that X and f are both open and closed.

Definition 3.1

A sequence in the set *X* is a function $f:N\to X$ from the set of natural numbers *N* to the set *X*. If $f(N)=\{a\}$ where $a\in X$, then f is called a constant sequence. If f is injective , then f is called a sequence of distinct points.

Example 3.1

$f(n)=n^2$ for each $n\in N$ is a sequence in R^1.

Definition 3.2

Let X be a topological space and $\{x_n\}$be a sequence in X . The sequence converges to a point “a“ in *X* if and only if given any open set $U(a)$ containing a there exists a natural number n_o such that if $n>n_o$ the point x_n is in $U(a)$.

Example 3.2

Consider the space R^1. The sequence $f(n)=1/n$ for each $n\in N$ in R^1 converges to $a=0$. To see this let U be any open set containing $a=0$.Then there exists a natural number n_o such that $V=\{x\in R^1 : -(1/n_o)<x<1/n_o\}\subset U$ .Thus for $n>n_o$, $f(n)=1/n\in V\subset U$ and hence $f(n)\to 0$.

Example 3.3

The sequence $f(n)=n$ in R^1 does not converge to any point in R^1 because for each $a\in R^1$ the open set $\{x\in R^1 : a-1<a<a+1\}$contains no more than two points of the set $f(n)=N$. Hence f cannot converge to a.

Lemma 3.1

Let $\{y_n\}$be a sequence in the closed set F. If $\{y_n\}$ converges to y then $y\in F$.

Proof

Assume $y\notin F$. Then $y\in X-F$. Since F is closed, its complement $X-F$ is open. Since $\{y_n\}\to y$ and y is in the open set $X-F$, there is an integer n_0 such that for $n\geq n_0$, $y_n\in X-F$. But $\{y_n\}_{n=1}^{\infty}\subset F$. Hence y must be in F.

Definition 4

A space $(X,\Im)$is called a T_1-space if for each pair of distinct points $x,y\in X$, there is an open set containing x but not y.

Lemma 4.1

If X is a T_1-space and $y\in X$,then $\{y\}$ is a closed subset of X.

Proof

We show that $\{y\}$ is closed by showing that $X-\{y\}$ is open.

Let x be any point in $X-\{y\}$. In particular $x\neq y$. Since X is a T_1-space , we can find an open set G_x such that $x\in G_x$ and $y\notin G_x$. Hence $G_x\subseteq X-\{y\}$.

If we do this for every point $x\in X-\{y\}$, we obtain $X-\{y\}=\bigcup_{x\neq y}G_x$.

Being the union of open sets $X-\{y\}$ is open. Hence $\{y\}$ is closed.

Definition 5

A Nagata space is a topological space $(X, \Im)$ together with a function $g : N \times N \to \Im$, where N is the set of positive integers, such that:

1. If x is in X, then x is in $g(n, x)$ for $n = 1,2,...$
2. If $g(n, x) \cap g(n, x_n)$ is not empty for all n in N, then the sequence $\{x_n\}$ converges to x.

Definition 6

A function $d : X \times X \to [0, \infty)$ is a distance function on X if :

1. $d(x, y) = d(y, x)$
2. $d(x, y) = 0$ if and only if $x = y$

Theorem

If X is a Nagata space, then X has a distance function.

Proof.

Let X be a Nagata space with Nagata function g.

For $\forall x, y \in X$, we define $w(x, y) = \{n : g(n, x) \cap g(n, y) \neq f\}$ where $g : N \times X \to \Im$.

Let $d(x, y) = \inf \{1/n+1 : n \in w(x, y)\}$. The function d is a distance function, if it satisfies the conditions mentioned in definition 6.

The first condition $d(x, y) = d(y, x)$ is clear.

For the second part that is $d(x, y) = 0 \Leftrightarrow x = y$, we first assume that $x = y$ and consequently $w(x, y) = w(x, x) = \{n : g(n, x) \cap g(n, x) \neq f\}$.

In other words $w(x, x) = \{1,2,3,...\}$ therefore $d(x, y) = d(x, x) = \inf \{1/n+1 : n \in w(x, x)\}$ $d(x, x) = \inf \{1/2, 1/3, 1/4,...\} \to 0$. Hence $x = y \Rightarrow d(x, y) = 0$.

Now we assume that $d(x, y) = 0$ therefore $d(x, y) = \inf \{1/n+1 : n \in w(x, y)\} = 0$

Since the $\inf \{1/n+1 : n \in w(x, y)\} = 0$, $w(x, y) = \{n : g(n, x) \cap g(n, y) \neq f\}$. That is,

$$g(1,x)\cap g(1,y)\neq f$$
$$g(2,x)\cap g(2,y)\neq f$$
$$g(3,x)\cap g(3,y)\neq f$$
$$.\vdots \qquad\qquad \vdots$$
$$\mathrm{g(n,\ x)}\cap \mathrm{g(n,\ y)}\neq f$$

Since $g(n,x)\cap g(n,y)$ is not empty for all n in N **,** referring to the second part of the Nagata space definition (definition 5), the sequence $\{y_n\}$ converges to x.

We know that the sequence $\{y_n = y\}_{n=1}^{\infty}$ is in the set $\{y\}$. Since the space is T_1, $\{y\}$ is closed by lemma 4.1, then lemma 3.1 implies that $x\in\{y\}$. But the set $\{y\}$ has a single member y. Hence $d(x,y)=0 \Rightarrow x=y$.

Bibliography

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