

Studying Calculus Without Topology

by

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Introduction

In his famous article “How To write Mathematics” P. R. Halmos wrote “Teachers of elementary mathematics in the U.S.A. frequently complain that all calculus books are bad. That is a case in point. Calculus books are bad because there is no such subject as calculus; it is not a subject because it is many subjects. What we call calculus nowadays in the union of a dab of logic and set theory, some axiomatic theory of complete ordered fields, analytic geometry and **topology**, the latter in both the “general” sense (limits and continuous functions) and the algebraic sense (orientation)”.

From studying calculus, we all aware of the importance of knowing when a continuous function attains a maximum or a minimum. For a function of a one variable $f(x)$, the often-stated result is that this happens whenever the function is defined on a closed interval $[a, b]$. The closed interval $[a, b]$ is a special case of a certain class of subsets of R^1 called compact sets. A famous theorem known by the Heine-Borel Theorem asserts that a subset A of R^n is compact if and only if A is closed and bounded. As stated in most of the advanced calculus textbooks, one can show that a continuous function $f : A \rightarrow R$, where A is closed and bounded subset of R^n , attains its maximum and minimum on A .

If f is a differentiable function defined on a nice closed and bounded subset, such as the closed interval $[a, b] \subset R$ or the disk $\{(x, y) : x^2 + y^2 \leq r\}$, there is a procedure for finding the maximum and minimum values of f . We first find the critical points located in the interior of the set, then we study the function along the boundary.

Without knowledge of some basic topological facts, the procedure can be misleading. It gives the impression that we always end up with a finite set of points where we need to

look for the absolute maximum and minimum of f . For instance if we let $A = \{1/n : n = 1, 2, \dots\} \cup \{0\}$, then a continuous function $f : A \rightarrow \mathbb{R}$ has an absolute maximum value $f(a_1)$ and an absolute minimum value $f(a_2)$ where a_1 and a_2 are points in A . To simplify our discussion, we consider the function $f(x) = x$. Since $f'(x) = 1$, the function has no critical points and we deduce that the absolute maximum and minimum values are located on the boundary. But the boundary of A is A itself and we have to investigate the function at every single point of A if we wish to locate the points where the maximum and minimum occur.

Another concept that needs to be clarified is the concept of continuity. When we felt that there is a need to define continuous functions over a closed interval $[a, b]$, we modified our definition to cover continuity at the end points. We simply adjusted the definition by requiring the existence of the limit from the left of b and from the right of a . What about the other types of closed bounded subsets? Is it possible to define continuity without limits?

These types of questions suggest that something is missing from our study of calculus.

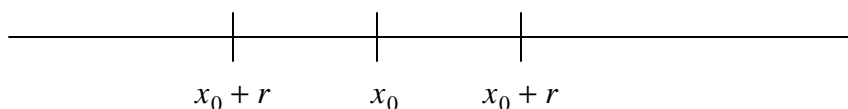
Distance Functions

If x and y are two real numbers, the distance between x and y is simply $|x - y|$ that is, the absolute value of $|x - y|$. In general, if $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ are points in \mathbb{R}^n we define the distance between x and y by $\|x - y\| = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$.

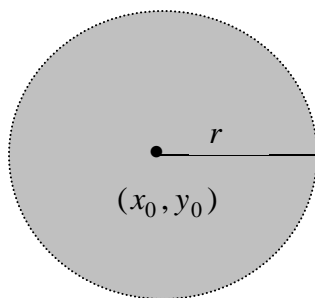
If $x_0 \in \mathbb{R}^n$ and r is a real number, we define $B(x_0, r) = \{x \in \mathbb{R}^n : \|x - x_0\| < r\}$. We call the set $B(x_0, r)$ an open ball with center x_0 and radius r .

Examples

If $n = 1$, then $B(x_0, r) = \{x \in \mathbb{R} : |x - x_0| < r\} = (x_0 - r, x_0 + r)$.



