GIBBS-APPELL'S EQUATIONS OF VARIABLE MASS NONLINEAR NONHOLONOMIC MECHANICAL SYSTEMS

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Abstract

In this paper, the Gibbs-Appell's equations of motion are extended to the most general variable mass nonholonomic mechanical systems. Then the Gibbs-Appell's equations of motion in terms of generalized coordinates or quasi-coordinates and an integral variational principle of variable mass nonlinear nonholonomic mechanical systems are obtained. Finally, an example is given.

Key words variable mass, nonholonomic system, Gibbs-Appell's equation, integral variational principle, quasi-velocity

I. Introduction

In 1879, Josiah Willard Gibbs[1] (1839 – 1903), an American mathematical physicist, first set up a new method for determining the motion of mechanical systems.

And this method independently discovered and developed by Appell[2], a famous French mathematician, in 1899. It was called the Gibbs-Appell method later.

The Appell's equations of motion are emphatically introduced in many textbooks. It's really a pity that most of them present the Gibbs-Appell method only as contents of secondary importance. As a matter of fact, Gibbs-Appell's equations are even simpler and more elegant in form, and they possess more practicability and more superiority.

In 1985, Mei Feng-xiang[3] presented Gibbs-Appell's motion equations for the linear nonholonomic mechanical systems in the form of generalized coordinates. In 1988, Edward A. Desloge[4] defined the concept of ideal constraint anew and studied the linear nonholonomic mechanical systems about Gibbs-Appell's equations of motion in terms of quasi-coordinates.

In this paper, the author attempted to extend Edward A. Desloge's idear to the mechanical systems with variable mass. Subsequently, the various forms and integral variational principle of Gibbs-Appell's equations for nonlinear nonholonomic mechanical systems with variable mass is derived. Finally, an example is given in order to explain the application of the new equations.

II. A New Motion Equations Form of Mechanical Systems with Variable Mass

Suppose that a mechanical system with variable mass is composed of N-particles, each of them with mass of M_γ. These particles are in motion with respect to the inertia coordinate system. Now let the configuration of every particle be characterized by the rectangular coordinates x_γ, y_γ and z_γ. Let $x_1=x_1, x_2=y_1, x_3=z_1, \cdots, x_{3N-2}=x_n, x_{3N-1}=y_N, x_{3N}=z_N$. Consider N-particles, which are subject to a set of known forces such as $G_{\gamma x}$, $G_{\gamma y}$, $G_{\gamma z}$; restraining counter-forces such as $H_{\gamma x}$, $H_{\gamma y}$, $H_{\gamma z}$, and counter thrusts such as $R_{\gamma x}$, $R_{\gamma y}$, $R_{\gamma z}$. If we let $f_1=G_{1x}$, $f_2=G_{1y}$,

$f_3=G_{1z}, \cdots, f_{3N-2}=G_{Nx}, f_{3N-1}=G_{Ny}, F_{3N}=G_{Nz}; F_1=H_{1x}, F_2=H_{1y}, F_3=H_{1z}, \cdots, F_{3N-2}=H_{Nx}, F_{3N-1}=H_{Ny}, F_{3N}=H_{Nz}; X_1^R=R_{1x}, X_2^R=R_{1y}, X_3^R=R_{1z}, \cdots, X_{3N-2}^R=R_{Nx}, X_{3N-1}^R=R_{Ny}, X_{3N}^R=R_{Nz}$.

Then the differential motion equations of a particle will be

$$m_i\ddot{x}_i=f_i+F_i+X_i^R \qquad (i=1,2,\cdots,3N) \tag{2.1}$$

Suppose that the motion of mechanical systems restrained by the following geometric constraints

$$g_a(x_i,t)=0 \qquad (a=1,2,\cdots,k) \tag{2.2}$$

We put generalized coordinates q_ω ($\omega=1,2,\cdots,n$) into x_i namely,

$$x_i=x_i(q_\omega,t) \qquad (\omega=1,2,\cdots,n;\ i=1,2,\cdots,3N)$$

where $n=3N-k$.

Thus, we have

$$\dot{x}_i=\sum_{\omega=1}^{n}\frac{\partial x_i}{\partial q_\omega}\dot{q}_\omega+\frac{\partial x_i}{\partial t} \tag{2.3}$$

$$\ddot{x}_i=\sum_{s=1}^{n}\sum_{\omega=1}^{n}\frac{\partial^2 x_i}{\partial q_\omega \partial q_s}\dot{q}_\omega\dot{q}_s+\sum_{\omega=1}^{n}\frac{\partial x_i}{\partial q_\omega}\ddot{q}_\omega+2\sum_{\omega=1}^{n}\frac{\partial^2 x_i}{\partial q_\omega \partial t}\dot{q}_\omega+\frac{\partial^2 x_i}{\partial t^2} \tag{2.4}$$

