A Covariant Approach to Geometry using Geometric Algebra

Anthony Lasenby **Joan Lasenby**
Richard Wareham

CUED/F-INFENG/TR-483

Abstract

This report aims to show that using the mathematical framework of conformal geometric algebra – a 5-dimensional representation of 3-dimensional space – we are able to provide an elegant covariant approach to geometry. In this language, objects such as spheres, circles, lines and planes are simply elements of the algebra and can be transformed and intersected with ease. In addition, rotations, translation, dilations and inversions all become *rotations* in our 5-dimensional space; we will show how this enables us to provide very simple proofs of complicated constructions. We give examples of the use of this system in computer graphics and indicate how it can be extended into an even more powerful tool – we also discuss its advantages and disadvantages as a programming language. Lastly, we indicate how the framework might possibly be used to unify all geometries, thus enabling us to deal simply with the projective and non-Euclidean cases.

Contents

"You know we all became mathematicians for the same reason: we were lazy."

— Max Rosenlicht

1 *Introduction*

It has long been known that going to a 4d, projective, description of 3d Euclidean space can have various advantages – particularly when intersections of planes and lines are required. Such projective descriptions are used extensively in computer vision where rotations and translations can be described by a single 4×4 matrix and non-linear projective transformations become linear. This 4d description of projective geometry fits very nicely into the Geometric Algebra framework and applications are given in [10, 12].

In [9] conformal geometry was briefly discussed and recently [8, 11, 15] the application of these ideas in which a 5d conformal space is used as the representation of 3d Euclidean space has been the subject of renewed interest. It appears that in this conformal space we are able to deal with incidence relations of lines and planes, but also have the ability to extend to circles and spheres. Below we will describe in detail the basics of this conformal representation, outline how it can be of use in specific problems in computer graphics and indicate how we can extend its applicability to conics and more exotic geometries. In particular we look at the advantages and disadvantages of using the rather redundant representation provided by the conformal algebra in real programming scenarios.

We start with the simplest formulation of Hestenes' conformal geometry work in 3d. However, unlike the treatments given in [9], we will use *reflection* as the key to *inversion* which will enable us to treat circles and spheres very easily and efficiently and will provide valuable insights later when we discuss other geometries. Throughout the report we will stress the *covariant* nature of the approach – by this we mean that the structure of equations and relationships must be invariant under the allowed transformations. Thus, under covariance, if we operate on an object and then transform it, this will be the same as transforming the object and then operating on it. We will assume that the reader is familiar with the basic ideas behind applied Clifford and Geometric Algebra (GA) – many short introductions now exist [17, 13, 2, 3, 4].

1.1 *The Basic Operations in Conformal Space*

The notation we use will follow the original notation given in [9]. Let x be a vector in a space denoted $\mathfrak{A}(p,q)$. The annotation (p,q) is termed the *signature* of the space. A given signature, (p,q), implies that we may construct an orthogonal basis for the space, $\{e_i\}$, $i = 1,\cdots,n = p+q$ where $e_i^2 = +1$ for $i = 1,\cdots,p$ and $e_i^2 = -1$ for $i = p+1,\cdots,n$; i.e. we take a general mixed signature space.

To perform geometric operations using GA we now extend the space to $\mathfrak{A}(p+1,q+1)$ via the inclusion of two additional orthogonal basis vectors, e and $\bar{e}$, such that

$$e^2 = +1, \quad \bar{e}^2 = -1$$

Note that if $x \in \mathfrak{A}(p,q)$, then $e\cdot x = \bar{e}\cdot x = 0$ since $e_i \cdot e = e_i \cdot \bar{e} = 0$ for $i = 1,\cdots,n$. We now compose vectors n and $\bar{n}$ as

$$n = e+\bar{e} \qquad \bar{n} = e-\bar{e}$$

These vectors will be useful later. It is easy to see that n and $\bar{n}$ are *null* vectors since

$$\begin{aligned} n^2 &= (e+\bar{e})\cdot(e+\bar{e}) = e^2 + 2e\cdot\bar{e} + \bar{e}^2 \\ &= 1+0-1 = 0 \end{aligned}$$

and

$$\begin{aligned} \bar{n}^2 &= (e-\bar{e})\cdot(e-\bar{e}) = e^2 - 2e\cdot\bar{e} + \bar{e}^2 \\ &= 1-0-1 = 0 \end{aligned}$$