If $n = 2$, we let $X_0 = (x_0, y_0)$ and $X = (x, y)$, then $B(X_0, r) = \{X \in \mathbb{R}^2 : \|X - X_0\| < r\}$.



If $n = 3$, then the open ball is the set of points (x, y, z) in R^3 such that $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < r^2$.

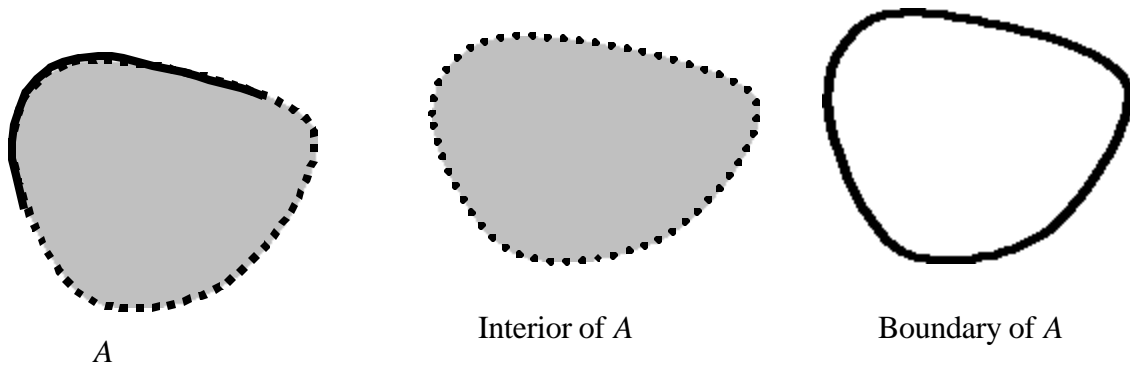
Open Sets and Closed sets

A subset $G \subset R^n$ is open if and only if for each $x \in G$ we can find a real number r such that $B(x, r) \subset G$. It follows that a subset of R^n is open if and only if it is the union of some open balls. The complement of an open set is a closed set. Therefore a set is closed if and only if its complement is open.

By definition, an open interval is an open set and an arbitrary union of open intervals is also an open set. Similarly an open disk in R^2 is an open set and an arbitrary union of open disks is also an open set in R^2 .

Interior and Boundary of a Set

If A is an arbitrary subset of R^n , then a point $x \in R^n$ is an interior point if and only if there is an open ball $B(x, r)$ such that $B(x, r) \subset A$. The union of all interior points is simply the interior of A . On the other hand a point $x \in R^n$ is a boundary point if and only if for any open ball $B(x, r)$ we have $B(x, r) \cap A \neq \emptyset$ and $B(x, r) \cap (R^n - A) \neq \emptyset$. The boundary of A is simply the union of all boundary points.

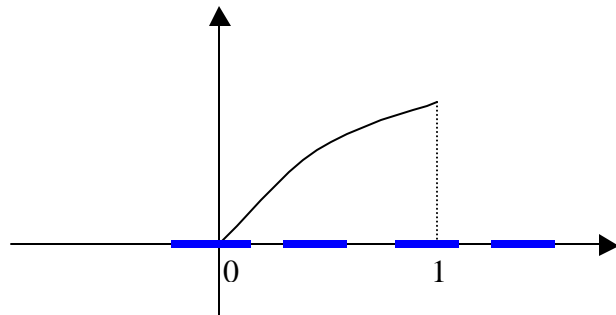


Continuous Functions

Now we are in a position to define continuity without limits. Let A be a subset of R^n and $f : A \rightarrow R$. The function f is continuous if and only if for every open interval I in R there is some open set V in R^n such that $f^{-1}(I) = V \cap A$.