In view of this, we obtain

$$\partial\dot{x}_i/\partial\dot{q}_\omega=\partial\ddot{x}_i/\partial\ddot{q}_\omega$$

Let

$$s_{i\omega}=\partial\dot{x}_i/\partial\dot{q}_\omega=\partial\ddot{x}_i/\partial\ddot{q}_\omega \tag{2.5}$$

We now define

$$\sum_{i=1}^{3N}F_i s_{i\omega}=0 \tag{2.6}$$

If we multiply both sides of Eq. (2.1) by $s_{i\omega}$, Sum over i, and take notice of Eq. (2.6), then we obtain.

$$\sum_{i=1}^{3N}m_i\ddot{x}_i s_{i\omega}=\sum_{i=1}^{3N}f_i s_{i\omega}+\sum_{i=1}^{3N}X_i^R s_{i\omega} \qquad (\omega=1,2,\cdots,n) \tag{2.7}$$

Eq. (2.7), which is suitable for holonomic and nonholonomic systems, is a motion equation of the new form in the mechanical system with variable mass.

If we make mass constant, then $\sum_{i=1}^{3N}X_i^R s_{i\omega}=0$ and Eq.(2.7) becomes

$$\sum_{i=1}^{3N}m_i\ddot{x}_i s_{i\omega}=\sum_{i=1}^{3N}f_i s_{i\omega} \tag{2.8}$$

Eq. (2.8) is the same as the result in Ref. [4].

III. Gibbs-Appell's Motion Equations for the Nonlinear Nonholonomic Mechanical Systems of First Order with Variable Mass in Terms of Generalized Coordinates

Assume that the configuration of a mechanical system with variable mass characterized by the n-generalized coordinates $q_1, q_2, \cdots, q_n$, but then its system is restrained by L-constraints, which are nonlinear as well as nonholonomic with first order and can be written as

$$\dot{q}_{\varepsilon+\beta}=\dot{q}_{\varepsilon+\beta}(q_{\omega},\ \dot{q}_{j},\ t) \qquad (\beta=1,2,\cdots,L;\varepsilon=n-L;j=1,2,\cdots,\varepsilon;\omega=1,2,\cdots,n) \quad (3.1)$$

The mass of various particles in a mechanical system can be written by

$$m_i=m_i(q_{\omega},\dot{q}_{\omega},t)$$

The restraining counter-forces F_i, acting on the mechanical system, satisfy with Eq. (2.6).

Differentiating Eq. (3 1) with respect to time t, we get

$$\ddot{q}_{\varepsilon+\beta}=\sum_{\omega=1}^{n}\frac{\partial\dot{q}_{\varepsilon+\beta}}{\partial q_{\omega}}\dot{q}_{\omega}+\sum_{j=1}^{\varepsilon}\frac{\partial\dot{q}_{\varepsilon+\beta}}{\partial\dot{q}_{j}}\ddot{q}_{j}+\frac{\partial\dot{q}_{\varepsilon+\beta}}{\partial t} \tag{3.2}$$

Therefore

$$\partial\dot{q}_{\varepsilon+\beta}/\partial\dot{q}_{j}=\partial\ddot{q}_{\varepsilon+\beta}/\partial\ddot{q}_{j}$$

Substituting Eqs. (3.1) and (3.2) into Eqs. (2.3) and (2.4) successively, we obtain easily

$$\frac{\partial(\dot{x}_i)}{\partial\dot{q}_j}=\frac{\partial(\ddot{x}_i)}{\partial\ddot{q}_j}=\frac{\partial x_i}{\partial q_j}+\sum_{\beta=1}^{L}\frac{\partial x_i}{\partial q_{\varepsilon+\beta}}\frac{\partial\dot{q}_{\varepsilon+\beta}}{\partial\dot{q}_j}$$

Let

$$s_{ij}=\partial(\dot{x}_i)/\partial\dot{q}_j=\partial(\ddot{x}_i)/\partial\ddot{q}_j \tag{3.3}$$

Then acceleration energy is

$$S=\frac{1}{2}\sum_{i=1}^{3N}m_i\ddot{x}_i^2 \tag{3.4}$$

Define

$$\tilde{\Psi}_j=\sum_{i=1}^{3N}X_i^R s_{ij} \tag{3.5}$$

and the generalized forces

$$\tilde{Q}_j=\sum_{i=1}^{3N}f_i s_{ij} \tag{3.6}$$

Let $\tilde{S}$ be the expression, which is obtained by means of the constraint relationship (3.2) in S, for cancelling $\ddot{q}_{\varepsilon+\beta}$, the dependent generalized acceleration. Then we have

$$\tilde{S}=\frac{1}{2}\sum_{i=1}^{3N}m_i(\ddot{x}_i)\cdot(\ddot{x}_i)$$

$$\frac{\partial\tilde{S}}{\partial\ddot{q}_i}=\sum_{i=1}^{3N}m_i(\ddot{x}_i)\frac{\partial(\ddot{x}_i)}{\partial\ddot{q}_j}=\sum_{i=1}^{3N}m_i\ddot{x}_i s_{ij} \tag{3.7}$$

