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Using Maple to Calculate Sums of Fractional Functions

by

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Consider the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} + \ldots \qquad (*)$$

We observe that

$$\frac{1}{n^2} \le \frac{1}{n-1} - \frac{1}{n} \quad (\text{for } n \ge 2)$$

which yields

$$\sum_{n=1}^{N} \frac{1}{n^2} \le 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) \ldots + \left(\frac{1}{N-1} - \frac{1}{N}\right) = 2 - \frac{1}{N}$$

and as *n* approaches infinity we obtain a well known result

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \le 2 \qquad (**)$$

That is, we approximated $(*)$ by decomposing the nth terms of the series as

$$\frac{1}{n^2} = u_n - u_{n+1}$$

for some positive sequence $\{u_n\}$. Hence the Nth partial sum of the series can be written in the form

$$\sum_{n=1}^{N} \frac{1}{n^2} = (u_1 - u_2) + (u_2 - u_3) + (u_3 - u_4) + \ldots + (u_N - u_{N+1}) = u_1 - u_{N+1}$$

and if

$$u_n \rightarrow 0$$

then

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = u_1 .$$

Now consider the sum

$$\sum_{n=1}^{\infty}\frac{1}{n^2+3n+1}=\frac{1}{5}+\frac{1}{11}+\frac{1}{19}+\dots$$

We shall use this series to illustrate how one can use Maple to find a good approximation for a series by finding a convergent sequence $u_n=\dfrac{an+b}{n^2+cn+d}$ such that

$$\frac{1}{n^2+3n+1}\le u_n-u_{n+1}$$

At the command line we define a function

```
> u:=n->(a*n+b)/(n^2+c*n+d);
```

$$u := n \to \frac{a\,n+b}{n^2+c\,n+d}$$

Then we define the difference

```
> h:=n->u(n)-u(n+1)-1/(n^2+3*n+1);
```

$$h := n \to u(n) - u(n+1) - \frac{1}{n^2+3\,n+1}$$

```
> h(n);
```

$$\frac{a\,n+b}{n^2+c\,n+d}-\frac{a\,(n+1)+b}{(n+1)^2+c\,(n+1)+d}-\frac{1}{n^2+3\,n+1}$$

We will compute a, b, c, d so that h(n) is as small as possible but positive.
But first we simplify the difference

```
> simplify(%);
```

$$\frac{1}{(n^2+c\,n+d)\,(n^2+2\,n+1+c\,n+c+d)\,(n^2+3\,n+1)}(a\,n-c\,n+b-d-n^2+4\,a\,n^3+4\,a\,n^2+7\,b\,n^2+5\,b\,n+b\,c+a\,n^4+2\,b\,n^3-a\,d-3\,a\,n\,d+3\,b\,c\,n-a\,n^2\,d+b\,c\,n^2-n^4-2\,n^3-2\,c\,n\,d-2\,n^3\,c-3\,n^2\,c-2\,n^2\,d-c^2\,n^2-c^2\,n-2\,d\,n-d\,c-d^2)$$

We are interested in the numerator

```
> numer(%);
```

$$a\,n-c\,n+b-d-n^2+4\,a\,n^3+4\,a\,n^2+7\,b\,n^2+5\,b\,n+b\,c+a\,n^4+2\,b\,n^3-a\,d-3\,a\,n\,d+3\,b\,c\,n-a\,n^2\,d+b\,c\,n^2-n^4-2\,n^3-2\,c\,n\,d-2\,n^3\,c-3\,n^2\,c-2\,n^2\,d-c^2\,n^2-c^2\,n-2\,d\,n-d\,c-d^2$$

```
> collect(%,n);
```

$$(a-1)n^4+(-2c+4a+2b-2)n^3+(-1+7b+bc-3c-c^2-ad-2d+4a)n^2$$
$$+(-2dc-c^2+5b+a-c+3bc-2d-3ad)n-dc-d^2+b-d-ad+bc$$

