An Introduction to Geometric Algebra

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History

Geometric algebra is the Clifford algebra of a finite dimensional vector space over real scalars cast in a form most appropriate for physics and engineering. This was done by David Hestenes (Arizona State University) in the 1960's. From this start he developed the geometric calculus whose fundamental theorem includes the generalized Stokes theorem, the residue theorem, and new integral theorems not realized before. Hestenes likes to say he was motivated by the fact that physicists and engineers did not know how to multiply vectors.

Researchers at Arizona State and Cambridge have applied these developments to classical mechanics,quantum mechanics, general relativity (guage theory of gravity), projective geometry, conformal geometry, etc.

Axioms of Geometric Algebra

Let $\mathcal{V}(p,q)$ be a finite dimensional vector space of signature (p,q) over $\Re$. Then $\forall a, b, c \in \mathcal{V}$ there exists a geometric product with the properties -

$$\begin{aligned} (ab)c &= a(bc) \\ a(b+c) &= ab + ac \\ (a+b)c &= ac + bc \\ aa &\in \Re \end{aligned}$$

If $a^2 \neq 0$ then $a^{-1} = \frac{1}{a^2}a$.

Why Learn This Stuff?

The geometric product of two (or more) vectors produces something "new" like the $\sqrt{-1}$ with respect to real numbers or vectors with respect to scalars. It must be studied in terms of its effect on vectors and in terms of its symmetries. It is worth the effort. Anything that makes understanding rotations in a N demensional space simple is worth the effort! Also, if one proceeds on to geometric calculus many diverse areas in mathematics are unified and many areas of physics and engineering are greatly simplified.

Inner, $\cdot$, and outer, $\wedge$, product of two vectors and their basic properties

$$a \cdot b = \frac{1}{2}(ab + ba) \tag{1}$$

$$a \wedge b = \frac{1}{2}(ab - ba) \tag{2}$$

$$ab = a \cdot b + a \wedge b \tag{3}$$

$$a \wedge b = -b \wedge a \tag{4}$$

$$\begin{gathered} c^2 = (a+b)^2 \\ c^2 = a^2 + ab + ba + b^2 \\ 2a \cdot b = c^2 - a^2 - b^2 \\ a \cdot b \in \Re \end{gathered} \tag{5}$$

$$a \cdot b = |a|\,|b| \cos(\theta) \text{ if } a^2, b^2 > 0 \tag{6}$$

Orthongonal vectors are defined by $a \cdot b = 0$.
For orthogonal vectors $a \wedge b = ab$.

Now compute $(a \wedge b)^2$

$$\begin{aligned}
(a \wedge b)^2 &= -(a \wedge b)(b \wedge a) && (7)\\
&= -(ab - a \cdot b)(ba - a \cdot b) && (8)\\
&= -\left(abba - (a \cdot b)(ab + ba) + (a \cdot b)^2\right) && (9)\\
&= -\left(a^2b^2 - (a \cdot b)^2\right) && (10)\\
&= -a^2b^2\left(1 - \cos^2(\theta)\right) && (11)\\
&= -a^2b^2\sin^2(\theta) && (12)
\end{aligned}$$

Thus in a Euclidian space, $a^2, b^2 > 0$, $(a \wedge b)^2 \leq 0$ and $a \wedge b$ is proportional to $\sin(\theta)$. If $e_{\|}$ and $e_{\perp}$ are any two orthonormal unit vectors in a Euclidian space then $\left(e_{\|}e_{\perp}\right)^2 = -1$. Who needs the $\sqrt{-1}$?

Outer, $\wedge$, product for r Vectors in terms of the geometric product

We define the outer product of r vectors to be

$$a_1 \wedge \ldots \wedge a_r = \frac{1}{r!} \sum_{i_1,\ldots,i_r} \varepsilon^{i_1 \ldots i_r}_{1 \ldots r} a_{i_1} \ldots a_{i_r} \tag{13}$$

Thus

$$\begin{aligned} a_1 \wedge \ldots \wedge (a_j + b_j) \wedge \ldots \wedge a_r &= \\ & a_1 \wedge \ldots \wedge a_j \wedge \ldots \wedge a_r + a_1 \wedge \ldots \wedge b_j \wedge \ldots \wedge a_r \end{aligned} \tag{14}$$

and

$$\begin{aligned} a_1 \wedge \ldots \wedge a_j \wedge a_{j+1} \wedge \ldots \wedge a_r &= \\ & -a_1 \wedge \ldots \wedge a_{j+1} \wedge a_j \wedge \ldots \wedge a_r \end{aligned} \tag{15}$$

The outer product of r vectors is called a blade of grade r.

Alternate Definition of Outer, $\wedge$, product for r Vectors

Let $e_1, e_2, \ldots, e_r$ be an orthogonal basis for the set of linearly independent vectors $a_1, a_2, \ldots, a_r$ so that we can write

$$a_i = \sum_j \alpha_{ij} e_j \tag{16}$$

Then

$$\begin{aligned} a_1 a_2 \ldots a_r &= \left(\sum_{j_1} \alpha_{1j_1} e_{j_1}\right)\left(\sum_{j_2} \alpha_{2j_2} e_{j_2}\right)\cdots\left(\sum_{j_r} \alpha_{rj_r} e_{j_r}\right) \\ &= \sum_{j_1,\ldots,j_r} \alpha_{1j_1}\alpha_{2j_2}\ldots\alpha_{rj_r} e_{j_1} e_{j_2}\ldots e_{j_r} \end{aligned} \tag{17}$$

Now define a blade of grade n as the geometric product of n orthogonal vectors. Thus the product $e_{j_1}e_{j_2}\ldots e_{j_r}$ in equation 17 could be a blade of grade r, $r-2$, $r-4$, etc. depending upon the number of repeated factors.

