

# Module 4

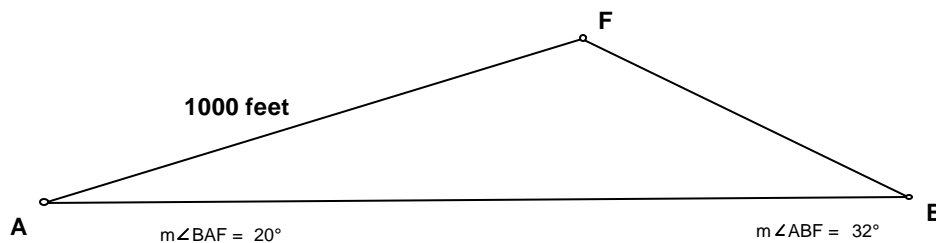
Atul Roy  
HT 222

## SOLVING ANY TRIANGLE

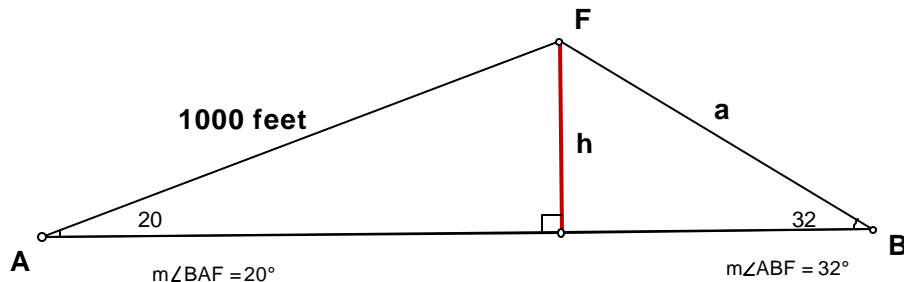
So far, you learnt the methods of solving right triangles. In this section, we are going to extend those methods of solving any triangle.

Instead of looking at the formulas directly, let us look at the examples that lead us towards those formulas. In case, you wish to go to the formulas directly, look at the equation (1) on the page 3.

Example 1: Alisha, Bob, and Frank live in a neighborhood. The distance between the houses of Alisha and Frank can be measured easily on the ground but the distance between Bob and Frank can not be easily measured.



Note that the problem would have been easily solvable, if we had a right triangle. Let us break the above triangle into two right triangles



Note that

$$\frac{h}{1000} = \sin 20^\circ \rightarrow h = 1000 \sin 20^\circ \cong 342 \text{ feet}$$

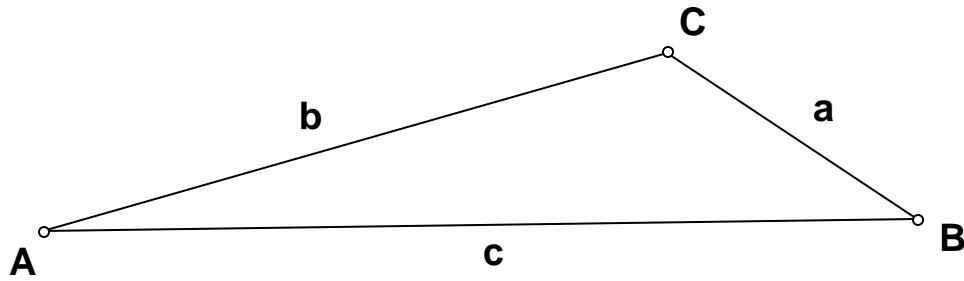
but would like the value of a

From the right triangle  $\triangle BNF$  note that

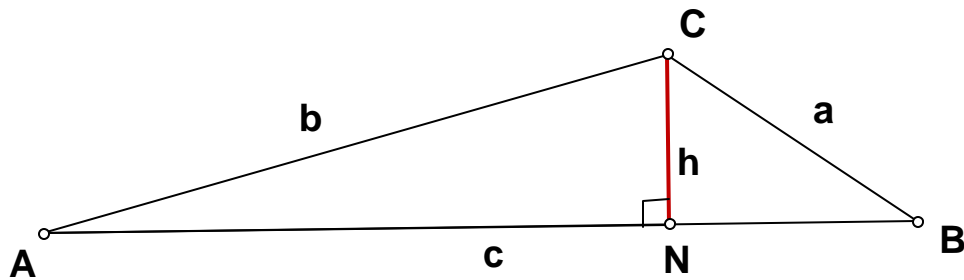
$$\frac{h}{a} = \sin 32^\circ \rightarrow a \sin 32^\circ = h \rightarrow a \sin 32^\circ = 342 \rightarrow a = \frac{342}{\sin 32^\circ} \cong 645.4 \text{ feet}$$

Therefore the distance between the houses of Bob and Frank is approximately 645.5 feet.

Let us use the above calculations to obtain a general formula for solving triangles. Remember our notations for the angles and the sides of a triangle say that the length of the side opposite to the angle A is a, angle B is b, and the angle C is c as shown in the following picture.



To write an equation giving us relation between these values, look at the diagram obtained by dropping a perpendicular from C to the segment AB.



From the right  $\triangle ANC$  note that  $\frac{h}{b} = \sin A$  or  $h = b \sin A$

From the right  $\triangle BNC$  note that  $\frac{h}{a} = \sin B$  or  $h = a \sin B$

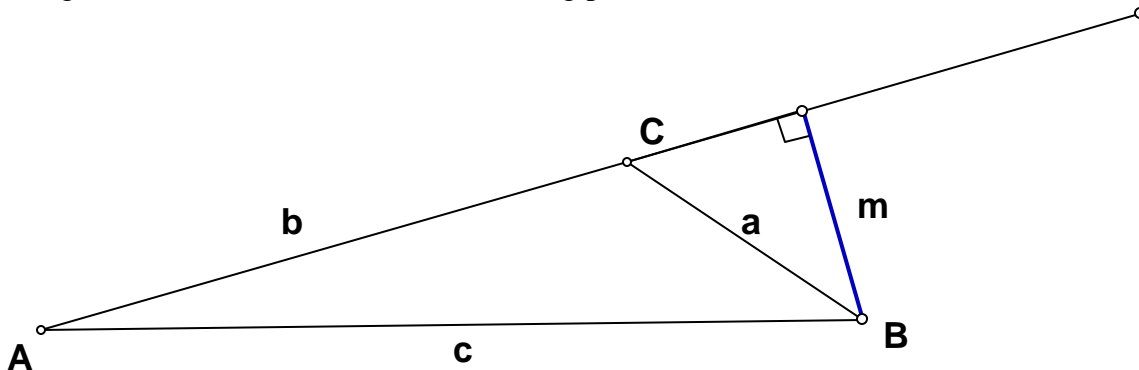
Therefore,

$$b \sin A = a \sin B$$

$$\frac{b \sin A}{ab} = \frac{a \sin B}{ab} \quad : \text{ divide both the sides by } ab$$