Putting (3.5) – (3.7) into (2.7), we get

$$\partial\tilde{S}/\partial\ddot{q}_j=\tilde{Q}_j+\tilde{\Psi}_j \tag{3.8}$$

Introduce Gibbs-Appell's function

$$\tilde{R}(q_\omega,\dot{q}_j,\ddot{q}_j,t)=\tilde{S}(q_\omega,\dot{q}_j,\ddot{q}_j,t)-\tilde{U}(q_\omega,\dot{q}_j,\ddot{q}_j,t) \tag{3.9}$$

where

$$\tilde{U}=\sum_{i=1}^{3N} f_i(\ddot{x}_i) \tag{3.10}$$

Then

$$\partial\tilde{R}/\partial\ddot{q}_j=\partial\tilde{S}/\partial\ddot{q}_j-\partial\tilde{U}/\partial\ddot{q}_j \tag{3.11}$$

and

$$\frac{\partial\tilde{U}}{\partial\ddot{q}_j}=\sum_{i=1}^{3N} f_i\frac{\partial(\ddot{x}_i)}{\partial\ddot{q}_j}=\sum_{i=1}^{3N} f_i s_{ij}=Q_j \tag{3.12}$$

Substituting Eqs. (3 8) and (3.12) into Eq. (3.11), we obtain

$$\partial\tilde{R}/\partial\ddot{q}_j=\tilde{\Psi}_j \qquad (j=1,2,\cdots,\varepsilon) \tag{3.13}$$

Eq. (3.13) is Gibbs-Appell's motion equation for the nonlinear nonholonomic mechanical system of first order with variable mass in the form of generalized coordinates.

IV. Gibbs-Appell's Motion Equations for the Nonlinear Nonholonomic Mechanical Systems of First Order with Variable Mass in Terms of Quasi-Coordinates

Assume the mechanical systems with variable mass is restrained by L-constraints, which are nonlinear as well as nonholonomic with first order and can be written as

$$E_\beta(q_\omega,\dot{q}_\omega,t)=0 \qquad (\beta=1,2,\cdots,L;\ \omega=1,2,\cdots,n) \tag{4.1}$$

The restraining counter-forces F_i, acting on the system, satisfy Eq. (2.6)

We choose $\dot{\Pi}_j$ and $\dot{\Pi}_{\varepsilon+\beta}$ as quasi-velocities

$$\left.\begin{aligned}&\dot{\Pi}_j=\dot{\Pi}_j(q_\omega,\dot{q}_\omega,t)\\&\dot{\Pi}_{\varepsilon+\beta}=E_\beta(q_\omega,q_\omega,t)=0\end{aligned}\right\}\quad\left(\begin{aligned}&j=1,2,\cdots,\varepsilon;\ \varepsilon=n-L\\&\omega=1,2,\cdots,n;\ \beta=1,2,\cdots,L\end{aligned}\right) \tag{4.2}$$

From (2.2), we get

$$\dot{q}_\omega=\dot{q}_\omega(q_r,\dot{\Pi}_j,t) \qquad (\omega,r=1,2,\cdots,n;\ j=1,2,\cdots,\varepsilon) \tag{4.3}$$

Differentiating the above expressions with respect to time t, we obtain

$$\ddot{q}_\omega=\sum_{j=1}^{\varepsilon}\frac{\partial\dot{q}_\omega}{\partial\dot{\Pi}_j}\ddot{\Pi}_j+A \tag{4.4}$$

where A is a term excluding $\ddot{\Pi}_j$

Thus, from Eqs (2.3) and (2.4), it is relatively easy to show that

$$\frac{\partial\dot{x}_i}{\partial\dot{\Pi}_j}=\frac{\partial\ddot{x}_i}{\partial\ddot{\Pi}_j}=\sum_{\omega=1}^{n}\frac{\partial x_i}{\partial q_\omega}\frac{\partial\dot{q}_\omega}{\partial\dot{\Pi}_j} \tag{4.5}$$

Let

$$s_{ij}=\partial\dot{x}_i/\partial\dot{\Pi}_j=\partial\ddot{x}_i/\partial\ddot{\Pi}_j \tag{4 6}$$

Now we define

$$P_j^* = \sum_{i=1}^{3N} f_i s_{ij} \tag{4.7}$$

$$\Phi_j^* = \sum_{i=1}^{3N} X_i^* s_{ij} \tag{4.8}$$

here P_j^* is the generalized force in terms of quasi-coordinates.
From Eq. (3.4), we obtain

$$\frac{\partial S}{\partial \ddot{\Pi}_j} = \sum_{i=1}^{3N} m_i \ddot{x}_i \frac{\partial \ddot{x}_i}{\partial \ddot{\Pi}_j} = \sum_{i=1}^{3N} m_i \ddot{x}_i s_{ij} \tag{4.9}$$

