The Small-Angle Formula

"Geometry is coeternal with God," Johanas Kepler.

Why We Need the Formula

The small-angle equation is useful for calculating the size of an object when its distance is known or for calculating the distance of an object when its size is known. The apparent size of an object, the size of the angle when you look at it, becomes smaller the farther away you are from the object. We all know this fact and use it every day to determine distance to nearly everything around us. When the angular size of a friend is small we know they are far away and when we wish to speak to them we speak loudly. When the angular size of a friend is large we know they are near us and when we wish to speak to them we speak softly. In astronomy we need to quantify this relationship—to abstract it into mathematics—so that we can use an equation to calculate the size of distant objects so they to can become familiar. Likewise, in astronomy we sometimes need to calculate how far away an object is when we are familiar with its size. We can alway measure the apparent size of the object, the angle that the edges of the object make with your eye.

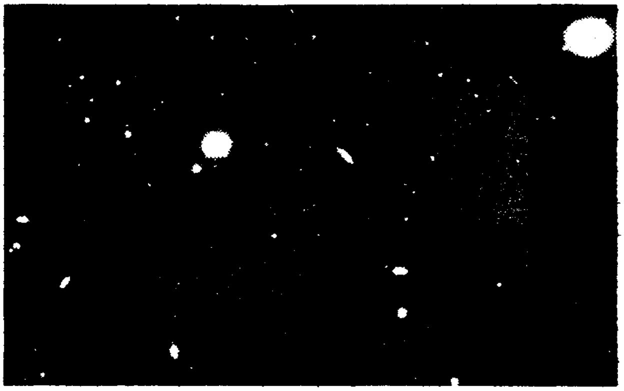


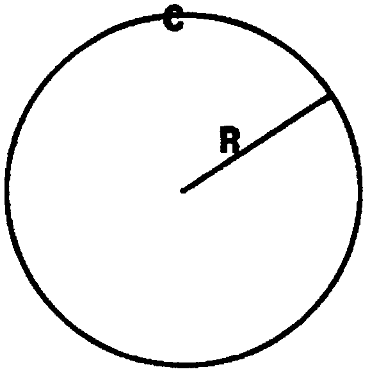
Figure 1. **Central part of the rich cluster of galaxies in the constellation Coma Berenices around 400 million ligh years away.**

Circumference of a Circle Formula

From very early times people have known that the relationship between the circumference of a circle and its radius is given by the sacred formula

$$C = 2\pi R\,,$$

where C is the circumference of a circle, π is a transcendental number which is approximately 22/7 or 3.1416[1], and R is its radius.

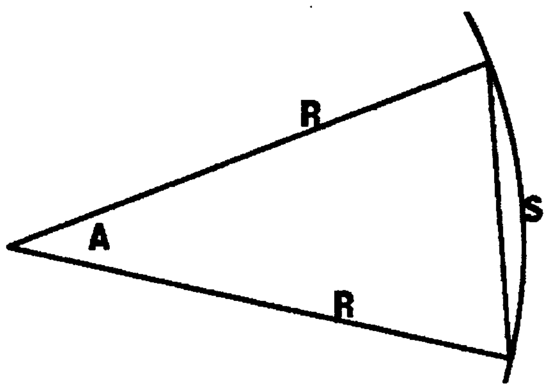


The Arc Formula

A corollary to the circumference of a circle formula is the arc length of a circle formulae

$$S = AR$$

where S is the length of an arc of a circle, A is the enclosed angle in radians, and R is the radius of the circle. Radians are the natural, simplest, way of measuring angles. There are 2π radians in a complete circle. See the diagram below.



If the angle A is small, then the arc length and chord are approximately equal—thus the name "small-angle formula."

1 The value of this transcendental number π has now been calculated to many thousands of decimal places. Many books have been written on methods of its calculation. Some people have even said that the history of π and the history of mathematics are the same. This is perhaps only a slight exaggeration. Fortunately for you, you only have to know a decent approximation for π to put it to good use.