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

Using similar calculations with the following picture



we may obtain

$$\frac{\sin A}{a} = \frac{\sin C}{c} \text{ which together with } \frac{\sin A}{a} = \frac{\sin B}{b}$$

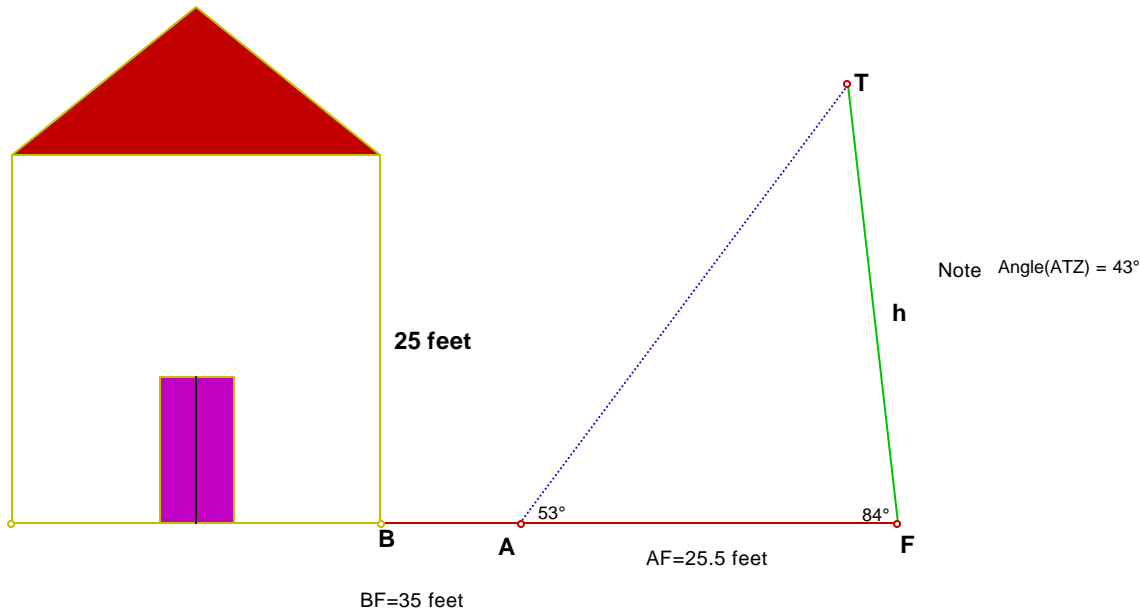
gives, the equation called the Law of Sines

$$\boxed{\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}} \text{ or } \boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}} \dots \text{equation(1)}$$

Note that for a triangle of non zero area, all the denominators involved above will be nonzero quantities.

Let us use this Law of Sines to work on the following examples.

Exercise 1: Suppose there is a tiled tree close to Amol's house. Amol's religious beliefs and the county ordinances do not allow cutting of this tree. Amol does the measurements shown in the following picture. Use the information to find the length of the tree.

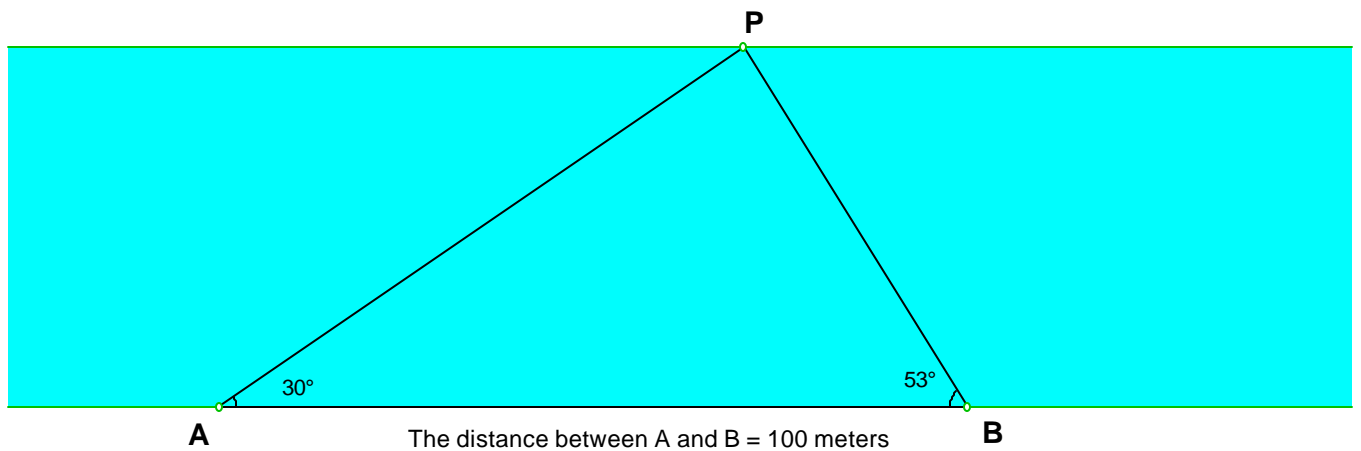


Note that  $\frac{h}{\sin 53^\circ} = \frac{25.5}{\sin 43^\circ}$

$$\rightarrow h = \frac{25.5}{\sin 43^\circ} \sin 53^\circ \cong 29.86 \text{ feet the answer}$$

**Amol does not have to worry about the tree falling on his house.**

**Example 2: Two countries that can not get along have been shelling mortars at each other for the last ten years. For the part of their common border there is a river running between them. One side spots a new enemy post P on the other side of the river. The soldier quickly measures the angles indicated in the picture from the spots A and B. Find the lengths of both AP and BP.**