Putting (4.7) – (4.9) into (2.7), we have

$$\partial S/\partial \ddot{\Pi}_j = P_j^* + \Phi_j^* \qquad (j=1,2,\cdots,\varepsilon) \tag{4.10}$$

We introduce Gibbs-Appell's function

$$R(q_\omega, \dot{\Pi}_j, \ddot{\Pi}_j, t) = S - U(q_\omega, \dot{\Pi}_j, \ddot{\Pi}_j, t) \tag{4.11}$$

where

$$U(q_\omega, \dot{\Pi}_j, \ddot{\Pi}_j, t) = \sum_{i=1}^{3N} f_i \ddot{x}_i \tag{4.12}$$

Thus

$$\partial R/\partial \ddot{\Pi}_j = \partial S/\partial \ddot{\Pi}_j - \partial U/\partial \ddot{\Pi}_j \tag{4.13}$$

Due to the cause

$$\frac{\partial U}{\partial \ddot{\Pi}_j} = \sum_{i=1}^{3N} f_i \frac{\partial \ddot{x}_i}{\partial \ddot{\Pi}_j} = \sum_{i=1}^{3N} f_i s_{ij} = P_j^* \tag{4.13}$$

Putting (4.10) and (4.13)' into (4.13), we obtain

$$\partial R/\partial \ddot{\Pi}_j = \Phi_j^* \qquad (j=1,2,\cdots,\varepsilon) \tag{4.14}$$

Eq (4.14) is Gibbs-Appell's motion equation for the nonlinear nonholonomic mechanical systems of first order with variable mass in the form of quasi-coordinates.

V. Gibbs-Appell's Motion Equations for the Nonlinear Nonholonomic Mechanical Systems of First Order with Variable Mass in Terms of Quasi-Velocities Associated with Quasi-Accelerations

Let the restraining counter-forces satisfy Eq (2.6), and assume the mechanical system with variable mass is restrained by L-nonholonomic constraints like Eq. (4.1) in form.

We choose $\dot{\Pi}_j$ as quasi-velocity with functional independence for each other

$$\dot{\Pi}_j = A_j(q_\omega, \dot{q}_\omega, t) \qquad (\omega=1,2,\cdots,n;\ j=1,2,\cdots,\varepsilon) \tag{5.1}$$

where $\varepsilon = n - L$.
Suppose that from (4.1) and (5.1) we get

$$\dot{q}_\omega = \varphi_\omega(q_k, \dot{\Pi}_j, t) \qquad (\omega, k=1,2,\cdots,n;\ j=1,2,\cdots,\varepsilon) \tag{5.2}$$

Differentiating the above expression with respect to time t, we obtain

$$\ddot{q}_\alpha=B_\alpha(q_k,\dot{\Pi}_j,\ddot{\Pi}_j,t) \tag{5.3}$$

where

$$B_\alpha=\sum_{k=1}^{n}\frac{\partial\varphi_\alpha}{\partial q_k}\varphi_k+\sum_{j=1}^{e}\frac{\partial\varphi_\alpha}{\partial\dot{\Pi}_j}\ddot{\Pi}_j+\frac{\partial\varphi_\alpha}{\partial t} \tag{5.4}$$

Take ε quasi-accelerations of functional independence for each other

$$\varepsilon_j=\lambda_j(q_\alpha,\dot{\Pi}_\gamma,\ddot{\Pi}_\gamma,t)\qquad(\gamma=1,2,\cdots,\varepsilon) \tag{5.5}$$

By inverse operation we get

$$\ddot{\Pi}_j=\psi_j(q_\alpha,\dot{\Pi}_\gamma,\varepsilon_\gamma,t) \tag{5.6}$$

Considering (5.2), (5.3) and (5.6), we prove easily the following relationship from (2.3) and(2.4)

$$\frac{\partial\dot{x}_i}{\partial\dot{\Pi}_j}=\frac{\partial\ddot{x}_i}{\partial\ddot{\Pi}_j}=\sum_{\alpha=1}^{n}\frac{\partial x_i}{\partial q_\alpha}\frac{\partial\varphi_\alpha}{\partial\dot{\Pi}_j} \tag{5.7}$$

$$\frac{\partial\ddot{x}_i}{\partial\varepsilon_\gamma}=\sum_{\alpha=1}^{n}\frac{\partial x_i}{\partial q_\alpha}\sum_{j=1}^{e}\frac{\partial B_\alpha}{\partial\ddot{\Pi}_j}\frac{\partial\psi_i}{\partial\varepsilon_\gamma}=\sum_{j=1}^{e}\frac{\partial\dot{x}_i}{\partial\dot{\Pi}_j}\frac{\partial\psi_j}{\partial\varepsilon_\gamma}=\sum_{j=1}^{e}\frac{\partial\ddot{x}_i}{\partial\ddot{\Pi}_j}\frac{\partial\psi_j}{\partial\varepsilon_\gamma} \tag{5.8}$$