We also note a number of useful identities for n, $\bar{n}$ and $x \in \mathfrak{A}(p,q)$

$$\begin{aligned} n\cdot\bar{n} &= (e+\bar{e})\cdot(e-\bar{e}) = e^2 - \bar{e}^2 = 2 \\ x\cdot n = x\cdot\bar{n} &= 0 \end{aligned}$$

It is equally easy to show that if we define the bivector $E = n\wedge\bar{n}$ then $E^2 = 4$

$$\begin{aligned} E^2 &= (n\wedge\bar{n})\cdot(n\wedge\bar{n}) \\ &= (n\cdot\bar{n})(n\cdot\bar{n}) - n^2\bar{n}^2 \\ &= 4 \end{aligned} \tag{1.1}$$

since $n\cdot\bar{n} = 2$ and $n^2 = \bar{n}^2 = 0$.

To perform geometric operations upon vectors, points and objects in $\mathfrak{A}(p,q)$ we map a point $x \in \mathfrak{A}(p,q)$ to a point $F(x) \in \mathfrak{A}(p+1,q+1)$ and find that the operations may be performed by simple algebraic manipulation of $F(x)$. The specific mapping used is the Hestenes ([9], p.302) representation

$$F(x) = -\frac{1}{2}(x-e)n(x-e) \tag{1.2}$$

where, substituting for $n = e+\bar{e}$ and using the fact that $\bar{e}\cdot x = 0 = \bar{e}\cdot e = n\cdot x$, it is not hard to rewrite this equation in terms of the null vectors as follows

$$F(x) = \frac{1}{2}(x^2n + 2x - \bar{n}) \tag{1.3}$$

which is similar to the form which is used in the more recent 'horosphere' formulations of the conformal framework [8]. We will see that there is some choice as to what factor we put in front of the $x^2n+2x-\bar{n}$ expression; we choose $\frac{1}{2}$ so that our normalisation condition for null vectors, which allows us to compute the reverse mapping $F(x) \mapsto x$ independent of the absolute scale of $F(x)$, becomes

$$F(x)\cdot n = -1$$

We will see that it will often be necessary to work with normalised 'unit' lines, planes, circles and spheres in order to apply the reflection etc. formulæ which will be so important in later sections. It is also worth noting that the mapping as it stands above is dimensionally inconsistent. In later chapters it will be shown how correcting this can allow one to extend the approach to non-Euclidean geometries.

It is somewhat trivial to show that $F(x)$ is always a null vector for any x by evaluating $[F(x)]^2$

$$\begin{aligned} [F(x)]^2 &= \frac{1}{4}(x^2n+2x-\bar{n})\cdot(x^2n+2x-\bar{n}) \\ &= -\frac{1}{2}x^2 n\cdot\bar{n} + x^2 \\ &= -x^2 + x^2 = 0 \end{aligned} \tag{1.4}$$

Thus we have mapped vectors in $\mathfrak{A}(p,q)$ into *null* vectors in $\mathfrak{A}(p+1,q+1)$ and this is precisely the horosphere construction. It also shows the need to impose a normalisation constraint upon the resultant null vectors as they remain null irrespective of absolute scale. More generally we can show that all null vectors in $\mathfrak{A}(p+1,q+1)$ must be the result of mapping some vector $x \in \mathfrak{A}(p,q)$ as above; any vector $X \in \mathfrak{A}(p+1,q+1)$ can be written as