First, note that  $\angle P = 180^\circ - 30^\circ - 53^\circ$

$$\rightarrow \angle P \cong 97^\circ$$

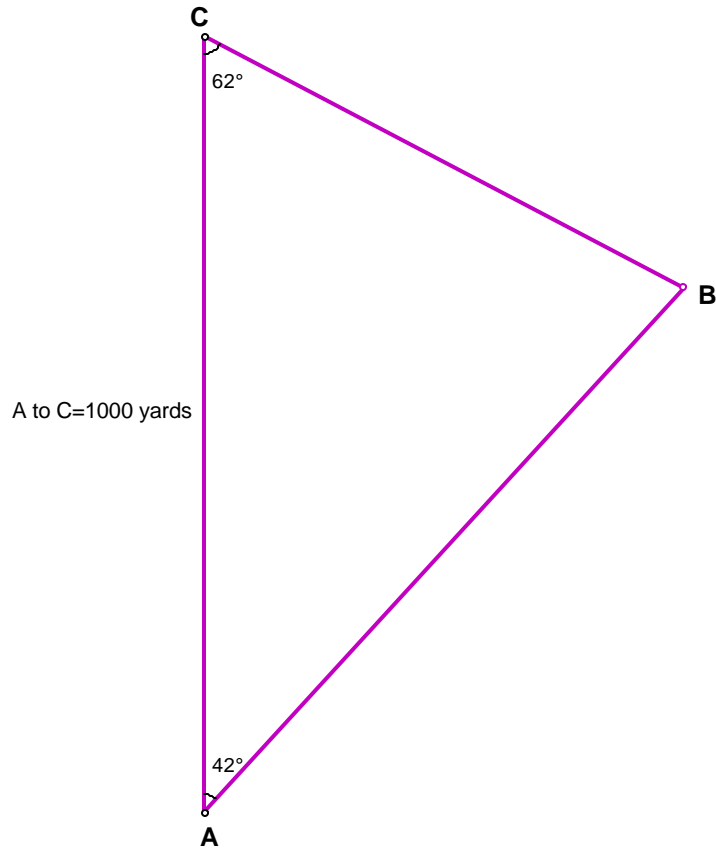
$$\frac{|AP|}{\sin 53^\circ} = \frac{|AB|}{\sin 97^\circ} \rightarrow |AP| = \frac{100}{\sin 97^\circ} \cdot \sin 53^\circ$$

$$\rightarrow |AP| \cong 80.46 \text{ meters}$$

$$\frac{|BP|}{\sin 30^\circ} = \frac{|AB|}{\sin 97^\circ} \rightarrow |BP| = \frac{100}{\sin 97^\circ} \cdot \sin 30^\circ$$

$$\rightarrow |BP| \cong 50.38 \text{ meters}$$

Example 3. Jessica ran from the point A to B then B to C as shown in the following picture. Find the total distance that Jessica ran.



First, Let us note that  $\angle C = 180^\circ - 62^\circ - 42^\circ$

$$\angle C = 76^\circ$$

We would like to find  $|AB| + |BC|$

$$\frac{|AB|}{\sin 62^\circ} = \frac{|AC|}{\sin 76^\circ} \rightarrow |AB| = \frac{1000}{\sin 76^\circ} \cdot \sin 62^\circ$$

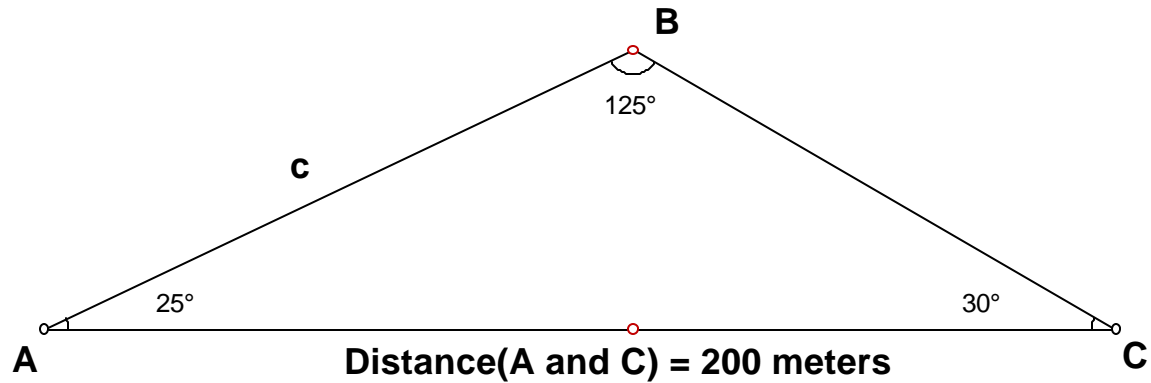
$$\rightarrow |AB| \cong 909.98 \text{ yards}$$

$$\frac{|BC|}{\sin 42^\circ} = \frac{|AC|}{\sin 76^\circ} \rightarrow |BC| = \frac{1000}{\sin 76^\circ} \cdot \sin 42^\circ$$

$$\rightarrow |BC| \cong 689.62 \text{ yards}$$

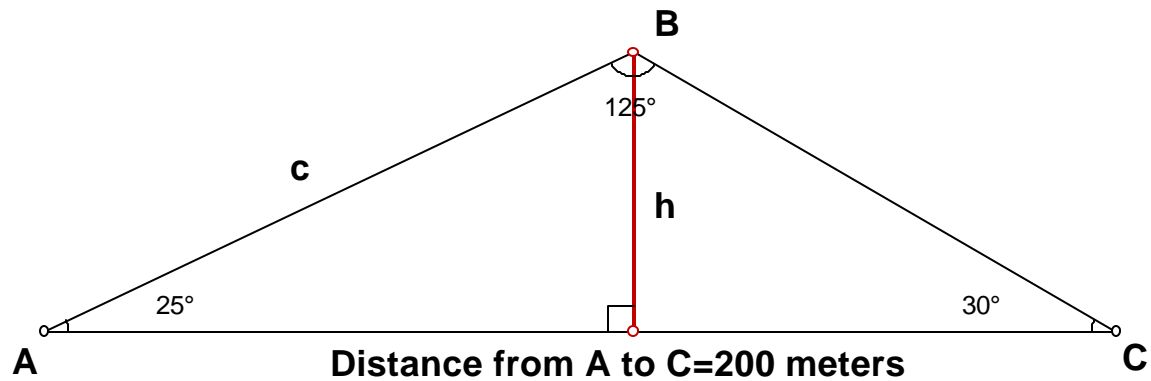
$$|AB| + |BC| \cong 909.98 + 689.62 = 1559.6 \text{ yards}$$

Example 4: Ahmed is standing at the point A and Charles is standing at the point C trying to estimate the height of the balloon B above the ground at a certain time. Find the height of the balloon with the information that is given to you in the following picture.