Let

$$s_{i\gamma}=\partial\ddot{x}_i/\partial\varepsilon_\gamma \tag{5.9}$$

We have

$$\sum_{i=1}^{3N}f_is_{i\gamma}=\sum_{i=1}^{3N}f_i\sum_{j=1}^{e}\frac{\partial\dot{x}_i}{\partial\dot{\Pi}_j}\frac{\partial\psi_j}{\partial\varepsilon_\gamma}=\sum_{j=1}^{e}\sum_{\alpha=1}^{n}Q_\alpha\frac{\partial\varphi_\alpha}{\partial\dot{\Pi}_j}\frac{\partial\psi_j}{\partial\varepsilon_\gamma}=P_\gamma^{**} \tag{5.10}$$

and

$$\sum_{i=1}^{3N}X_i^{*}s_{i\gamma}=\sum_{j=1}^{e}\sum_{\alpha=1}^{n}G_\alpha\frac{\partial\varphi_\alpha}{\partial\dot{\Pi}_j}\frac{\partial\psi_j}{\partial\varepsilon_\gamma}=G_\gamma^{**} \tag{5.11}$$

where

$$G_\alpha=\sum_{i=1}^{3N}X_i^{*}\frac{\partial x_i}{\partial q_\alpha}$$

Let S^* and S^{**} be the expressions of acceleration in terms of quasi-velocities and quasi-accelerations successively, we have

$$\sum_{i=1}^{3N}m_i\ddot{x}_is_{i\gamma}=\sum_{i=1}^{3N}m_i\ddot{x}_i\sum_{j=1}^{e}\frac{\partial\ddot{x}_i}{\partial\ddot{\Pi}_j}\frac{\partial\psi_j}{\partial\varepsilon_\gamma}=\sum_{j=1}^{e}\frac{\partial S^*}{\partial\ddot{\Pi}_j}\frac{\partial\psi_j}{\partial\varepsilon_\gamma}=\frac{\partial S^{**}}{\partial\varepsilon_\gamma} \tag{5.12}$$

Putting (5.10) – (5.12) into (2.7), we obtain

$$\partial S^{**}/\partial\varepsilon_\gamma=P_\gamma^{**}+G_\gamma^{**} \tag{5.13}$$

Now we introduce Gibbs-Appell's function

$$R^{**}(q_\alpha,\dot{\Pi}_j,\varepsilon_\gamma,t)=S^{**}(q_\alpha,\dot{\Pi}_j,\varepsilon_\gamma,t)-U^{**}(q_\alpha,\dot{\Pi}_j,\varepsilon_\gamma,t) \tag{5.14}$$

where

$$U^{**}=\sum_{i=1}^{3N} f_i \ddot{x}_i$$

Thus

$$\partial R^{**}/\partial \varepsilon_\gamma=\partial S^{**}/\partial \varepsilon_\gamma-\partial U^{**}/\partial \varepsilon_\gamma \tag{5.15}$$

and

$$\frac{\partial U^{**}}{\partial \varepsilon_\gamma}=\sum_{i=1}^{3N} f_i \frac{\partial \ddot{x}_i}{\partial \varepsilon_\gamma}=\sum_{i=1}^{3N} f_i s_{i\gamma}=P_\gamma^{**} \tag{5.16}$$

Putitng (5.13) and (5.16) into (5.15), we get

$$\partial R^{**}/\partial \varepsilon_\gamma=G_\gamma^{**} \quad (\gamma=1,2,\cdots,\varepsilon) \tag{5.17}$$

Eq. (5.17) is Gibbs-Appell's motion equation for the nonlinear nonholonomic mechanical system of first order with variable mass in the form of quasi-velocities associated with quasi-accelerations.

VI. Gibbs-Appell's Equations of Relative Motion for the nonlinear Nonholonomic Mechanical Systems of First Order with Variable Mass in Terms of Generalized Coordinates

6.1 A new equations form of relative motion in the mechanical system with variable mass

Let the mechanical system with variable mass be composed of N-particles M_i $(i=1,2,\cdots,N)$. Suppose that the configuration of the above system characterized by the n-generalized coordinates $q_1, q_2, \cdots, q_n$. Therefore, the radius rector of particle M_i corresponding to the motional coordinate system $Ox'y'z'$ fixed to the carrier will be $\bar{r}'_i$