If there are no repeated factors in the product we have that

$$e_{j_1}\ldots e_{j_r} = \varepsilon^{j_1\ldots j_r}_{1\ldots r} e_1 \ldots e_r \tag{18}$$

Due to the fact that interchanging two adjacent orthogonal vectors in the geometric product will reverse the sign of the product and we can define the outer product of r vectors as

$$\begin{aligned} a_1 \wedge \ldots \wedge a_r &= \sum_{j_1,\ldots,j_r} \varepsilon^{j_1\ldots j_r}_{1\ldots r} \alpha_{1j_1} \ldots \alpha_{rj_r} e_1 \ldots e_r & (19) \\ &= \det(\alpha)\, e_1 \ldots e_r & (20) \end{aligned}$$

Thus the outer product of r independent vectors is the part of the geometric product of the r vectors that is of grade r. Equation 19 is equivalent to equation 13.

This can be proved by substituting equation 17 into equation 13 to get

$$a_1 \wedge \ldots \wedge a_r = \frac{1}{r!} \sum_{i_1,\ldots,i_r} \sum_{j_1,\ldots,j_r} \varepsilon^{i_1\ldots i_r}_{1\ldots r} \alpha_{i_1 j_1} \ldots \alpha_{i_r j_r} e_{j_1} \ldots e_{j_r} \tag{21}$$

$$= \frac{1}{r!} \sum_{i_1,\ldots,i_r} \sum_{j_1,\ldots,j_r} \varepsilon^{i_1\ldots i_r}_{1\ldots r} \varepsilon^{j_1\ldots j_r}_{1\ldots r} \alpha_{i_1 j_1} \ldots \alpha_{i_r j_r} e_1 \ldots e_r \tag{22}$$

$$= \frac{1}{r!} \sum_{j_1,\ldots,j_r} \varepsilon^{j_1\ldots j_r}_{1\ldots r} \varepsilon^{j_1\ldots j_r}_{1\ldots r} \det(\alpha)\, e_1 \ldots e_r \tag{23}$$

$$= \det(\alpha)\, e_1 \ldots e_r \tag{24}$$

We go from equation 22 to equation 23 by noting that $\sum\limits_{i_1,\ldots,i_r} \varepsilon^{i_1\ldots i_r}_{1\ldots r} \alpha_{i_1 j_1} \ldots \alpha_{i_r j_r}$ is just $\det(\alpha)$ with the columns permuted.

Multiplying $\det(\alpha)$ by $\varepsilon^{j_1 \ldots j_r}_{1 \ldots r}$ gives the correct sign for the determinant with the columns permuted.

Useful Relation's

1. For a set of r orthogonal vectors, $e_1, \ldots, e_r$

$$e_1 \wedge \ldots \wedge e_r = e_1 \ldots e_r \tag{25}$$

2. For a set of r linearly independent vectors, $a_1, \ldots, a_r$, there exists a set of r orthongonal vectors, $e_1, \ldots, e_r$, such that

$$a_1 \wedge \ldots \wedge a_r = e_1 \ldots e_r \tag{26}$$

   If the vectors, $a_1, \ldots, a_r$, are not linearly independent then

$$a_1 \wedge \ldots \wedge a_r = 0 \tag{27}$$

The product $a_1 \wedge \ldots \wedge a_r$ is call a "blade" of grade r. The dimension of the vector space is the highest grade any blade can have.

Projection Operator

A multivector, the basic element of the geometric algebra, is made of of a sum of scalars, vectors, blades. A mutivector is homogenous (pure) if all the blades in it are of the same grade. The grade of a scalar is 0 and the grade of a vector is 1. The general multivector A is decomposed with the grade projection operator $\langle A \rangle_r$ as (N is dimension of the vector space):

$$A = \sum_{r=0}^{N} \langle A \rangle_r \tag{28}$$

As an example consider ab, the product of two vectors. Then

$$ab = \langle ab \rangle_0 + \langle ab \rangle_2 \tag{29}$$

We define $\langle A \rangle \equiv \langle A \rangle_0$ for any multivector A

Basis Blades

The geometric algebra of a vector space, $\mathcal{V}(p,q)$, is denoted $\mathcal{G}(p,q)$ or $\mathcal{G}(\mathcal{V})$ where (p,q) is the signature of the vector space (first p unit vectors square to $+1$ and next q unit vectors square to -1, dimension of the space is $p+q$). Examples are:

p	q	Type of Space
3	0	3D Euclidian
1	3	Relativistic Space Time
4	1	3D Conformal Geometry