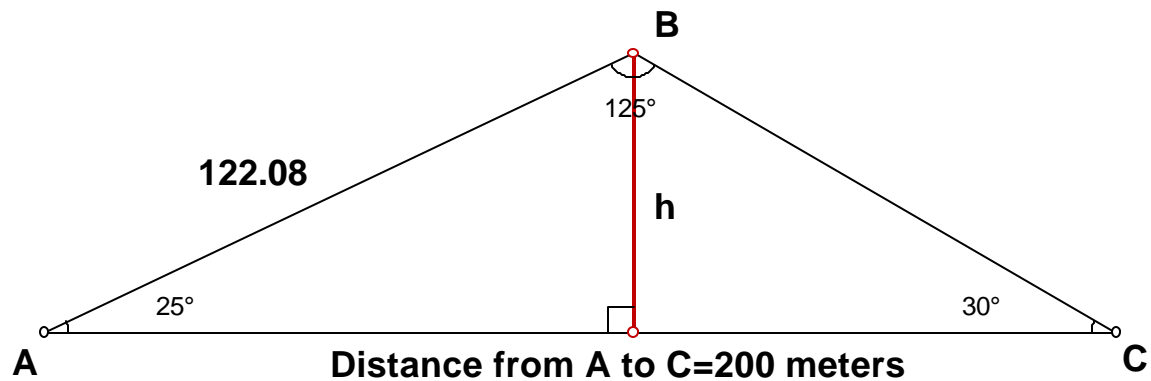


Let us look at the following picture.

$$\frac{c}{\sin 30^\circ} = \frac{200}{\sin 125^\circ} \rightarrow c = \frac{200 \sin 30^\circ}{\sin 125^\circ} \cong 122.08 \text{ meters}$$



We want the value of h.



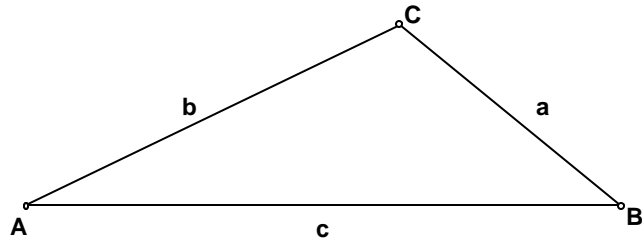
We want the value of h,

$$\frac{h}{122.08} = \sin 25^\circ \rightarrow h = 122.08 \sin 25^\circ \cong 51.59 \text{ meters } \text{The answer}$$

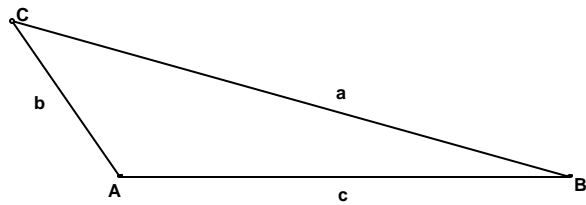
Upcoming Attractions: When you take calculus, you will determine the rate the balloon is rising by using the rate of change of its angle of elevation.

## Solving Triangles

Remember the following notations for the vertices and the lengths of the sides of a triangle.

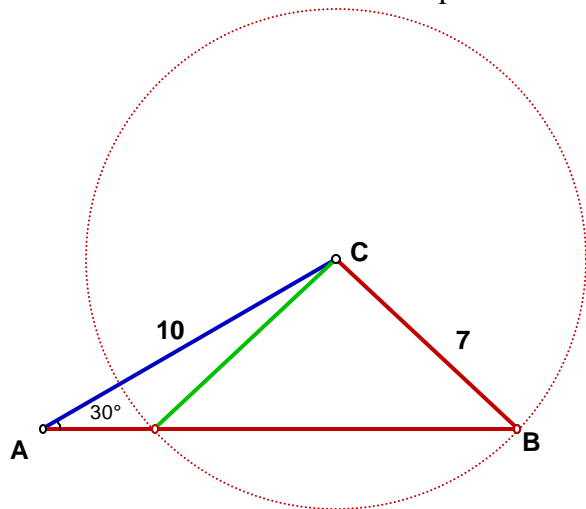


or

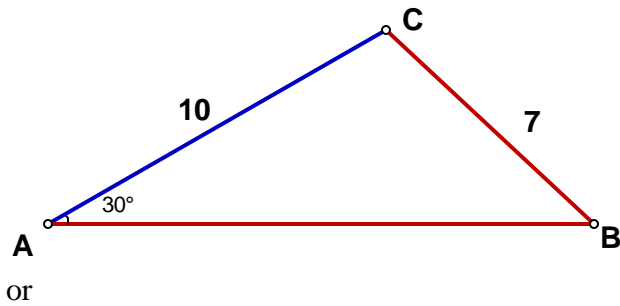


Example 1. Solve  $\triangle ABC$ , if  $\angle A = 30^\circ$ ,  $b = 10$ ,  $a = 7$

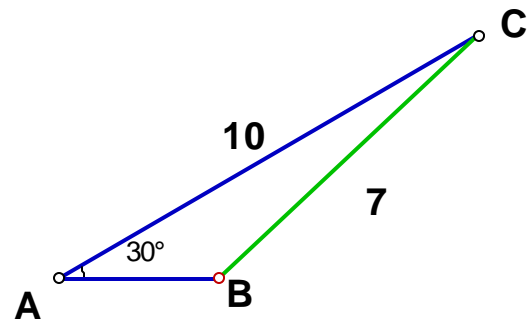
**Solution:** Note that there are two possibilities.



giving rise to the triangles



or

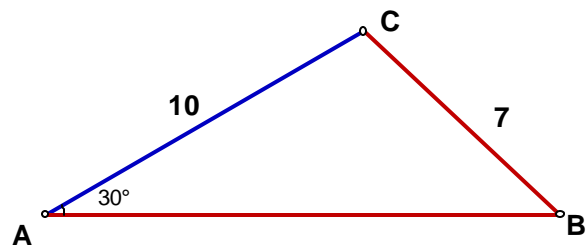


$$\frac{\sin B}{10} = \frac{\sin 30^\circ}{7} \rightarrow \sin B = \frac{10 \sin 30^\circ}{7} \cong .714286$$

$$\rightarrow B = \sin^{-1} .71286 \rightarrow B = 45.58^\circ \text{ or } 180^\circ - 45.58^\circ = 134.42^\circ$$

Solution I:

$B = 45.58^\circ$  corresponds to

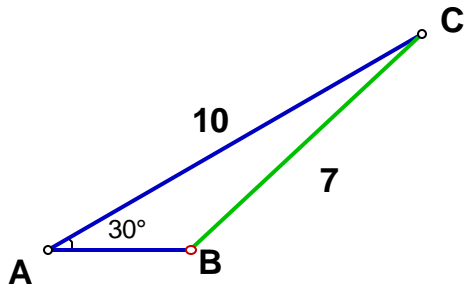


therefore  $C = 180^\circ - 30^\circ - 45.58^\circ = 104.42^\circ$

$$\frac{c}{\sin 104.42^\circ} = \frac{7}{\sin 30^\circ} \rightarrow c = \frac{7 \sin 104.42^\circ}{\sin 30^\circ} \cong 13.56$$

Solution II:

$B = 132.42^\circ$  corresponds to



giving  $C = 180^\circ - 30^\circ - 134.42^\circ = 15.58^\circ$

$$\frac{c}{\sin 15.58} = \frac{7}{\sin 30^\circ} \rightarrow c = \frac{7 \sin 15.58^\circ}{\sin 30^\circ} \cong 3.76$$