Here

$$\bar{r}'_i=\bar{r}'_i(q_1,q_2,\cdots,q_n,t) \tag{6.1}$$

and the differential equation of relative motion of M_i can be written by

$$m_i\bar{a}_{ir}=-m_i\bar{a}_{ie}-m_i\bar{a}_{ic}+\bar{G}_i+\bar{H}_i+\bar{R}_i \qquad (i=1,2,\cdots,N) \tag{6.2}$$

where

$$\bar{a}_{ie}=\bar{a}_0+\bar{\omega}\times(\bar{\omega}\times\bar{r}'_i)+\dot{\bar{\omega}}\times\bar{r}'_i, \quad \bar{a}_{ic}=2\bar{\omega}\times\dot{\bar{r}}'_i$$

Let

$$\bar{s}_{i\omega}=\partial\dot{\bar{r}}'_i/\partial\dot{q}_\omega=\partial\ddot{\bar{r}}'_i/\partial\ddot{q}_\omega \tag{6.3}$$

Assume the restraining counter-forces $\bar{H}_i$ satisfy Eq. (6.4)

$$\sum_{i=1}^{N}\bar{H}_i\cdot\bar{s}_{i\omega}=0 \tag{6.4}$$

Use dot product in both sides of Eq. (6.2) by $\bar{s}_{i\omega}$, sum over i, and take notice of Eq. (6.4), we have

$$\sum_{i=1}^{N} m_i\bar{a}_{ir}\cdot\bar{s}_{i\omega}=-\sum_{i=1}^{N} m_i\bar{a}_{ie}\cdot\bar{s}_{i\omega}-\sum_{i=1}^{N} m_i\bar{a}_{ic}\bar{s}_{i\omega}+\sum_{i=1}^{N}\bar{G}_i\cdot\bar{s}_{i\omega}+\sum_{i=1}^{N}\bar{R}_i\cdot\bar{s}_{i\omega} \tag{6.5}$$

Eq. (6.5) is differential motion equation of relative motion in the variable mass mechanical system with a new form.

$\sum_{i=1}^{N} \bar{R}_i \cdot \bar{s}_{i\omega}=0$, when mass is regarded as a constant, and (6.5) may be written as

$$\sum_{i=1}^{N} m_i \bar{a}_{ir} \cdot \bar{s}_{i\omega} = -\sum_{i=1}^{N} m_i \bar{a}_{ie} \cdot \bar{s}_{i\omega} - \sum_{i=1}^{N} m_i \bar{a}_{ic} \cdot \bar{s}_{i\omega} + \sum_{i=1}^{N} \bar{G}_i \cdot \bar{s}_{i\omega} \tag{6.6}$$

6.2 Gibbs-Appell's equations of relative motion for the nonlinear nonholonomic mechanical systems of first order with variable mass in terms of generalized coordinates

Let the restraining counter-forces satisfy Eq. (6.4) and suppose the mechanical system with variable mass is restrained by L-nonlinear nonholonomic constraints of first order like Eq. (3.1) in form.

In accordance with (3.1) and (3.2), using (6.1) it is easy to show that

$$\frac{\partial(\dot{\bar{r}}_i')}{\partial \dot{q}_j} = \frac{\partial(\ddot{\bar{r}}_i')}{\partial \ddot{q}_j} = \frac{\partial \bar{r}_i'}{\partial q_j} + \sum_{\beta=1}^{L} \frac{\partial \bar{r}_i'}{\partial q_{\varepsilon+\beta}} \frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_j} \tag{6.7}$$

Let

$$\bar{s}_{ij} = \partial(\dot{\bar{r}}_i')/\partial \dot{q}_j = \partial(\ddot{\bar{r}}_i')/\partial \ddot{q}_j \tag{6.8}$$

The equation of acceleration energy of relative motion can be written as

$$S_r = \frac{1}{2}\sum_{i=1}^{N} m_i \ddot{\bar{r}}_i' \cdot \ddot{\bar{r}}_i'$$

where S_r is the acceleration energy.

Now define

$$\sum_{i=1}^{N} \bar{R}_i \cdot \bar{s}_{ij} = \tilde{\Psi}_j, \quad \sum_{i=1}^{N} \bar{G}_i \cdot \bar{s}_{ij} = \tilde{Q}_j \tag{6.9}$$

Letting $\tilde{S}_r$ be the expression, which is obtained by means of Eq. (3.2) in S_r for cancelling $\ddot{q}_{\varepsilon+\beta}$, the dependent generalized acceleration, we have

$$\tilde{S}_r = \frac{1}{2}\sum_{i=1}^{N} m_i (\ddot{\bar{r}}_i') \cdot (\ddot{\bar{r}}_i')$$

$$\frac{\partial \tilde{S}_r}{\partial \ddot{q}_j} = \sum_{i=1}^{N} m_i (\ddot{\bar{r}}_i') \cdot \frac{\partial(\ddot{\bar{r}}_i')}{\partial \ddot{q}_j} = \sum_{i=1}^{N} m_i \ddot{\bar{r}}_i' \cdot \bar{s}_{ij} \tag{6.10}$$

