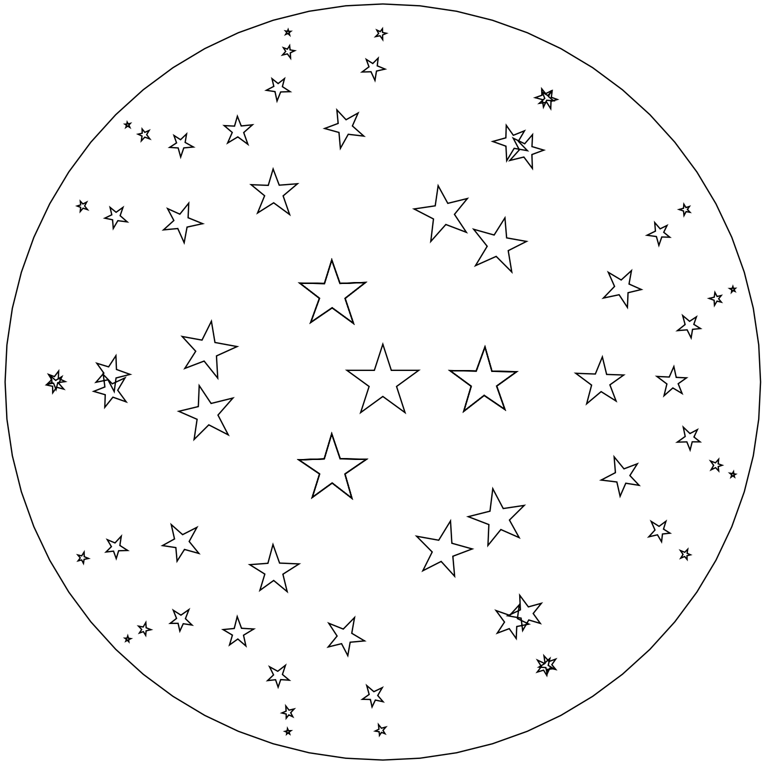
Conformal Geometry via Geometric Algebra