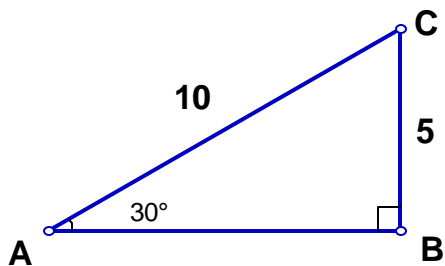
Example 2: Solve  $\triangle ABC$ , if  $\angle A = 30^\circ, b = 10, a = 5$

**Solution:**

$$\frac{\sin B}{10} = \frac{\sin 30^\circ}{5} \rightarrow \sin B = \frac{10 \sin 30^\circ}{5} \rightarrow \sin B = 1$$

$\rightarrow B = 90^\circ$  just note that  $180^\circ - 90^\circ = 90^\circ$

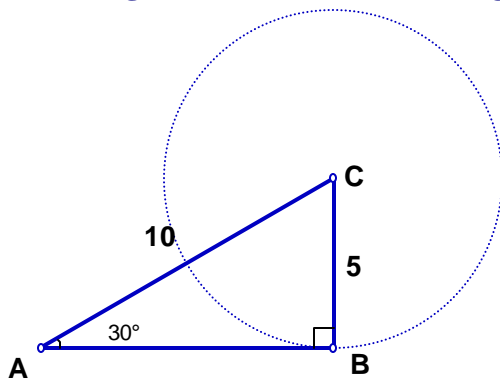
**Therefore there is only one triangle that can be a solution**



$$C = 180^\circ - 30^\circ = 60^\circ$$

$$c = \sqrt{10^2 - 5^2} = \sqrt{75} = 5\sqrt{3}$$

**Note the geometric reason for having one solution only.**



Example 3: Solve  $\triangle ABC$ , if  $\angle A=30^\circ, b=10, a=12$

**Solution:**

$$\frac{\sin B}{10} = \frac{\sin 30^\circ}{12} \rightarrow \sin B = \frac{10 \sin 30^\circ}{12} \rightarrow \sin B = \frac{5}{12}$$

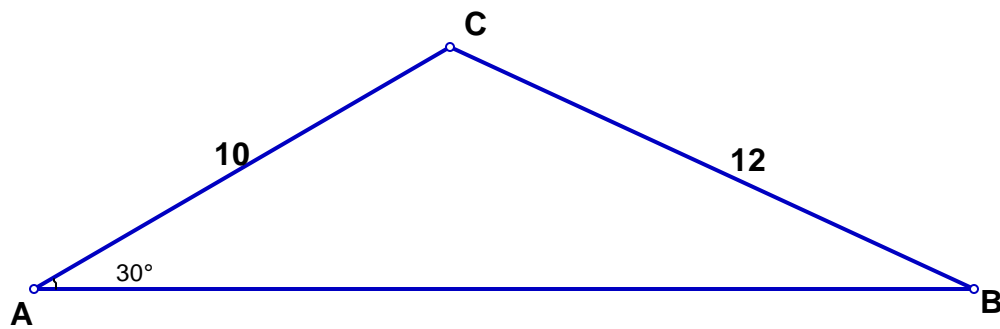
$$\rightarrow B = 24.62^\circ \text{ or } 180 - 24.62 = 155.38^\circ$$

Note that  $B=155.38^\circ$  is not acceptable here because

$$\text{then } C=180 - 30 - 155.38 = -5.38^\circ$$

Therefore there will be only triangle as a solution

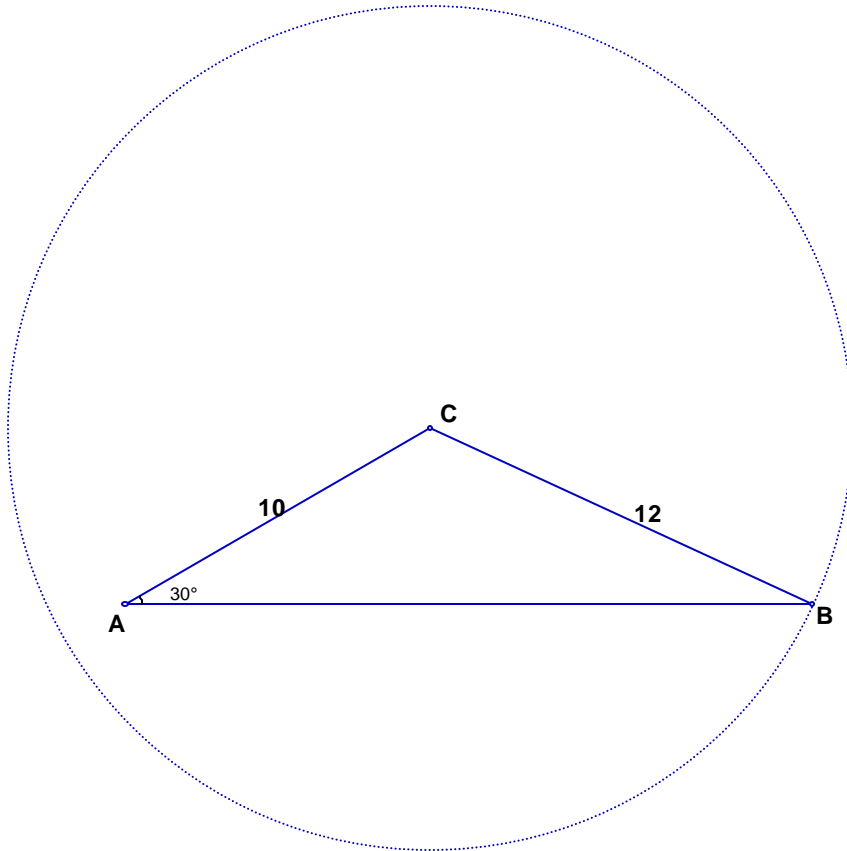
with  $B=24.62^\circ$ .



$$C = 180 - 30 - 24.62 = 125.38^\circ$$

$$\frac{c}{\sin 125.38^\circ} = \frac{12}{\sin 30^\circ} \rightarrow c = \frac{12 \sin 125.38^\circ}{\sin 30^\circ} \rightarrow c \cong 19.52$$

Geometrical reason for having only one solution



There is only one possible answer with  $A = 30^\circ$ ,  $a=12$ , and  $b=10$ .