Substituting Eqs. (6.9) and (6.10) into Eq. (6.5), and considering the result in Ref. [3] we obtain

$$\frac{\partial \tilde{S}_r}{\partial \ddot{q}_j} = \tilde{Q}_j + \tilde{\Psi} - \frac{\partial}{\partial q_j}(V^0 + V^\omega) - \sum_{\beta=1}^{L} \frac{\partial}{\partial q_{\varepsilon+\beta}}(V^0 + V^\omega) \frac{\partial \dot{q}_{\varepsilon+\beta}}{\partial \dot{q}_j}$$

$$+ \tilde{Q}_j^{\dot{\omega}} + \tilde{\Gamma}_j \quad (j=1,2,\cdots,\varepsilon) \tag{6.11}$$

Introduce Gibbs-Appell's function

$$\tilde{R}_r(q_\omega,\dot{q}_j,\ddot{q}_j,t)=\tilde{S}_r(q_\omega,\dot{q}_j,\ddot{q}_j,t)-\tilde{U}(q_\omega,\dot{q}_j,\ddot{q}_j,t) \tag{6.12}$$

where

$$\tilde{U}=\sum_{i=1}^{N}\bar{G}_i\cdot(\ddot{\bar{r}}'_i)$$

$$\partial\tilde{R}_r/\partial\ddot{q}_j=\partial\tilde{S}_r/\partial\ddot{q}_j-\partial\tilde{U}/\partial\ddot{q}_j \tag{6.13}$$

$$\frac{\partial\tilde{U}}{\partial\ddot{q}_j}=\sum_{i=1}^{N}\bar{G}_i\cdot\frac{\partial(\ddot{\bar{r}}'_i)}{\partial\ddot{q}_j}=\sum_{i=1}^{N}\bar{G}_i\cdot\bar{s}_{ij}=\tilde{Q}_j \tag{6.14}$$

Putting (6.11) and(6.14) into (6.13), we obtan

$$\frac{\partial\tilde{R}_r}{\partial\ddot{q}_j}=\tilde{\Psi}_j-\frac{\partial}{\partial q_j}(V^0+V^\omega)-\sum_{\beta=1}^{L}\frac{\partial}{\partial q_{\varepsilon+\beta}}(V^0+V^\omega)\frac{\partial\dot{q}_{\varepsilon+\beta}}{\partial\dot{q}_j}+\tilde{Q}_j^{\dot{\omega}}+\tilde{\Gamma}_j$$

$$(j=1,2,\cdots,\varepsilon) \tag{6.15}$$

Equations (6.15) are Gibbs-Appell's equations of relative motion for the nonlinear nonholonomic mechanical systems of first order with variable mass in terms of generalized coordinates.

VII. The Integral Variational Principle in Velocity Space

Eq. (4.14) is Gibbs-Appell's equations of motion for the nonholonomic mechanical systems with variable mass in the form of quasi-coordinates, namely,

$$\partial R/\partial\ddot{\Pi}_j=P_j^* \tag{7.1}$$

Define a special variational notation by δ^* in order that when mass is regarded as constant[13], we outght to adopt δ^* . Now consider δ^*R, the variation of this sort for R, as follows

$$\delta^*R\triangleq\sum_{j=1}^{\cdot}\frac{\partial R}{\partial\dot{\Pi}_j}\delta\dot{\Pi}_j+\sum_{j=1}^{\cdot}\frac{\partial R}{\partial\ddot{\Pi}_j}\delta\ddot{\Pi}_j \tag{7.2}$$

If we multiply both sides of Eq. (7.1) by $\delta\ddot{\Pi}_j$, sum over j, then we have

$$\sum_{j=1}^{\cdot}\frac{\partial R}{\partial\ddot{\Pi}_j}\delta\ddot{\Pi}_j=\sum_{j=1}P_j^*\delta\ddot{\Pi}_j \tag{7.3}$$

Now putting (7.3) into (7.2), we get

$$-\delta^*R+\sum_{j=1}^{\cdot}\frac{\partial R}{\partial\dot{\Pi}_j}\partial\dot{\Pi}_j+\sum_{j=1}^{\cdot}P_j^*\delta\ddot{\Pi}_j=0 \tag{7.4}$$

The Suslov's exchange relation in velocity space becomes

$$\frac{d}{dt}\delta\dot{\Pi}_j=\delta\ddot{\Pi}_j \tag{7.5}$$

Integrating integrand (7.4) between the closed interval $[t_0, t_1]$, we obtain

$$\int_{t_0}^{t_1}\left(-\delta^*R+\sum_{j=1}^{\cdot}\frac{\partial R}{\partial\dot{\Pi}_j}\delta\dot{\Pi}_j+\sum_{j=1}^{\cdot}P_j^*\delta\ddot{\Pi}_j\right)dt=0 \tag{7.6}$$