If the orthonormal basis set of the vector space is $e_1, \ldots, e_N$, the basis of the geometric algebra (multivector space) is formed from the geometric products (since we have choosen an orthonormal basis) of the basis vectors. For grade r multivectors the basis blades are all the combinations of basis vectors products taken r at a time from the set of N vectors. Thus the number basis blades of r rank are $\binom{N}{r}$, the binonomial expansion coefficient and the total dimension of the multivector space is the sum of $\binom{N}{r}$ over r which is 2^N. Thus the basis blades for $\mathcal{G}(3,0)$ are:

Grade			
0	1	2	3
1	e_1	e_1e_2	$e_1e_2e_3$
	e_2	e_1e_3	
	e_3	e_2e_3	

The multiplication table for the $\mathcal{G}(3,0)$ basis blades is

	1	e_1	e_2	e_3	e_1e_2	e_1e_3	e_2e_3	$e_1e_2e_3$
1	1	e_1	e_2	e_3	e_1e_2	e_1e_3	e_2e_3	$e_1e_2e_3$
e_1	e_1	1	e_1e_2	e_1e_3	e_2	e_3	$e_1e_2e_3$	e_2e_3
e_2	e_2	$-e_1e_2$	1	e_2e_3	$-e_1$	$-e_1e_2e_3$	e_3	$-e_1e_3$
e_3	e_3	$-e_1e_3$	$-e_2e_3$	1	$e_1e_2e_3$	$-e_1$	$-e_2$	e_1e_2
e_1e_2	e_1e_2	$-e_2$	e_1	$e_1e_2e_3$	-1	$-e_2e_3$	e_1e_3	$-e_3$
e_1e_3	e_1e_3	$-e_3$	$-e_1e_2e_3$	e_1	e_2e_3	-1	$-e_1e_2$	e_2
e_2e_3	e_2e_3	$e_1e_2e_3$	$-e_3$	e_2	$-e_1e_3$	e_1e_2	-1	$-e_1$
$e_1e_2e_3$	$e_1e_2e_3$	e_2e_3	$-e_1e_3$	e_1e_2	$-e_3$	e_2	$-e_1$	-1

Note that the squares of all the grade 2 and 3 basis blades are -1. The highest rank basis blade (in this case $e_1e_2e_3$) is ususally denoted by I and is called the pseudoscalar.

The multiplication table for the $\mathcal{G}(1,3)$ basis blades is (Part I)

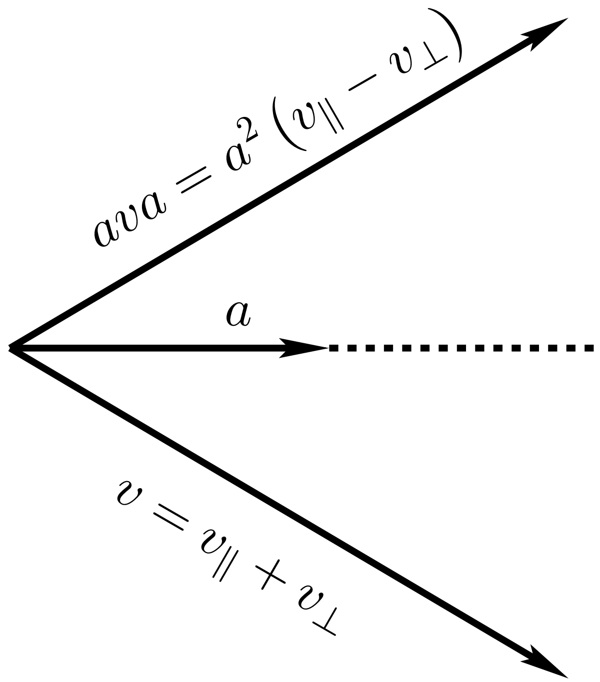
	1	γ_0	γ_1	γ_2	γ_3	$\gamma_0\gamma_1$	$\gamma_0\gamma_2$	$\gamma_1\gamma_2$
1	1	γ_0	γ_1	γ_2	γ_3	$\gamma_0\gamma_1$	$\gamma_0\gamma_2$	$\gamma_1\gamma_2$
γ_0	γ_0	1	$\gamma_0\gamma_1$	$\gamma_0\gamma_2$	$\gamma_0\gamma_3$	γ_1	γ_2	$\gamma_0\gamma_1\gamma_2$
γ_1	γ_1	$-\gamma_0\gamma_1$	-1	$\gamma_1\gamma_2$	$\gamma_1\gamma_3$	γ_0	$-\gamma_0\gamma_1\gamma_2$	$-\gamma_2$
γ_2	γ_2	$-\gamma_0\gamma_2$	$-\gamma_1\gamma_2$	-1	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2$	γ_0	γ_1
γ_3	γ_3	$-\gamma_0\gamma_3$	$-\gamma_1\gamma_3$	$-\gamma_2\gamma_3$	-1	$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$
$\gamma_0\gamma_1$	$\gamma_0\gamma_1$	$-\gamma_1$	$-\gamma_0$	$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_3$	1	$-\gamma_1\gamma_2$	$-\gamma_0\gamma_2$
$\gamma_0\gamma_2$	$\gamma_0\gamma_2$	$-\gamma_2$	$-\gamma_0\gamma_1\gamma_2$	$-\gamma_0$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2$	1	$\gamma_0\gamma_1$
$\gamma_1\gamma_2$	$\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2$	γ_2	$-\gamma_1$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_2$	$-\gamma_0\gamma_1$	-1
$\gamma_0\gamma_3$	$\gamma_0\gamma_3$	$-\gamma_3$	$-\gamma_0\gamma_1\gamma_3$	$-\gamma_0\gamma_2\gamma_3$	$-\gamma_0$	$\gamma_1\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$
$\gamma_1\gamma_3$	$\gamma_1\gamma_3$	$\gamma_0\gamma_1\gamma_3$	γ_3	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_1$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_2\gamma_3$
$\gamma_2\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$	γ_3	$-\gamma_2$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_3$	$\gamma_1\gamma_3$
$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2$	$\gamma_1\gamma_2$	$\gamma_0\gamma_2$	$-\gamma_0\gamma_1$	$\gamma_0\gamma_1\gamma_2\gamma_3$	γ_2	$-\gamma_1$	$-\gamma_0$
$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_1\gamma_3$	$\gamma_1\gamma_3$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_1$	γ_3	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_2\gamma_3$
$\gamma_0\gamma_2\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_2$	$\gamma_1\gamma_2\gamma_3$	γ_3	$\gamma_0\gamma_1\gamma_3$
$\gamma_1\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_2\gamma_3$	$\gamma_1\gamma_3$	$-\gamma_1\gamma_2$	$\gamma_0\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_3$	$-\gamma_3$
$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_3$	$-\gamma_0\gamma_1\gamma_2$	$\gamma_2\gamma_3$	$-\gamma_1\gamma_3$	$-\gamma_0\gamma_3$