Ancient Secrets

Naturally $S = AR$ is too simple, so no one measures angles in radians. If they did, everyone would probably understand mathematics and see some of the beauty in mathematical abstraction. Some Sumerian priests, about 5,000 years ago, who worshiped gods that few people now remembers. The priests (astrologers) lived near the Tigress and Euphrates river in what is today lower Iraq (just north of Kuwait) and were keepers of the sacred multiplication tables. They liked to do fractions in parts per sixty instead of parts per ten, decimals, (like we use today) so they decided to measure angles in degrees so that there would be $360 = 60 \times 6$ degrees in a circle.[2] There are 360 degrees in a complete circle. So there are 360 degrees in 2π radians, a complete circle. You see, these same priests did all the mathematics in their society. They computed the rents, interest on loans, figured out land measures, and advised the ruler according to the stars, astrology. They also did most of the writing. They could write and figure. Many ordinary people, the middle class business men and warriors, could read a little and count. However, in Egypt if you could count you were considered a magician for a while. The priests held this monopoly, through thousands of years, through the eras of Sumer, Babylon, and Assyria. These Sumerian priest were so good at controlling the mathematical secrets that they lasted many thousands of years and several changes in the spoken language. Cuneiform, their form of writing was syllabic, and enabled them to write any spoken language. If you want to be in on the ancient secrets, which are still used today then you will need to learn how to convert units. This is a simple procedure, but if you don't know the magic, you will be left out and have to depend upon magicians. **Better to be a magician than to be dependent upon one!**

Unit Conversions Applied to the Arc Formula

With a suitable change in units A can be expressed in units other than radians. The useful conversion factors involve that fact that 360 degrees = 2π radians, 1 degree = 60 arc minutes, and 1 arc minute = 60 arc seconds. Remember these Babylonian priests like to do fractions in parts per sixty.[3] If a° is the angle measured in degrees then A in radians is

$$A = a^\circ \frac{2\pi \text{ radians}}{360 \text{ degrees}} .$$

Substituting this equation for A the arc formula, $S = AR$, can be rewritten as

$$S = a^\circ \frac{2\pi}{360} R \approx \frac{a^\circ}{57.2956} R .$$

2 The earliest approximations for π was 3, therefore the approximation for 2π was 6.

3 That is why we have 60 minutes in an hour and 60 seconds in a minute of time.

If a' is the angle measured in arc minutes then A in radians is

$$A = a' \frac{2\pi \text{ radians}}{360 \text{ degrees}} \frac{1 \text{ degrees}}{60 \text{ arc minutes}} .$$

Substituting this equation for A the arc formula, $S = AR$, can be rewritten as

$$S = a' \frac{2\pi}{360 \times 60} R \approx \frac{a'}{3,437.74} R .$$

If a'' is the angle measured in arc seconds then A in radians is

$$A = a'' \frac{2\pi \text{ radians}}{360 \text{ degrees}} \frac{1 \text{ degrees}}{60 \text{ arc minutes}} \frac{1 \text{ arc minute}}{60 \text{ arc seconds}} .$$

Substituting this equation for A the arc formula, $S = AR$, can be rewritten as

$$S = \frac{a'' 2\pi}{360 \times 60 \times 60} R \approx \frac{a''}{206,264} R .$$

Notice in the A unit conversion equations the units cancel so that A always comes out in radians. This is no accident. This is by design. The angular scale on astronomical photographs will usually be in degrees, arc minutes (called minutes for short), or arc seconds (called seconds for short). That the code. Know it. Use it. Understand where it came from and you understand something about history, human nature, and astronomy a human activity.

Calulating Distance When Size Is Known

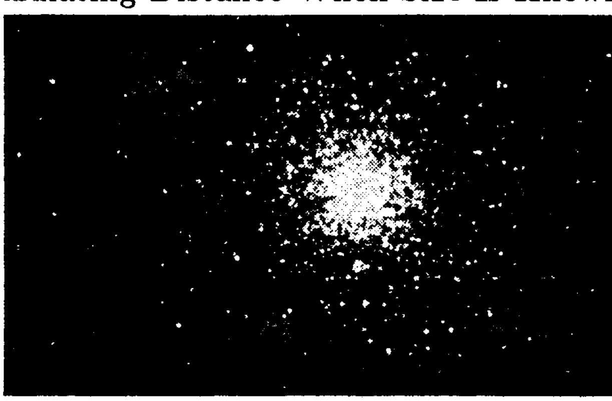


Figure 4. M92 a bright globular star cluster in the constellation Hercules.

The typical mean diameter for a globular cluster is 40 light years. If the scale of the photograph is 15 arc minutes across the bottom, how far away is the globular star cluster M92?

If the typical disk galaxy is 100,000 light years across and the scale of the Coma Berneice cluster of galaxy pictures is 20 arc minutes across the bottom, how far away are the disk galaxies?