Alan Bromborsky
Army Research Lab (Retired)
brombo@comcast.net

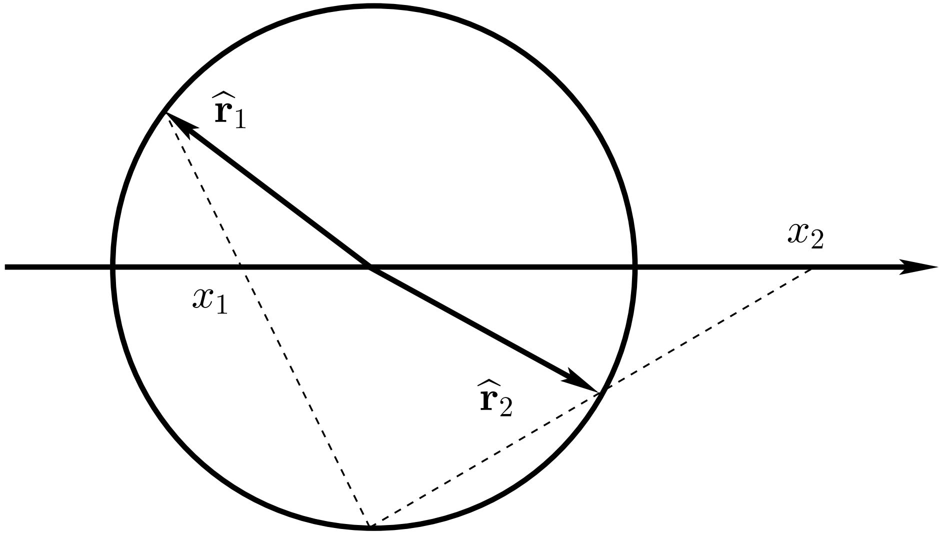
September 9, 2007

Hyperbolic Translation



Conformal Geometry

In Conformal Geometry the vector space $\mathcal{V}(p,q)$ is enlarged to $\mathcal{V}(p+1,q+1)$ with a mapping function $F:\mathcal{V}(p,q)\rightarrow\mathcal{V}(p+1,q+1)$ such that $F(x)^2=0\;\forall x\in\mathcal{V}(p,q)$. $F(x)$ is a null vector. The motivation for this is that translations, rotations, dialations, and inversions in $\mathcal{V}(p,q)$ can all be encoded as rotations in $\mathcal{V}(p+1,q+1)$. The geometric algebra of $\mathcal{G}(p+1,q+1)$ makes handeling the rotations very simple. The starting point of conformal geometric is the stereographic projection of the line as show:



The points x_1 and x_2 are mapped into the unit vectors $\widehat{\mathbf{r}}_1$ and $\widehat{\mathbf{r}}_2$. A little algebra shows that for a general point x on the line

$$\widehat{\mathbf{r}} = \frac{2x}{1+x^2}e_1 + \frac{1-x^2}{1+x^2}e_2 \tag{1}$$

But this representation is not homogenous since $\widehat{\mathbf{r}}^2 \neq 0$. Rename e_2 to e to distinguish it from the basis vector(s) of the space being described (in this case the line) and scale and add the vector $\bar{e}$ to equation 1 to get

$$X = 2xe_1 + \left(1 - x^2\right) e + \left(1 + x^2\right) \bar{e} \tag{2}$$

where

$$e^2 = 1, \qquad \bar{e}^2 = -1, \qquad e \cdot \bar{e} = 0$$

and $X^2 = 0$ and the vector space $\mathcal{V}\left(1,0\right)$ has been extended to $\mathcal{V}\left(2,1\right)$. To put equation 2 into standard form define

$$
\begin{gathered}
n = e + \bar{e}, \qquad \bar{n} = e - \bar{e} \\
n^2 = \bar{n}^2 = 0, \qquad n \cdot \bar{n} = 2 \\
X = 2xe_1 + x^2 n - \bar{n}
\end{gathered}
$$

Then the general mapping $F : \mathcal{V}(p,q) \rightarrow \mathcal{V}(p+1,q+1)$ is given by

$$
F(x) = x^2 n + 2x - \bar{n},\ \forall x \in \mathcal{V}(p,q) \tag{3}
$$

Note that

$$
\begin{aligned}
F(x) \cdot F(y) &= \left(x^2 n + 2x - \bar{n}\right) \cdot \left(y^2 n + 2y - \bar{n}\right) \\
&= -2x^2 - 2y^2 + 4x \cdot y \\
&= -2(x-y)^2 \qquad (4)
\end{aligned}
$$

so that the inner product in $\mathcal{V}(p+1,q+1)$ encodes the distance between two points in $\mathcal{V}(p,q)$. Note that $F(x) \cdot F(y) = 0$ implies that $x = y$, such are the wonders of null vectors.

Conformal Transformations

Conformal transformations consist of translations, rotatations, dilations, and inversions in $\mathcal{V}\left(p,q\right)$. In the conformal space $\mathcal{V}\left(p+1,q+1\right)$ all these transformations can be represented by rotations and reflections.

1. Translations: $a \in \mathcal{V}\left(p,q\right)$ is the translation vector then
$$F\left(x'\right) = e^{\frac{na}{2}} F\left(x\right) e^{-\frac{na}{2}} \text{ where } x' = x + a \tag{5}$$
2. Rotations: B is a unit bivector in $\mathcal{G}\left(p,q\right)$ and ϕ the rotation angle then
$$F\left(x'\right) = e^{\frac{B\phi}{2}} F\left(x\right) e^{-\frac{B\phi}{2}} \tag{6}$$
3. Dilations: $\alpha \in \Re$ then $x' = e^{-\alpha} x$
$$F\left(x'\right) = e^{\frac{\alpha e\bar{e}}{2}} F\left(x\right) e^{-\frac{\alpha e\bar{e}}{2}} \tag{7}$$