**Example 4.** Solve  $\triangle ABC$ , if  $\angle A=30^\circ$ ,  $b = 10$ ,  $a = 4$

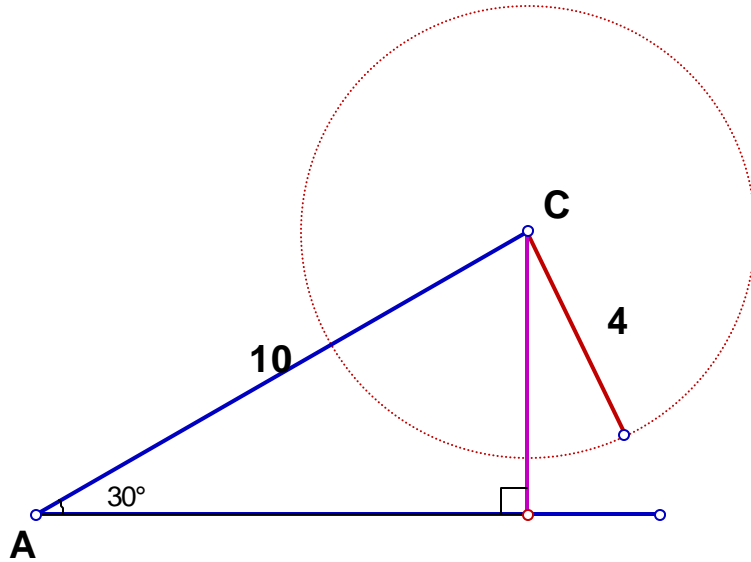
**Solution:**

$$\frac{\sin B}{10} = \frac{\sin 30^\circ}{4} \rightarrow \sin B = \frac{10 \sin 30^\circ}{4} \rightarrow \sin B = \frac{5}{4}$$

$\rightarrow$  no solution for B.

therefore we do not have any such triangle available.

**Geometric reason for such an answer:**



YOU MAY REFER TO THE TEXT BOOK TO SEE HOW DOES THE RELATION BETWEEN  $b\sin A$  and  $a$  AFFECT THE AVAILABILITY OF THE SOLUTIONS.

Exercises:

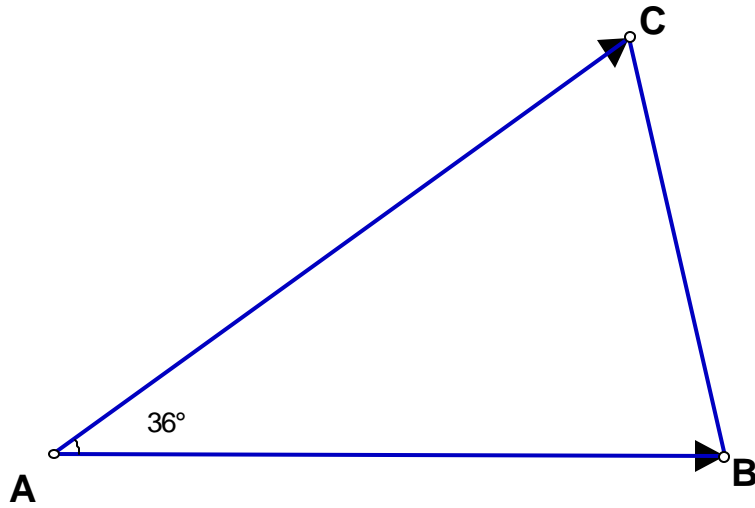
Solve the following triangles

1. Solve  $\triangle ABC$  , if  $\angle A=60^\circ, b = 10, a = 12$
2. Solve  $\triangle ABC$  , if  $\angle A=60^\circ, b = 10, a = 9$
3. Solve  $\triangle ABC$  , if  $\angle A=30^\circ, b = 10, a = 8$
4. Solve  $\triangle ABC$  , if  $\angle A=40^\circ, b = 8, a = 6$

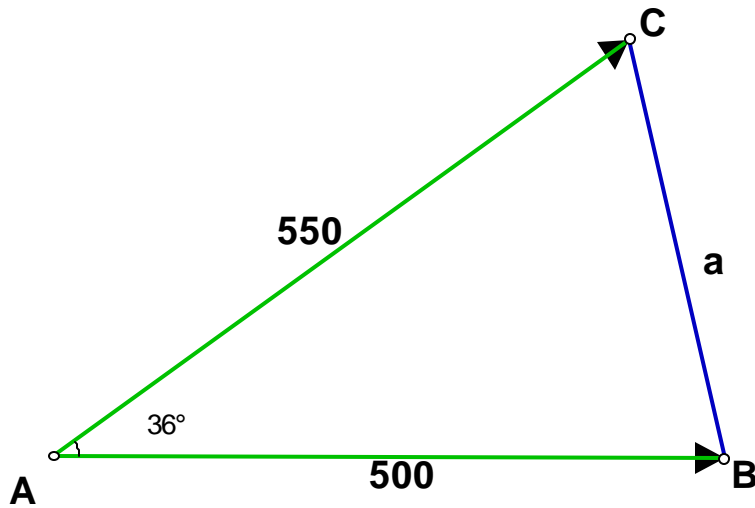
### The Law of Cosines:

The Law of Sines may not always work. Let us look at the following example.

Two airplanes B and C leave at the same time from the airport A in different directions as shown in the picture below. Suppose B has been at a constant rate of 500 miles per hour and C has been flying at a constant rate of 550 miles per hour. If A, B, and C can be assumed in the same plane during this time and that they fly along a straight line, calculate the distance between B and C after one hour.



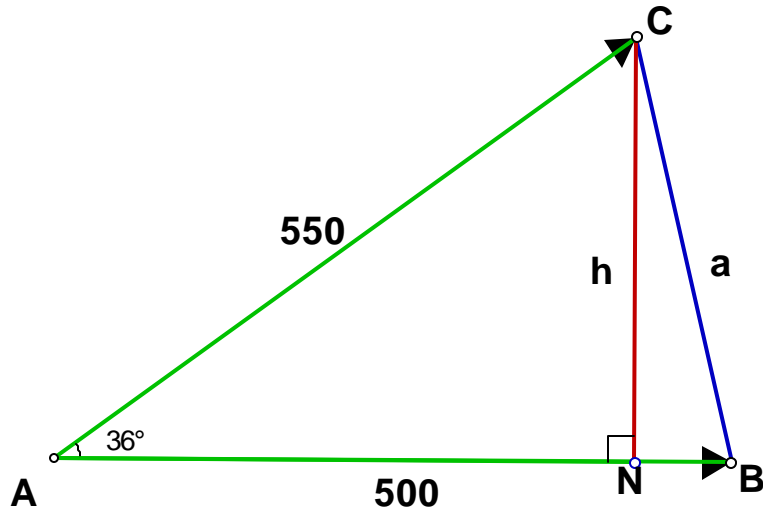
**Solution:** After one hour, the picture will be the following and we have to calculate the length of the blue segment.



In order to be able to find  $a$  by using the law of sines, we need the measure of another angle. Therefore we can not use the law of sines in this case.

Let us use a different method to work on this problem.