The multiplication table for the $\mathcal{G}(1,3)$ basis blades is (Part II)

	$\gamma_0\gamma_3$	$\gamma_1\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$
1	$\gamma_0\gamma_3$	$\gamma_1\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$
γ_0	γ_3	$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2$	$\gamma_1\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$
γ_1	$-\gamma_0\gamma_1\gamma_3$	$-\gamma_3$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_2$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_2\gamma_3$	$\gamma_0\gamma_2\gamma_3$
γ_2	$-\gamma_0\gamma_2\gamma_3$	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_3$	$-\gamma_0\gamma_1$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_3$	$\gamma_1\gamma_3$	$-\gamma_0\gamma_1\gamma_3$
γ_3	γ_0	γ_1	γ_2	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_1$	$-\gamma_0\gamma_2$	$-\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2$
$\gamma_0\gamma_1$	$-\gamma_1\gamma_3$	$-\gamma_0\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$	γ_2	γ_3	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_2\gamma_3$	$\gamma_2\gamma_3$
$\gamma_0\gamma_2$	$-\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_3$	$-\gamma_1$	$\gamma_1\gamma_2\gamma_3$	γ_3	$\gamma_0\gamma_1\gamma_3$	$-\gamma_1\gamma_3$
$\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_2\gamma_3$	$-\gamma_1\gamma_3$	$-\gamma_0$	$\gamma_0\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_3$	$-\gamma_3$	$-\gamma_0\gamma_3$
$\gamma_0\gamma_3$	1	$\gamma_0\gamma_1$	$\gamma_0\gamma_2$	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_1$	$-\gamma_2$	$-\gamma_0\gamma_1\gamma_2$	$\gamma_1\gamma_2$
$\gamma_1\gamma_3$	$-\gamma_0\gamma_1$	-1	$\gamma_1\gamma_2$	$-\gamma_0\gamma_2\gamma_3$	$-\gamma_0$	$\gamma_0\gamma_1\gamma_2$	γ_2	$\gamma_0\gamma_2$
$\gamma_2\gamma_3$	$-\gamma_0\gamma_2$	$-\gamma_1\gamma_2$	-1	$\gamma_0\gamma_1\gamma_3$	$-\gamma_0\gamma_1\gamma_2$	$-\gamma_0$	$-\gamma_1$	$-\gamma_0\gamma_1$
$\gamma_0\gamma_1\gamma_2$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_3$	-1	$\gamma_2\gamma_3$	$-\gamma_1\gamma_3$	$-\gamma_0\gamma_3$	$-\gamma_3$
$\gamma_0\gamma_1\gamma_3$	$-\gamma_1$	$-\gamma_0$	$\gamma_0\gamma_1\gamma_2$	$-\gamma_2\gamma_3$	-1	$\gamma_1\gamma_2$	$\gamma_0\gamma_2$	γ_2
$\gamma_0\gamma_2\gamma_3$	$-\gamma_2$	$-\gamma_0\gamma_1\gamma_2$	$-\gamma_0$	$\gamma_1\gamma_3$	$-\gamma_1\gamma_2$	-1	$-\gamma_0\gamma_1$	$-\gamma_1$
$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2$	γ_2	$-\gamma_1$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_2$	$\gamma_0\gamma_1$	1	$-\gamma_0$
$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_1\gamma_2$	$\gamma_0\gamma_2$	$-\gamma_0\gamma_1$	γ_3	$-\gamma_2$	γ_1	γ_0	-1

Reflections

We wish to show that $a, v \in \mathcal{V} \rightarrow ava \in \mathcal{V}$ and v is reflected about a if $a^2 = 1$.