We set a equal to 1

```
> subs(a=1,%);
```

$$(-2c+2+2b)n^3+(3+7b+bc-3c-c^2-3d)n^2+(-2dc-c^2+5b+1-c+3bc-5d)n-dc$$
$$-d^2+b-2d+bc$$

We set b equal to c-1

```
> subs(b=c-1,%);
```

$$-1+(-4+4c+(c-1)c-c^2-3d)n^2+(-2dc-c^2+4c-4+3(c-1)c-5d)n-dc-d^2+c-2d$$
$$+(c-1)c$$

```
> simplify(%);
```

$$-1-4n^2+3n^2c-3n^2d-2cnd+2c^2n+cn-4n-5dn-dc-d^2-2d+c^2$$

```
> collect(%,n);
```

$$(-4+3c-3d)n^2+(-2dc+2c^2+c-4-5d)n-1-dc-d^2-2d+c^2$$

We set c equal to d+4/3

```
> subs(c=d+4/3,%);
```

$$-1+\left(-2d\left(d+\frac{4}{3}\right)+2\left(d+\frac{4}{3}\right)^2-4d-\frac{8}{3}\right)n-d\left(d+\frac{4}{3}\right)-d^2-2d+\left(d+\frac{4}{3}\right)^2$$

```
> simplify(%);
```

$$\frac{7}{9}-\frac{4}{3}dn+\frac{8}{9}n-d^2-\frac{2}{3}d$$

```
> collect(%,n);
```

$$\left(-\frac{4}{3}d+\frac{8}{9}\right)n+\frac{7}{9}-d^2-\frac{2}{3}d$$

Intuitively we choose d = 2/3

```
> subs(d=2/3,%);
```

$$\frac{-1}{9}$$

But then the difference h(n) will be negative! So we choose d a little smaller. Let’s try…

```
> for i from .57 by 0.01 to .67 do
```

```
        print(subs(d=i,f))
    end do;
>
```

$$0.1288888889\,n + 0.0728777778$$

$$0.1155555556\,n + 0.0547111111$$

$$0.1022222223\,n + 0.0363444445$$

$$0.0888888889\,n + 0.0177777778$$

$$0.0755555556\,n - 0.0009888889$$

$$0.0622222223\,n - 0.0199555555$$

$$0.0488888889\,n - 0.0391222222$$

$$0.0355555556\,n - 0.0584888889$$

$$0.0222222223\,n - 0.0780555555$$

$$0.0088888889\,n - 0.0978222222$$

$$-0.0044444444\,n - 0.1177888889$$

For h(n) ≥ 0 we can choose d = 0.6. Then

```
> u(n);
```

$$\frac{n + \frac{14}{15}}{n^2 + \frac{29}{15}n + \frac{3}{5}}$$

```
> simplify(%);
```

$$\frac{15\,n + 14}{15\,n^2 + 29\,n + 9}$$

```
> h(n);
```

$$\frac{n + \frac{14}{15}}{n^2 + \frac{29}{15}n + \frac{3}{5}} - \frac{n + \frac{29}{15}}{(n+1)^2 + \frac{29}{15}n + \frac{38}{15}} - \frac{1}{n^2 + 3\,n + 1}$$

```
> simplify(%);
```

$$\frac{4\,(5\,n + 1)}{(15\,n^2 + 29\,n + 9)\,(15\,n^2 + 59\,n + 53)\,(n^2 + 3\,n + 1)}$$

So we approximate

$$\frac{1}{n^2+3n+1} \le \frac{15n+14}{15n^2+29n+9} - \frac{15(n+1)+14}{15(n+1)^2+29(n+1)+9}$$

$$\sum_{n=1}^{N} \frac{1}{n^2+3n+1} \le \frac{15(1)+14}{15(1)^2+29(1)+9} - \frac{15(N+1)+14}{15(N+1)^2+29(N+1)+9}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2+3n+1} \le \frac{15(1)+14}{15(1)^2+29(1)+9} = \frac{29}{53}$$