As before, we shall drop a perpendicular segment from  $C$  to meet  $AB$  at  $N$ .



Note that

Note that  $h = 550 \sin 36^\circ \rightarrow h \cong 323.28$  miles

$AN = 550 \cos 36^\circ \rightarrow AN \cong 444.96$  miles

Therefore  $NB = 500 - 444.96 \cong 55.04$  miles

giving  $a = \sqrt{232.28^2 + 55.04^2} \rightarrow a \cong 238.71$  miles.

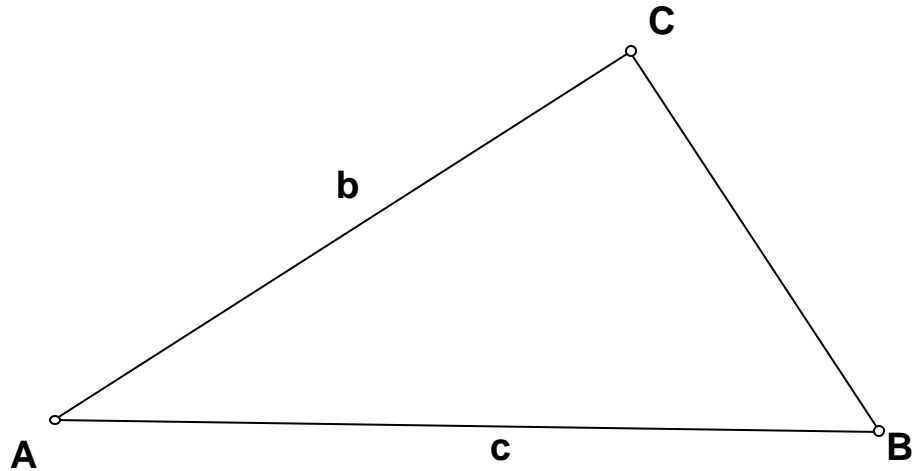
The planes will be approximately 238.71 miles apart after one hour.

**Let us proceed along the lines of the above calculations to derive what is called the Law of Cosines.**

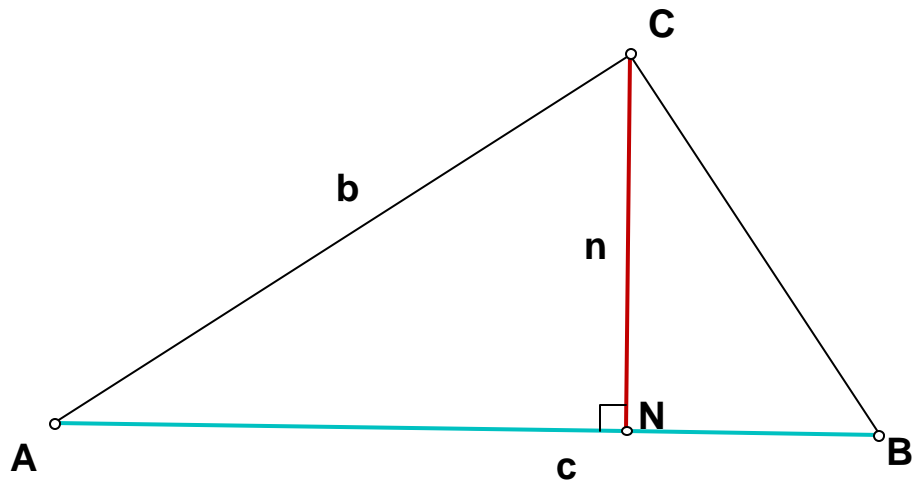
**Here we have to solve a triangle if A, b and c are given.**

**i.e. to find a, B and C. Note that once we determine a the rest is easily available by the Law of Sines.**

**Let us proceed.**



As before we drop a perpendicular segment CN.



Note that  $n = |CN| = b \sin A$

$|AN| = b \cos A$

$|NB| = c - b \cos A$

Note that  $\triangle CNB$  is a right triangle, therefore, we can use the Pythagorean Theorem, and conclude that

$$|BC|^2 = |CN|^2 + |NB|^2$$

or

$$a^2 = (b \sin A)^2 + (c - b \cos A)^2$$

$$\rightarrow a^2 = b^2 \sin^2 A + c^2 - 2cb \cos A + b^2 \cos^2 A$$

$$\rightarrow a^2 = b^2 \sin^2 A + b^2 \cos^2 A + c^2 - 2cb \cos A$$

$$\rightarrow a^2 = b^2(\sin^2 A + \cos^2 A) + c^2 - 2cb \cos A$$

$$\rightarrow a^2 = b^2(1) + c^2 - 2cb \cos A$$

$$\rightarrow a^2 = b^2 + c^2 - 2cb \cos A$$

A law of cosines

Similarly, we can obtain

$$b^2 = c^2 + a^2 - 2ca \cos B$$

or

$$c^2 = a^2 + b^2 - 2ab \cos C$$

## LET US RESTATE THE FOLLOWING ARE CALLED THE LAWS OF COSINES

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = c^2 + a^2 - 2ca \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

**We could this law to solve the airplane problem.**

**Example 1: Solve the  $\triangle ABC$ , given that  $b=5, c=3$ , and  $A=60^\circ$ .**

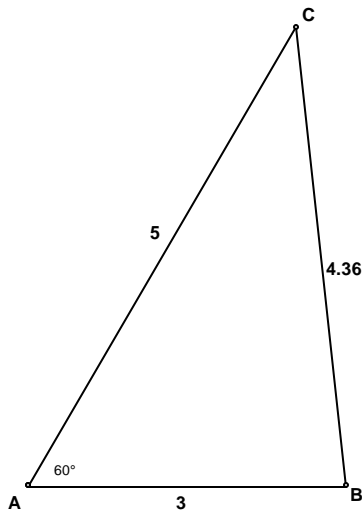
$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\rightarrow a^2 = 5^2 + 3^2 - 2 \cdot 5 \cdot 3 \cos 60^\circ$$

$$\rightarrow a^2 = 25 + 9 - 15$$

$$\rightarrow a^2 = 19$$

$$\rightarrow a \cong 4.36$$

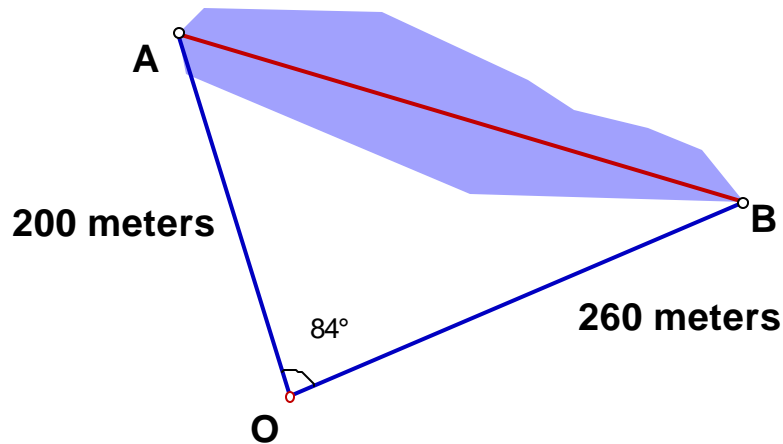


$$\frac{\sin C}{3} = \frac{\sin 60^\circ}{436} \rightarrow \sin C = \frac{3}{436} \sin 60^\circ$$

$$\rightarrow \sin C \cong .596 \rightarrow C = \sin^{-1}.596 \rightarrow C = 36.58^\circ$$

$$\rightarrow B = 180 - 60 - 36.58 = 83.42^\circ$$

Example 2: Find the length AB across the lake as shown in the following figure.



