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Abstract

In this paper we approximate the series $\sum_{n=1}^{\infty} c_n a_n$, $c_n = 4n^2 \sum_{n=m}^{\infty} \frac{1}{m(m+1)^2}$. The approximation of such a series is related to the study of Carleman’s inequality, which has found many applications in mathematics. In particular we show that $\sum_{n=1}^{\infty} c_n \le 2 \sum_{n=1}^{\infty} \frac{3n}{3n+1} a_n$ and discuss the strength of the approximation.

The following Carleman’s inequality is well known

$$(*) \qquad \sum_{n=1}^{\infty} \left(\frac{a_1^{1/p} + a_2^{1/p} + \ldots + a_n^{1/p}}{n} \right)^p \le \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n$$

where $\{a_n\}$ is a non-negative sequence such that $\sum_{n=1}^{\infty} a_n < \infty$, and $p > 1$.

We are interested in the case where $p = -1$ and one can ask the question whether (*) remains true in this case; that is

$$(**) \qquad \sum_{n=1}^{\infty} \frac{n}{1/a_1 + 1/a_2 + \ldots + 1/a_n} \le 2 \sum_{n=1}^{\infty} a_n$$

We shall answer this question positively and strengthen the inequality (**).

In [1] the authors developed a technique to study Carleman’s inequality for negative power number and later on achieved the following inequality:

$$\sum_{n=1}^{\infty}\frac{n}{1/a_1+1/a_2+\ldots+1/a_n}\le\sum_{n=1}^{\infty}c_n a_n \quad \text{where} \quad c_n=4n^2\sum_{m=n}^{\infty}\frac{1}{m(m+1)^2}.$$

We observe that

$$\frac{1}{m(m+1)^2}=\frac{1}{2}\left(\frac{2m}{m^2(m+1)^2}\right)\le\frac{1}{2}\left(\frac{2m+1}{m^2(m+1)^2}\right)=\frac{1}{2}\left(\frac{1}{m^2}-\frac{1}{(m+1)^2}\right)$$

and consequently,

$$\begin{aligned}c_n&=4n^2\sum_{m=n}^{\infty}\frac{1}{m(m+1)^2}\\&\le 4n^2\,\frac{1}{2}\sum_{m=n}^{\infty}\left(\frac{1}{m^2}-\frac{1}{(m+1)^2}\right)\\&=2n^2\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}+\frac{1}{(n+1)^2}-\frac{1}{(n+2)^2}\ldots\right)\\&=2\end{aligned}$$

that is $c_n\le 2$ and hence $\sum_{n=1}^{\infty}\frac{n}{1/a_1+1/a_2+\ldots+1/a_n}\le 2\sum_{n=1}^{\infty}a_n$. That is the Carleman's inequality holds for $p=-1$.

In general, if for some positive numbers *a* and *b*

$$\frac{1}{m(m+1)^2}\le\frac{1}{am^2+bm}-\frac{1}{a(m+1)^2+b(m+1)}$$

then

$$\begin{aligned}c_n&=4n^2\sum_{m=n}^{\infty}\frac{1}{m(m+1)^2}\\&\le 4n^2\sum_{m=n}^{\infty}\left(\frac{1}{am^2+bm}-\frac{1}{a(m+1)^2+b(m+1)}\right)\\&=4n^2\left(\frac{1}{an^2+bn}-\frac{1}{a(n+1)^2+b(n+1)}+\frac{1}{a(n+1)^2+b(n+1)}-\ldots\right)\\&=\frac{4n^2}{an^2+bn}\\&=\frac{4n}{an+b}\end{aligned}$$

that is

$$(***) \qquad c_n \le \frac{4n}{an+b}$$

and hence

$$\sum_{n=1}^{\infty} \frac{n}{1/a_1 + 1/a_2 + \ldots + 1/a_n} \le 2\sum_{n=1}^{\infty} \frac{2n}{an+b} a_n$$

Consider the two inequalities:

(1) $$\sum_{n=1}^{\infty} \frac{n}{1/a_1 + 1/a_2 + \ldots + 1/a_n} \le 2\sum_{n=1}^{\infty} a_n$$

(2) $$\sum_{n=1}^{\infty} \frac{n}{1/a_1 + 1/a_2 + \ldots + 1/a_n} \le 2\sum_{n=1}^{\infty} \frac{2n}{an+b} a_n$$

Remark 1.