1. Decompose $v = v_{\|} + v_{\perp}$ where $v_{\|}$ is the part of v parallel to a and $v_{\perp}$ is the part perpendicular to a.

2. $av = av_{\|} + av_{\perp} = v_{\|}a - v_{\perp}a$ since a and $v_{\perp}$ are orthogonal.

3. $ava = a^2 \left(v_{\|} - v_{\perp}\right)$ is a vector since a^2 is a scalar.

4. ava is the reflection of v about the direction of a if $a^2 = 1$.

5. Thus $a_1 \ldots a_r v a_r \ldots a_1 \in \mathcal{V}$ and produces a composition of reflections of v if $a_1^2 = \ldots = a_r^2 = 1$.

Rotations, Part 1

First define the reverse of a product of vectors. If $R = a_1 \ldots a_s$ then the reverse is $R^\dagger = (a_1 \ldots a_s)^\dagger = a_r \ldots a_1$, the order of multiplication is reversed. Then let $R = ab$ so that

$$RR^\dagger = (ab)(ba) = ab^2a = a^2b^2 = R^\dagger R \tag{30}$$

Let $RR^\dagger = 1$ and calculate $\left(RvR^\dagger\right)^2$, where v is an arbitrary vector.

$$\left(RvR^\dagger\right)^2 = RvR^\dagger RvR^\dagger = Rv^2R^\dagger = v^2RR^\dagger = v^2 \tag{31}$$

Thus $RvR^\dagger$ leaves the length of v unchanged.

Now we must also prove $Rv_1R^\dagger \cdot Rv_2R^\dagger = v_1 \cdot v_2$. Since $Rv_1R^\dagger$ and $Rv_2R^\dagger$ are both vectors we can use the definition of the dot product for two vectors

$$
\begin{aligned}
Rv_1R^\dagger \cdot Rv_2R^\dagger &= \frac{1}{2}\left(Rv_1R^\dagger Rv_2R^\dagger + Rv_2R^\dagger Rv_1R^\dagger\right) \\
&= \frac{1}{2}\left(Rv_1v_2R^\dagger + Rv_2v_1R^\dagger\right) \\
&= \frac{1}{2}R\left(v_1v_2 + v_2v_1\right)R^\dagger \\
&= R\left(v_1 \cdot v_2\right)R^\dagger \\
&= v_1 \cdot v_2RR^\dagger \\
&= v_1 \cdot v_2
\end{aligned}
$$

Thus the transfomation $RvR^\dagger$ preserves both length and angle and must be a rotation. The normal designation for R is a rotor.

Rotations, Part 2

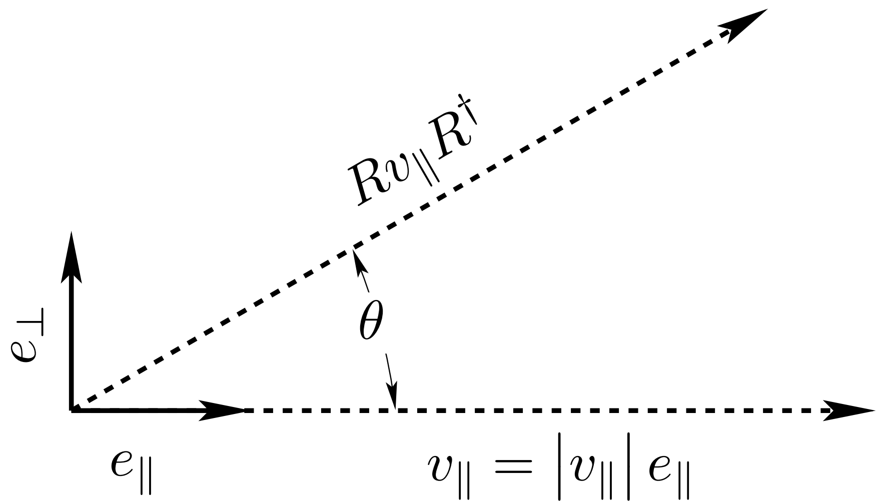
The general rotation can be represented by $R = e^{\frac{\theta}{2}u}$ where u is a unit bivector in the plane of the rotation and θ is the rotation angle in the plane. The two possible non-degenerate cases are $u^2 = \pm 1$

$$e^{\frac{\theta}{2}u} = \left\{ \begin{array}{lrl} \text{(Euclidian plane)} & u^2 = -1: & \cos\left(\frac{\theta}{2}\right) + u\sin\left(\frac{\theta}{2}\right) \\ \text{(Minkowski plane)} & u^2 = 1: & \cosh\left(\frac{\theta}{2}\right) + u\sinh\left(\frac{\theta}{2}\right) \end{array} \right\} \quad (32)$$

Decompose $v = v_{\|} + \left(v - v_{\|}\right)$ where $v_{\|}$ is the projection of v into the plane defined by u. Note the $v - v_{\|}$ is orthogonal to all vectors in the u plane. Now let $u = e_{\perp}e_{\|}$ where $e_{\|}$ is parallel to $v_{\|}$ and of course $e_{\perp}$ is in the plane u and orthogonal to $e_{\|}$. $v - v_{\|}$ anticommutes with $e_{\|}$ and $e_{\perp}$ and $v_{\|}$ anticommutes with $e_{\perp}$ (it is left to the viewer to show $RR^{\dagger} = 1$).