4. Inversion: $x' = \dfrac{x}{x^2}$

$$F\left(x'\right) = eF\left(x\right)e \tag{8}$$

Composition of transformations is as expected. If $T\left(a\right) = e^{\frac{na}{2}}$, $R\left(B,\phi\right) = e^{\frac{B\phi}{2}}$, and $D\left(\alpha\right) = e^{\frac{\alpha e\bar{e}}{2}}$ and we wish to translate, rotate, dilate, and invert (in that order) the composite transformation f is

$$f = eD\left(\alpha\right)R\left(B,\phi\right)T\left(a\right) \tag{9}$$

and

$$F\left(x'\right) = fF\left(x\right)f^{\dagger} \tag{10}$$

where

$$f^{\dagger} = T\left(a\right)^{\dagger}R\left(B,\phi\right)^{\dagger}D\left(\alpha\right)^{\dagger}e^{\dagger},\ \text{Note that } e^{\dagger} = e \tag{11}$$

Geometric Primitives

Geometric primitives are formed from the repeated application of the exterior product to null vectors in $\mathcal{V}(p+1,q+1)$ (conformal space).

1. Points: If A and B are null vectors in the conformal space then the solutions to

$$G \wedge X = A \wedge B \wedge X = 0,\ X^2 = 0 \text{ are exactly } A \text{ and } B \tag{12}$$

2. Circles/Lines: If A, B, and C are null vectors in the conformal space then the solutions to

$$L \wedge X = A \wedge B \wedge C \wedge X = 0,\ X^2 = 0 \text{ are circles} \tag{13}$$

   with points a, b, and c on the circle. If $C = n$ the circle becomes a straight line (n is the point at infinity).

The circle parameters are

$$\rho^2 = -\frac{L^2}{(L \wedge n)^2} \;\; \text{radius, } X_0 = LnL \;\; \text{center.} \tag{14}$$

3. Spheres/Planes: If A, B, C, and D are null vectors in the conformal space then the solutions to

$$P \wedge X = A \wedge B \wedge C \wedge D \wedge X = 0, \;\; X^2 = 0 \text{ are spheres} \tag{15}$$

with points a, b, c, and d on the sphere. If $D = n$ the sphere becomes a plane (n is the point at infinity). The sphere parameters are

$$\rho^2 = \frac{P^2}{(P \wedge n)^2} \;\; \text{radius, } X_0 = PnP \;\; \text{center.} \tag{16}$$

Intersections

The dual operation for pure grade multivectors (denoted by *) in the conformal space is defined by

$$C^* = IC \tag{17}$$

where I is the pseudo-scalar for the conformal space. The intersection of two lines and/or two circles defined by tri-vectors L_1 and L_2 (2D geometry) is given by the bi-vector

$$B = \left(L_1^* \wedge L_2^*\right)^* . \tag{18}$$

B represents zero, one, or two points depending upon the sign of B^2. Likewise if one has a line/circle represents by the trivector L and a plane/sphere represented by the quad-vector P (3D geometry) the points of intersection are given by the bi-vector B

$$B = (L^* \wedge P^*) \tag{19}$$

Finally (for 3D space) the intersection of two planes/spheres denoted by quad-vectors P_1 and P_2 is given by the tri-vector L (line or circle)

$$L = (P_1^* \wedge P_2^*) \tag{20}$$

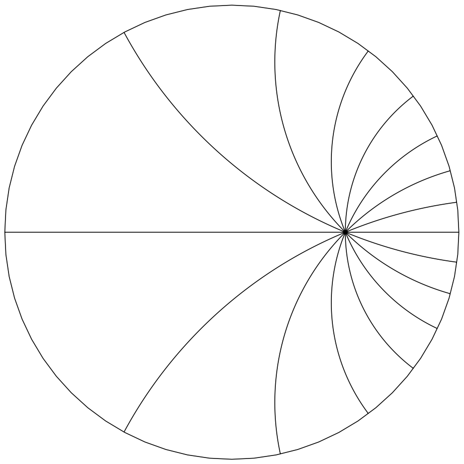
The intersection of lines/circles in 3D space does not have such simple formulas and must be approached indirectly by analyzing the intersection of planes defined by circles and their intersection with each other or with lines.