$$X = an + bx + c\bar{n}$$

where $x = x^i e_i$, $i = 1, \cdots, p+q$ and thus $x \in \mathfrak{A}(p,q)$. We can say that $X \cdot n = 2c$ and $X \cdot \bar{n} = 2a$ (since n is null and $x \cdot n = 0$). Therefore a and c are uniquely determined. However we can also write our general X as $X = bx^i e_i + \alpha e + \bar{\alpha}\bar{e}$ for suitable scalars α and $\bar{\alpha}$ and have $X \cdot e_j = bx^j$ ($j = 1, \cdots, p+q$). So, whilst the product bx^j is uniquely defined by X, b and x^j individually are not. Now suppose that X is null so that $X^2 = 0$

$$\begin{aligned} X^2 &= (an + bx + c\bar{n}) \cdot (an + bx + c\bar{n}) \\ &= 2acn \cdot \bar{n} + b^2 x^2 \\ &= b^2 x^2 + 4ac = 0 \end{aligned} \tag{1.5}$$

From this we are easily able to see that any null vector can be written in the form

$$\lambda(x^2 n + 2x - \bar{n}) \tag{1.6}$$

since if $(an + bx + c\bar{n}) = \lambda(x^2 n + 2x - \bar{n})$ we have that

$$c = -\lambda \quad \lambda x^2 = a \ \text{ and } \ 2\lambda = b$$

and we can then eliminate λ from these last two equations to give the condition $b^2 x^2 = -4ac$. This is precisely the condition given in equation 1.5.

These results may now be used to provide a projective mapping between $\mathfrak{A}(p,q)$ and $\mathfrak{A}(p+1,q+1)$. Specifically that the family of null vectors $\lambda(x^2 n + 2x - \bar{n})$, in $\mathfrak{A}(p+1,q+1)$ are taken to correspond to the single point $x \in \mathfrak{A}(p,q)$. If x is the origin then we see that $F(x) = -\bar{n}$ and we therefore may associate null vectors parallel to $\bar{n}$ with the origin. We will see later that when we *invert* $\bar{n}$ we obtain n, thus suggesting that we associate null vectors parallel to n with the *point at infinity* (the usual result of inverting the origin).

When we look at the inner product of null vectors in $\mathfrak{A}(p+1,q+1)$ we discover something very interesting. If A and B in $\mathfrak{A}(p+1,q+1)$ represent the points a and b in $\mathfrak{A}(p,q)$, then

$$\begin{aligned} A\cdot B &= F(a)\cdot F(b) \\ &= \frac{1}{4}(a^2n+2a-\bar{n})\cdot(b^2n+2b-\bar{n}) \\ &= -\frac{1}{2}a^2+a\cdot b-\frac{1}{2}b^2 \\ &= -\frac{1}{2}(a-b)^2 \end{aligned} \tag{1.7}$$

and we see that $A\cdot B$ is proportional to the Euclidean distance between points a and b. This property of the conformal space and its relationship to *distance geometry* [5] is discussed at more length in [8]. The mapping and properties described here were outlined originally in [9].

"The art of doing mathematics consists in finding that special case which contains all the germs of generality."
— David Hilbert

2 Transformations in the Conformal Model

As suggested by naming the 5d geometry, *the conformal model*, we will be especially interested in *Special Conformal Transformations* in $\mathfrak{A}(p,q)$, and we shall see later that these are represented by elements in $\mathfrak{A}(p+1,q+1)$ termed *rotors*. In this chapter we look at the operations of rotation and reflection more closely. We shall assume the reader knows about rotors in conventional Geometric Algebra and their methods of application.

2.1 Rotations

In normal 3d GA rotations are performed with elements of the algebra termed *rotors*. In this section we aim to show that these rotors may be used unchanged on the null vectors we obtain by performing our projective mapping. Let $x \mapsto Rx\tilde{R}$ with $x \in \mathfrak{A}(p,q)$ and R be a rotor in the Clifford algebra over $\mathfrak{A}(p,q)$. Consider what happens when R acts upon $F(x)$; i.e. the nature of $RF(x)\tilde{R}$

$$RF(x)\tilde{R} = \frac{1}{2}R(x^2 n + 2x - \bar{n})\tilde{R} = \frac{1}{2}[x^2 Rn\tilde{R} + 2Rx\tilde{R} - R\bar{n}\tilde{R}]$$