Euclidian Case

For the case of $u^2 = -1$



$$
\begin{aligned}
RvR^{\dagger} \quad = \quad & \left(\cos\left(\frac{\theta}{2}\right) + e_{\perp}e_{\|}\sin\left(\frac{\theta}{2}\right)\right)\left(v_{\|} + \left(v - v_{\|}\right)\right) \times \\
& \left(\cos\left(\frac{\theta}{2}\right) + e_{\|}e_{\perp}\sin\left(\frac{\theta}{2}\right)\right) \qquad (33)
\end{aligned}
$$

Since $v - v_{\|}$ anticommutes with $e_{\|}$ and $e_{\perp}$ it commutes with R and

$$RvR^{\dagger} = Rv_{\|}R^{\dagger} + \left(v - v_{\|}\right) \tag{34}$$

So that we only have to evaluate

$$Rv_{\|}R^{\dagger} = \left(\cos\left(\frac{\theta}{2}\right) + e_{\perp}e_{\|}\sin\left(\frac{\theta}{2}\right)\right) v_{\|} \left(\cos\left(\frac{\theta}{2}\right) + e_{\|}e_{\perp}\sin\left(\frac{\theta}{2}\right)\right) \tag{35}$$

Since $v_{\|} = \left|v_{\|}\right| e_{\|}$

$$Rv_{\|}R^{\dagger} = \left|v_{\|}\right| \left(\cos\left(\theta\right) e_{\|} + \sin\left(\theta\right) e_{\perp}\right) \tag{36}$$

and the component of v in the u plane is rotated correctly.

Minkowski Case

For the case of $u^2 = 1$ there are two possiblities, $v_{\|}^2 > 0$ or $v_{\|}^2 < 0$. In the first case $e_{\|}^2 = 1$ and $e_{\perp}^2 = -1$. In the second case $e_{\|}^2 = -1$ and $e_{\perp}^2 = 1$. Again $v - v_{\|}$ is not affected by the rotation so that we need only evaluate

$$\begin{aligned} R v_{\|} R^{\dagger} &= \left(\cosh\left(\frac{\theta}{2}\right) + e_{\perp} e_{\|} \sinh\left(\frac{\theta}{2}\right)\right) v_{\|} \times \\ &\quad \left(\cosh\left(\frac{\theta}{2}\right) + e_{\|} e_{\perp} \sinh\left(\frac{\theta}{2}\right)\right) \end{aligned} \tag{37}$$

Note that in this case $\left|v_{\|}\right| = \sqrt{\left|v_{\|}^2\right|}$ and

$$R v_{\|} R^{\dagger} = \left\{ \begin{array}{l} v_{\|}^2 > 0 : \left|v_{\|}\right| \left(\cosh\left(\theta\right) e_{\|} + \sinh\left(\theta\right) e_{\perp}\right) \\ v_{\|}^2 < 0 : \left|v_{\|}\right| \left(\cosh\left(\theta\right) e_{\|} - \sinh\left(\theta\right) e_{\perp}\right) \end{array} \right\} \tag{38}$$

Spinors

The general definition of a spinor is a even (all odd grade components are zero) multivector, ψ, such that $\psi v \psi^\dagger \in \mathcal{V}\ \forall v \in \mathcal{V}$. The most important case of spinors is for $\mathcal{G}(1,3)$ (relativistic quantum mechanics). We can write the general spinor for $\mathcal{G}(1,3)$ as

$$\psi = \alpha + \beta u + \gamma I \tag{39}$$

where u is a unit bivector (note that for $\mathcal{G}(1,3)$ $I^\dagger = I$ and v commutes with I since the grade of I is even). Then

$$\begin{aligned} \psi v \psi^\dagger &= (\alpha + \beta u + \gamma I)\, v \left(\alpha + \beta u^\dagger + \gamma I^\dagger\right) \\ &= (\alpha + \gamma I)^2\, v + \beta^2 u v u^\dagger \\ &= \left(\alpha^2 + \gamma^2 I^2 + \alpha\gamma I\right) v + \beta^2 u v u^\dagger \end{aligned} \tag{40}$$

Thus $\psi v \psi^{\dagger} \in \mathcal{V} \rightarrow \alpha\gamma = 0$. Either $\alpha = 0$ or $\gamma = 0$. If $\gamma = 0$ the the spinor can be expressed as $(\rho > 0)$

$$\psi = \sqrt{\rho} e^{u\frac{\theta}{2}} \tag{41}$$

so that $\psi\psi^{\dagger} = \rho$ and ψ rotates v and dialates it by a factor of ρ. If $\alpha = 0$ then

$$\psi = \sqrt{\rho} I e^{Iu\frac{\theta}{2}} = \left\{ \begin{array}{rl} u^2 = -1: & \sqrt{\rho} I \left(\cosh\left(\frac{\theta}{2}\right) + Iu \sinh\left(\frac{\theta}{2}\right)\right) \\ u^2 = 1: & \sqrt{\rho} I \left(\cos\left(\frac{\theta}{2}\right) + Iu \sin\left(\frac{\theta}{2}\right)\right) \end{array} \right\} \tag{42}$$

In this case $\psi\psi^{\dagger} = -\rho$ and ψ rotates v, dialates it by a factor of ρ, and reflects it. The case of a spinor for $\mathcal{G}(p, q)$ in general is a bit more complicated.