The second inequality gives a better approximation provided that $\frac{2n}{an+b} < 1$ for all n and this holds whenever $a \ge 2$.

Lemma

The inequality $\frac{1}{m(m+1)^2} \le \frac{1}{am^2+bm} - \frac{1}{a(m+1)^2 + b(m+1)}$ is equivalent to

$$\frac{a(a-2)m^2 + \left[(2a-1)b + a(a-3)\right]m + (b-1)(b+a)}{m(m+1)^2(am+b)(am+b+a)} \le 0.$$

We observe that the denominator of $\frac{a(a-2)m^2 + \left[(2a-1)b + a(a-3)\right]m + (b-1)(b+a)}{m(m+1)^2(am+b)(am+b+a)}$ is always positive. Consequently $a(a-2)m^2 + \left[(2a-1)b + a(a-3)\right]m + (b-1)(b+a) \le 0$ for all m and this is possible only if $0 \le a \le 2$. In view of Remark 1, we deduce that $a = 2$.

Remark 2.

The second inequality gives a better approximation provided that $\frac{2n}{an+b} = \frac{2n}{2n+b}$ and the larger the value of b the better the approximation is.

Substituting 2 for a in the inequality

$$a(a-2)m^2+\left[(2a-1)b+a(a-3)\right]m+(b-1)(b+a)\leq 0,\ \text{for all } m$$

We obtain

$$(3b-2)m+(b-1)(b+2)\leq 0, \text{for all } m$$

and this is possible only if $3b-2\leq 0$ or $b\leq 2/3$.

In view of Remark 2, $b=2/3$ yields the best approximation of c_n.

Conclusion

From (***) we have

$$c_n \leq \frac{4n}{an+b}$$

or,

$$c_n = 4n^2\sum_{m=n}^{\infty}\frac{1}{m(m+1)^2}\leq\frac{4n}{2n+2/3}$$

or,

$$c_n = 4n^2\sum_{m=n}^{\infty}\frac{1}{m(m+1)^2}\leq\frac{6n}{3n+1}$$

How good our approximation is?

We shall answer this question by comparing c_n and $\dfrac{6n}{3n+1}$ for certain values of n.

First, we note that

$$\begin{aligned}
c_n &= 4n^2\sum_{m=n}^{\infty}\frac{1}{m(m+1)^2}\\
&= 4n^2\left(\sum_{m=1}^{\infty}\frac{1}{m(m+1)^2}-\sum_{m=1}^{n-1}\frac{1}{m(m+1)^2}\right)\\
&= 4n^2\left(\sum_{m=1}^{\infty}\left(\frac{1}{m(m+1)}-\frac{1}{(m+1)^2}\right)-\sum_{m=1}^{n-1}\frac{1}{m(m+1)^2}\right)\\
&= 4n^2\left(1-(\pi^2/6-1)-\sum_{m=1}^{n-1}\frac{1}{m(m+1)^2}\right)\\
&= 4n^2\left(2-\pi^2/6-\sum_{m=1}^{n-1}\frac{1}{m(m+1)^2}\right)
\end{aligned}$$

n	c_n	$\frac{6n}{3n+1}$
1	1.4202637	1.5
2	1.6810549	1.7143
3	1.7823736	1.8
4	1.8353308	1.8462
5	1.8677044	1.875
6	1.8894944	1.8947
7	1.9051451	1.9091
8	1.9169242	1.92
9	1.9261072	1.9286
10	1.9334657	1.9355
20	1.9666833	1.9672
30	1.9777827	1.978
50	1.9866677	1.9868
70	1.9904766	1.9905
90	1.9925928	1.9926
150	1.9955556	1.9956

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