$$|AB|^2 = 200^2 + 260^2 - 2 \cdot 200 \cdot 260 \cos 84^\circ$$

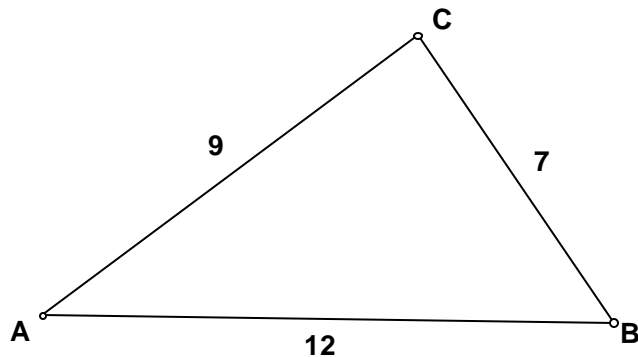
$$\rightarrow |AB|^2 \cong 96729.04$$

$$\rightarrow |AB| \cong \sqrt{96729.04}$$

$$\rightarrow |AB| \cong 31102 \text{ meters}$$

Example 3: Solve the  $\triangle ABC$  if  $a=7, b=9$ , and  $c=12$ .

In this case, we have to determine the angles.



For the angle A, note that

$$7^2 = 9^2 + 12^2 - 2 \cdot 9 \cdot 12 \cos A$$

$$\rightarrow 2 \cdot 9 \cdot 12 \cos A = 9^2 + 12^2 - 7^2$$

$$\rightarrow \cos A = \frac{9^2 + 12^2 - 7^2}{2 \cdot 9 \cdot 12}$$

$$\rightarrow \cos A = \frac{81 + 144 - 49}{216}$$

$$\rightarrow \cos A = \frac{22}{27}$$

$$\rightarrow A = \cos^{-1}\left(\frac{22}{27}\right)$$

$$\rightarrow A \cong 35.43^\circ$$

To determine B, note that

$$9^2 = 7^2 + 12^2 - 2 \cdot 7 \cdot 12 \cos B$$

$$\rightarrow 81 = 49 + 144 - 168 \cos B$$

$$\rightarrow 168 \cos B = 49 + 144 - 81$$

$$\rightarrow \cos B = \frac{49 + 144 - 81}{168}$$

$$\rightarrow B = \cos^{-1}\left(\frac{49 + 144 - 81}{168}\right)$$

$$\rightarrow B = \cos^{-1}\left(\frac{2}{3}\right)$$

$$\rightarrow B \cong 48.19^\circ$$

$$\rightarrow A = 180 - 48.19 - 35.43$$

$$\rightarrow A = 96.38^\circ$$

Example 4: Solve the  $\triangle ABC$ , it is given that  $a=5$ ,  $b=7$ , and  $c=20$ .

Let us make an attempt to find the angle A.

$$5^2 = 7^2 + 20^2 - 2 \cdot 7 \cdot 20 \cos A$$

$$\rightarrow 25 = 49 + 400 - 280 \cos A$$

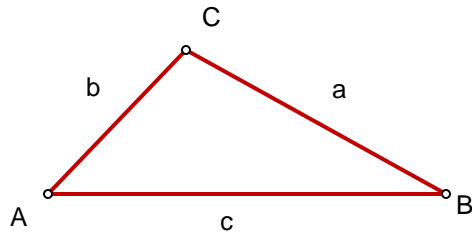
$$\rightarrow 280 \cos A = 449 - 25$$

$$\rightarrow \cos A = \frac{424}{280}$$

such an A is not possible because the cosine of an angle can not be more than 1.

Therefore such a triangle is not possible.

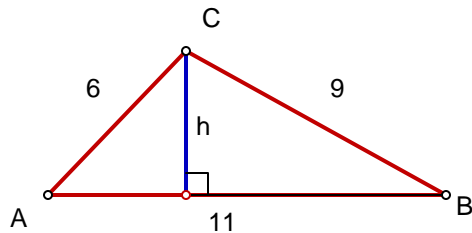
Example 5: We can use the law of cosines to determine the areas of triangles if the lengths of the sides are given. For example, let us try to determine the area of a triangle, for which  $a=9$ ,  $b=6$ ,  $c=11$ .



First, let us recall that the area of the triangle equals

$$\frac{1}{2} \text{base} \times \text{altitude}$$

Therefore, in this case, the area is



$$\frac{1}{2} \times 11 \times h$$

But,  $h=6\sin A$

Let us first determine the value of A, by using a Law of Cosines

$$9^2 = 6^2 + 11^2 - 2 \times 6 \times 11 \times \cos A$$

$81 = 36 + 121 - 132 \cos A$ , solve the equation for  $\cos A$  to obtain,

$$\cos A = \frac{36 + 121 - 81}{132}$$

$$\cos A = \frac{76}{132} \rightarrow \cos A = \frac{19}{33} \rightarrow A = \cos^{-1}\left(\frac{19}{33}\right)$$

$$\rightarrow A \cong 54.85^\circ$$

Therefore  $h = 6 \sin 54.85^\circ \cong 4.91$

The area of this triangle is  $\frac{1}{2} \times 11 \times h = \frac{11 \times 4.91}{2} \cong 27$  square units.

**The law of cosines can be used to derive the following formula to obtain the area of a triangle if the lengths of the sides are given. (This is called Heron's Formula)**

$$\text{If } s = \frac{a+b+c}{2}$$

The area of  $\triangle ABC = \sqrt{s(s-a)(s-b)(s-c)}$

**Let us use this to obtain the area of the triangle in the example 5,**

$$s = \frac{6+9+11}{2} = 13$$

$$\sqrt{13(13-6)(13-9)(13-11)} \cong 27 \text{ square units}$$

Hope this helps you in working on the exercises in the sections 8.1 and 8.2.