Non Euclidian Geometry

The unit circle in 2D space, $\mathcal{G}\left(3,1\right)$, is defined by the tri-vector Ie. The angle of intersection of the unit circle with a general line L that is perpendicular to the unit circle is given by

$$(Ie)\cdot L = I\left(e\wedge L\right) = 0. \tag{21}$$

This implies $L = A\wedge B\wedge e$.



With typical $L's$ (D-lines) as shown.

We now need a non-euclidian distance function. The first step is to determine a rotor that maps D-lines into D-lines. For this we require that the rotor $R = e^{\frac{\alpha\widehat{U}}{2}}$ map $e = ReR^{\dagger}$ where $U = |U|\,\widehat{U}$ is a bi-vector. e must be an invarient point of the transformation.

Thus we need

$$e = \left(c\left(\frac{\alpha}{2}\right) + s\left(\frac{\alpha}{2}\right)\widehat{U}\right) e \left(c\left(\frac{\alpha}{2}\right) - s\left(\frac{\alpha}{2}\right)\widehat{U}\right) \tag{22}$$

If $U^2 > 0$ $c\left(\right) = \cosh\left(\right)$ and $s\left(\right) = \sinh\left(\right)$ or if $B^2 < 0$ $c\left(\right) = \cos\left(\right)$ and $s\left(\right) = \sin\left(\right)$. Equation 22 is true if $eU = Ue$. If we let $U = Le$ we have $Le = eL$ which implies $Ue = eU = L$ thus

$$U = Le = \left(A \wedge B \wedge e\right) e \text{ and } \widehat{U} = \frac{U}{|U|} \tag{23}$$

Note that $U^2 = L^2$. After some laborious algebra we find that

$$\begin{aligned} L^2 &= (A\cdot B)^2 - 2\,(A\cdot B)\,(A\cdot e)\,(B\cdot e) \\ &= 4\,(a-b)^2\left((a-b)^2 + \left(1-a^2\right)\left(1-b^2\right)\right) \end{aligned} \tag{24}$$

So that $L^2 > 0 \ \forall \ a^2, b^2 < 1$ or $\forall \ a^2, b^2 > 1$ and we must use the hyperbolic functions in expanding $e^{\frac{\alpha\widehat{U}}{2}}$. Now we must determine what α correspondes to the transport of A to B along the D-line connecting A and B. With our definition of $R(\alpha)$ we have $e = R(\alpha)\,eR(\alpha)^\dagger$. Now consider the equation

$$X(\alpha) = R(\alpha)\,AR(\alpha)^\dagger \tag{25}$$

We have $X(0) = A$ and must determine if there is an α such that $B = X(\alpha)$. Or equivalently is there a solution to $B\cdot R(\alpha)\,AR(\alpha)^\dagger = 0$ since B is a null vector.

After a lot of algebra we determine that

$$\begin{aligned} d\left(A,B\right) &= 2\sinh^{-1}\left(\sqrt{-\frac{A\cdot B}{2\left(A\cdot e\right)\left(B\cdot e\right)}}\right) \\ &= 2\sinh^{-1}\left(\sqrt{\frac{\left|a-b\right|^{2}}{\left(1-a^{2}\right)\left(1-b^{2}\right)}}\right). \end{aligned} \tag{26}$$

For three points $A,B,$ and C on the same D-line we have

$$d\left(A,C\right)=d\left(A,B\right)+d\left(B,C\right) \tag{27}$$

since

$$R\left(d\left(A,C\right)\right)=R\left(d\left(A,B\right)\right)R\left(d\left(B,C\right)\right). \tag{28}$$