Quaternions

Any multivector $A \in \mathcal{G}(3,0)$ may be written as

$$A = \alpha + a + B + \beta I \tag{43}$$

where $\alpha, \beta \in \Re$, $a \in \mathcal{V}(3,0)$, B is a bivector, and I is the unit pseudoscalar. The quaternions are the multivectors of even grades

$$A = \alpha + B \tag{44}$$

B can be represented as

$$B = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k} \tag{45}$$

where $\mathbf{i} = e_2e_3$, $\mathbf{j} = e_1e_3$, and $\mathbf{k} = e_1e_2$, and

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \tag{46}$$

The quaternions form a subalgebra of $\mathcal{G}(3,0)$ since the geometric product of any two quaternions is also a quaternion since the geometric product of two even grade multivector components is a even grade multivector. For example the product of two grade 2 multivectors can only consist of grades 0, 2, and 4, but in $\mathcal{G}(3,0)$ we can only have grades 0 and 2 since the highest possible grade is 3.

Expansion of geometric product and generalization of $\cdot$ and $\wedge$

If A_r and B_s are respectively grade r and s pure grade multivectors then

$$A_r B_s = \left\langle A_r B_s \right\rangle_{|r-s|} + \left\langle A_r B_s \right\rangle_{|r-s|+2} + \cdots + \left\langle A_r B_s \right\rangle_{\min(r+s, 2N-(r+s))} \quad (47)$$

$$A_r \cdot B_s = \left\langle A_r B_s \right\rangle_{|r-s|} \quad (48)$$

$$A_r \wedge B_s = \left\langle A_r B_s \right\rangle_{r+s} \quad (49)$$

Thus if $r + s > N$ then $A_r \wedge B_s = 0$, also note that these formulas are the most efficient way of calculating $A_r \cdot B_s$ and $A_r \wedge B_s$.

Using equations 26 and 47 we can prove that for a vector a and a grade r multivector B_r

$$a \cdot B_r = \frac{1}{2}\left(aB_r - (-1)^r B_r a\right) \tag{50}$$

$$a \wedge B_r = \frac{1}{2}\left(aB_r + (-1)^r B_r a\right) \tag{51}$$

If equations 50 and 51 are true for a grade r blade they are also true for a grade r multivector (superposition of grade r blades). By equation 26 let $B_r = e_1 \ldots e_r$ where the $e's$ are orthogonal and expand a

$$a = a_\perp + \sum_{j=1}^{r} \alpha_j e_j \tag{52}$$

where $a_\perp$ is orthognal to all the $e's$. Then

$$aB_r = \sum_{j=1}^{r} (-1)^{j-1} \alpha_j e_1 \cdots \breve{e}_j \cdots e_r + a_\perp e_1 \ldots e_r$$

$$= \quad a \cdot B_r + a \wedge B_r \tag{53}$$

Now calculate

$$\begin{aligned}
B_r a &= \sum_{j=1}^{r} (-1)^{r-j} \alpha_j e_1 \cdots \breve{e}_j \cdots e_r - (-1)^{r-1} a_\perp e_1 \ldots e_r \\
&= (-1)^{r-1} \left(\sum_{j=1}^{r} (-1)^{j-1} \alpha_j e_1 \cdots \breve{e}_j \cdots e_r - a_\perp e_1 \ldots e_r \right) \\
&= (-1)^{r-1} (a \cdot B_r - a \wedge B_r)
\end{aligned} \tag{54}$$

Adding and subtracting equations 53 and 54 gives equations 50 and 51.

Duality and the Pseudoscalar

If $e_1, \ldots, e_n$ is an orthonormal basis for the vector space the the pseudoscalar I is defined by

$$I = e_1 \ldots e_n \tag{55}$$

Since one can tranform one orthonormal basis to another by an orthogonal transformation the I's for all orthonormal bases are equal to within a ± 1 scale factor with depends on the ordering of the basis vectors.

If A_r is a pure r grade multivector $(A_r = \langle A_r \rangle_r)$ then

$$A_r I = \langle A_r I \rangle_{n-r} \tag{56}$$

or $A_r I$ is a pure $n - r$ grade multivector. Further by the symmetry properties of I we have

$$IA_r = (-1)^{n(r-1)} A_r I \tag{57}$$

I can also be used to exchange the $\cdot$ and $\wedge$ products as follows using equations 50 and 51

$$\begin{aligned} a \cdot (A_r I) &= \frac{1}{2}\left(aA_r - (-1)^{n-r} A_r I a\right) &(58)\\ &= \frac{1}{2}\left(aA_r - (-1)^{n-r} (-1)^{n-1} A_r I a\right) &(59)\\ &= \frac{1}{2}\left(aA_r + (-1)^{r} A_r a\right) I &(60)\\ &= a \wedge A_r I &(61) \end{aligned}$$