Let us illustrate this definition by examining a classic example of a continuity of a function over a closed interval. Let $f : [0,1] \rightarrow R$ be the function $f(x) = \sqrt{x}$ and let $I = (a, b)$ be an arbitrary open interval in R . Note that there are five possibilities:

- $f^{-1}(I) = \mathbf{\emptyset}$
- $f^{-1}(I) = (a^2, b^2), 0 < a < b < 1$
- $f^{-1}(I) = (a^2, 1], 0 < a, \text{ and } b > 1$
- $f^{-1}(I) = [0, b^2), a < 0 < b < 1.$
- $f^{-1}(I) = [0, 1], a < 0 \text{ and } b > 1.$



In the first case we can take $V = \mathbf{\emptyset}$ and in the second case we let $V = (a^2, b^2)$. For the remaining cases we can take V to be $(a^2, 1+k)$, $(-k, b^2)$, and $(-k, 1+k)$ respectively, where k is any positive real number.

Critical Points

Let f be a differentiable function defined on a set U , and P is a point in U .

We say P is a *critical point* (or *stationary point*) of f if all the partial derivatives of f are 0 at P , or one of these partial derivatives does not exist.

In three variables, the point $P(x_1, x_2, x_3)$ is a critical point if and only if

$$D_1f(x_1, x_2, x_3) = 0, D_2f(x_1, x_2, x_3) = 0, D_3f(x_1, x_2, x_3) = 0$$

In other words, the three partial derivatives must be equal to zero when evaluated at the point $P = (x_1, x_2, x_3)$.

In n variables, the point $P(x_1, x_2, \dots, x_n)$ is a critical point if and only if $D_1f(P) = 0, \dots, D_nf(P) = 0$. Or more concisely $\nabla f(P) = 0$.

Maximum and Minimum of a Continuous Function

A function $f(x_1, x_2, \dots, x_n)$ has a local maximum at $a = (a_1, a_2, \dots, a_n)$ if and only if

$$f(x_1, x_2, \dots, x_n) \leq f(a_1, a_2, \dots, a_n) \text{ for all points } (x_1, x_2, \dots, x_n) \text{ in some open ball } B(a, r).$$

A function $f(x_1, x_2, \dots, x_n)$ has a local minimum at $a = (a_1, a_2, \dots, a_n)$ if and only if

$$f(x_1, x_2, \dots, x_n) \geq f(a_1, a_2, \dots, a_n) \text{ for all points } (x_1, x_2, \dots, x_n) \text{ in some open ball } B(a, r).$$

A function $f(x_1, x_2, \dots, x_n)$ has an absolute maximum at $a = (a_1, a_2, \dots, a_n)$ if and only if

$$f(x_1, x_2, \dots, x_n) \leq f(a_1, a_2, \dots, a_n) \text{ for all points } (x_1, x_2, \dots, x_n) \text{ in the domain.}$$

A function $f(x_1, x_2, \dots, x_n)$ has an absolute minimum at $a = (a_1, a_2, \dots, a_n)$ if and only if

$$f(x_1, x_2, \dots, x_n) \geq f(a_1, a_2, \dots, a_n) \text{ for all points } (x_1, x_2, \dots, x_n) \text{ in the domain.}$$

Result

If f has an absolute maximum or an absolute minimum at a , and a is not a boundary point, then f has a local maximum/ minimum at a .

Proof

Assume that f has an absolute maximum at a . Then $f(a_1, \dots, a_n) \geq f(x_1, \dots, x_n)$ for all $x = (x_1, x_2, \dots, x_n)$ in the domain D of f . If $a = (a_1, a_2, \dots, a_n)$ is not a boundary point, then there is an open ball $B(a, r)$ that is contained in the domain D . By definition f has a local maximum at $a = (a_1, a_2, \dots, a_n)$.

No one is claiming that the above result is a fundamental finding, it is just an example to illustrate the importance of topology in the study of calculus. The point is topology makes the study of calculus precise and gives a sense how to generalize elementary concepts to higher dimensional spaces and even abstract spaces.

Calculus textbooks made us feel that calculus I is the study of functions over an interval and calculus III is the study of functions over a disk. After studying basic topological concepts, I realized that calculus courses ought to be preceded by a brief topology course where concepts such as open and closed sets, boundary, metric spaces, connected and simply connected spaces, and of course continuity are introduced.

Bibliography

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