Provide

$$\delta\dot{\Pi}_j|_{t_0}=0 \qquad \delta\dot{\Pi}_j|_{t_1}=0$$

With the aid of integration by parts, we get

$$\int_{t_0}^{t_1}\left[-\delta^*R+\sum_{j=1}^{s}\left(\frac{\partial R}{\partial\dot{\Pi}_j}-p_j^*\right)\delta\dot{\Pi}_j\right]dt=0 \tag{7.7}$$

$p_j^*=0$, when mass is regarded as a constant, the above expression becomes

$$\int_{t_0}^{t_1}\left[-\delta^*R+\sum_{j=1}^{s}\frac{\partial R}{\partial\dot{\Pi}_j}\delta\dot{\Pi}_j\right]dt=0 \tag{7.8}$$

VIII. Example

Figure 1 shows a plane tracing curve that allows that point Q to be moved along the horizontal axis Ox according to the known law $OQ=\xi(t)$, and allows the particle P to be moved in the vertical plane. In this case the particle P possesses variable mass $m=m(t)$, and the arrow of $\bar{V}_P$ always points to point Q. Let us now construct the differential motion equation[5] of particle P as follows

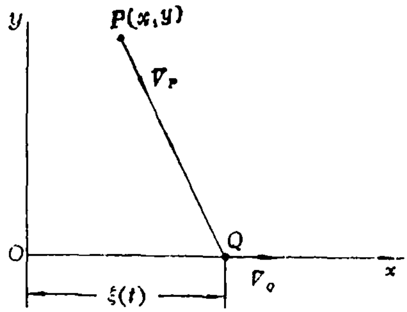

Fig. 1

If we let x and y be the rectangular coordinates of particle P, and adopt x together with y as generalized coordinates, then the nonholonomic constrained equations of particle P are

$$\dot{y}=\frac{y}{x-\xi}\dot{x} \tag{8.1}$$

here $i=1,2;\ j=1$

and

$$\ddot{y}=\frac{\dot{y}\dot{x}(x-\xi)-y\dot{x}(\dot{x}-\dot{\xi})}{(x-\xi)^2}+\frac{y\ddot{x}}{x-\xi} \tag{8.2}$$

Gibbs-Appell's functions are

$$R=S-U=m(\ddot{x}^2+\ddot{y}^2)/2+mg\ddot{y} \tag{8.3}$$

$$\tilde{R}=\frac{1}{2}m\left\{\ddot{x}^2+\left[\frac{\dot{y}\dot{x}(x-\xi)-y\dot{x}(\dot{x}-\dot{\xi})}{(x-\xi)^2}+\frac{y\ddot{x}}{x-\xi}\right]^2\right\}+mg\left[\frac{\dot{y}\dot{x}(x-\xi)-y\dot{x}(\dot{x}-\dot{\xi})}{(x-\xi)^2}+\frac{y\ddot{x}}{x-\xi}\right] \tag{8.4}$$

$$\frac{\partial\tilde{R}}{\partial\ddot{x}}=m\ddot{x}\left[1+\frac{y^2}{(x-\xi)^2}\right]+\frac{my^2\dot{x}\dot{\xi}}{(x-\xi)^3}+\frac{mgy}{x-\xi} \tag{8.5}$$

Let the relative velocity of separate corpuscule be

$$\bar{u}=\eta(t)\dot{\bar{r}}-\dot{\bar{r}} \tag{8.6}$$

where $\eta(t)$ is a given function, we have

$$\widetilde{\Psi}=\sum_{i=1}^{2}X_i^* s_{i1}=\dot{m}[\eta(t)\dot{x}-\dot{x}]\frac{\partial\dot{x}}{\partial\dot{x}}+\dot{m}[\eta(t)\dot{y}-\dot{y}]\frac{\partial(\dot{y})}{\partial\dot{x}}$$

$$=\dot{m}[\eta(t)\dot{x}-\dot{x}]+\dot{m}\dot{x}[\eta(t)-1]\frac{y^2}{(x-\xi)^2}=\dot{m}\dot{x}[\eta(t)-1]\left[1+\frac{y^2}{(x-\xi)^2}\right] \quad (8.7)$$

Now puttng (8.6) and (8.7) into (3.13), we obtain

$$m\ddot{x}\left[1+\frac{y^2}{(x-\xi)^2}\right]+\frac{m\dot{x}\dot{\xi}y^2}{(x-\xi)^3}=-mg\frac{y}{x-\xi}+\dot{m}\dot{x}[\eta(t)-1]\left[1+\frac{y^2}{(x-\xi)^2}\right] \quad (8.8)$$

From the example, it is visible that the operational process of the Gibbs-Appell method is simpler and therefore Gibbs-Appell's equations possess a clear superiority in solving problems.

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