More generally if A_r and B_s are pure grade multivectors with $r+s \leq n$ we have using equation 48 and 56

$$A_r \cdot (B_s I) = \langle A_r B_s I \rangle_{|r-(n-s)|} \tag{62}$$

$$= \langle A_r B_s I \rangle_{n-(r+s)} \tag{63}$$

$$= \langle A_r B_s \rangle_{r+s} I \tag{64}$$

$$= A_r \wedge B_s I \tag{65}$$

Reciprocal Frames

Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be a set of linearly independent vectors that span the vector space that are not necessarily orthogonal. These vectors define the frame (frame vectors are shown in bold face since they are almost always associated with a particular coordinate system) with volume element

$$E_n \equiv \mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_n \tag{66}$$

So that $E_n \propto I$. The reciprocal frame is the set of vectors $\mathbf{e}^1, \ldots, \mathbf{e}^n$ that statisfy the relation

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j, \quad \forall i, j = 1, \ldots, n \tag{67}$$

The $\mathbf{e}^i$ are constructed as follows

$$\mathbf{e}^j = (-1)^{j-1}\, \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \ldots \wedge \breve{\mathbf{e}}_j \wedge \ldots \wedge \mathbf{e}_n E_n^{-1} \tag{68}$$

So that the dot product is (using equation 61 since $E_n^{-1} \propto I$)

$$\begin{aligned}
\mathbf{e}_i \cdot \mathbf{e}^j &= (-1)^{j-1}\, \mathbf{e}_i \cdot \left(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \ldots \wedge \breve{\mathbf{e}}_j \wedge \ldots \wedge \mathbf{e}_n E_n^{-1}\right) && (69)\\
&= (-1)^{j-1} \left(\mathbf{e}_i \wedge \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \ldots \wedge \breve{\mathbf{e}}_j \wedge \ldots \wedge \mathbf{e}_n\right) E_n^{-1} && (70)\\
&= 0, \quad \forall i \neq j && (71)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{e}_1 \cdot \mathbf{e}^1 &= \mathbf{e}_1 \cdot \left(\mathbf{e}_2 \wedge \ldots \wedge \mathbf{e}_n E_n^{-1}\right) && (72)\\
&= \left(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \ldots \wedge \mathbf{e}_n\right) E_n^{-1} && (73)\\
&= 1 && (74)
\end{aligned}$$

Linear Transformations

Let f be a linear transformation $f : \mathcal{V} \rightarrow \mathcal{V}$ with $f(\alpha a + \beta b) = \alpha f(a) + \beta f(b)$ $\forall a, b \in \mathcal{V}$ and $\alpha, \beta \in \Re$. Then define the action of f on a blade of the geometric algebra by

$$f(a_1 \wedge \ldots \wedge a_r) = f(a_1) \wedge \ldots \wedge f(a_1) \tag{75}$$

and the action of f on any two $A, B \in \mathcal{G}(\mathcal{V})$ by

$$f(\alpha A + \beta B) = \alpha f(A) + \beta f(B) \tag{76}$$

Since any multivector A can be expanded as a sum of blades $f(A)$ is defined. This has many consequences. Consider the following definition for the determinant of f, $\det(f)$.

$$f(I) = \det(f)\, I \tag{77}$$

First show that this definition is equivalent to the standard definition of the determinant (again $e_1, \ldots, e_N$ is an orthonormal basis for $\mathcal{V}$).

$$f(e_r) = \sum_{s=1}^{N} a_{rs} e_s \tag{78}$$

Then

$$\begin{aligned} f(I) &= \left(\sum_{s_1=1}^{N} a_{1s_1} e_s\right) \wedge \ldots \wedge \left(\sum_{s_N=1}^{N} a_{Ns_N} e_s\right) \\ &= \sum_{s_1,\ldots,s_N} a_{1s_1} \ldots a_{Ns_N} e_{s_1} \ldots e_{s_N} \end{aligned} \tag{79}$$

But

$$e_{s_1} \ldots e_{s_N} = \varepsilon_{1\ldots N}^{s_1\ldots s_N} e_1 \ldots e_N \tag{80}$$

so that

$$f(I) = \sum_{s_1,\ldots,s_N} \varepsilon_{1\ldots N}^{s_1\ldots s_N} a_{1s_1}\ldots a_{Ns_N} I \tag{81}$$

or

$$\det(f) = \sum_{s_1,\ldots,s_N} \varepsilon_{1\ldots N}^{s_1\ldots s_N} a_{1s_1}\ldots a_{Ns_N} \tag{82}$$

which is the standard definition. Now compute the determinant of the product of the linear transformations f and g

$$\begin{aligned} \det(fg)\, I &= fg(I) \\ &= f(g(I)) \\ &= f(\det(g)\, I) \\ &= \det(g)\, f(I) \\ &= \det(g)\det(f)\, I \end{aligned} \tag{83}$$

or

$$\det(fg) = \det(f)\det(g) \tag{84}$$

Do you have any idea of how miserable that is to prove from the standard definition of determinant?

Another linear algebraic relation in geometric algebra is

$$f^{-1}(A) = \frac{I\underline{f}\left(I^{-1}A\right)}{\det(f)} \quad \forall A \in \mathcal{G}(\mathcal{V}) \tag{85}$$

where $\underline{f}$ is the adjoint transformation defined by $a \cdot \underline{f}(b) = b \cdot f(a)$ $\forall a, b \in \mathcal{V}$ and you have an explicit formula for the inverse of a linear transformation!

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