and since R is a rotor it contains only even blades and therefore commutes with n and $\bar{n}$ so

$$RF(x)\tilde{R} = \frac{1}{2}(\hat{x}^2 n + 2\hat{x} - \bar{n}) \tag{2.1}$$

where $\hat{x} = Rx\tilde{R}$. We have therefore shown that rotors in $\mathfrak{A}(p,q)$ remain rotors in $\mathfrak{A}(p,+1q+1)$ in that they retain their action about the point x represented by $F(x)$. To summarise

$$x \mapsto Rx\tilde{R} \qquad \Leftrightarrow \qquad F(x) \mapsto F(Rx\tilde{R}) \tag{2.2}$$

2.2 *Inversions*

In the usual 3d geometric algebra we can reflect a vector a in a plane with unit normal n by 'sandwiching' the vector between the normal, $-nan$ [13]. Sandwiching the object *to be reflected* between the object *in which we wish to reflect* is a very general prescription in GA and one which will be used heavily in later parts of the report. In this section we look at how inversions are brought about by this same reflection operation.

In this report by 'inversion' we mean the mapping $x \mapsto \frac{x}{x^2}$ or, equivalently, $x \mapsto x^{-1}$. Firstly, we look at the properties of the reflection in e of various vectors

$$-ene = -ee\bar{n} = -\bar{n}$$

since $ne = (e + \bar{e})e = (e^2 + \bar{e}e) = (e^2 - e\bar{e}) = e\bar{n}$. Similarly, we can show that a number of reflection properties hold

$$-ene = -\bar{n} \tag{2.3}$$

$$-e\bar{n}e = -n \tag{2.4}$$

$$-exe = x \tag{2.5}$$

and finally we may observe what happens to $F(x)$ under reflection in e

$$\begin{aligned} -eF(x)e &= -e\frac{1}{2}(x^2 n + 2x - \bar{n})e \\ &= \frac{1}{2}\left[-x^2\bar{n} + 2x + n\right] \\ &= x^2 \frac{1}{2}\left[\frac{1}{x^2}n + 2\frac{x}{x^2} - \bar{n}\right] \\ &= x^2 F\left(\frac{x}{x^2}\right) \end{aligned} \tag{2.6}$$

We have, therefore, shown that the inversion operation in $\mathfrak{A}(p,q)$ can be performed via the reflection in e of the projection into $\mathfrak{A}(p,+1q+1)$.

$$x \mapsto \frac{x}{x^2} \qquad \Leftrightarrow \qquad F(x) \mapsto -\frac{eF(x)e}{x^2} = F\left(\frac{x}{x^2}\right) \tag{2.7}$$

Since the absolute scale of $F(x)$ is irrelevant, as we always rescale to impose our normalisation constraint, we can omit the scaling by x^{-2}. It is also irrelevant, by the same logic, whether we take $-e(\cdot)e$ or $e(\cdot)e$ as the reflection and henceforth we will use $e(\cdot)e$ for convenience. We reiterate here that later in the report the concept of reflection of a quantity in an object being brought about by this method of 'sandwiching' is crucial to much of our code for fast 3d manipulations.

2.3 *Translations*

Translation along a vector a is defined as the mapping $x \mapsto x+a$ for some $x, a \in \mathfrak{A}(p,q)$. In this section we will show that this is performed by applying a rotor $R = T_a = \exp\left(\frac{na}{2}\right)$ to $F(x)$. To begin with, consider the usual power series expansion of the exponential which we may immediately simplify thus

$$R = T_a = \exp\left(\frac{na}{2}\right) = 1 + \frac{na}{2} + \frac{1}{2}\left(\frac{na}{2}\right)^2 + \cdots = 1 + \frac{na}{2} \tag{2.8}$$

since n is null, $an = -na$ and therefore the higher order terms are all zero. We now see how R acts on the vectors n, $\bar{n}$ and x.

$$\begin{aligned} Rn\tilde{R} &= \left(1+\frac{na}{2}\right) n \left(1+\frac{an}{2}\right) \\ &= n + \frac{1}{2}nan + \frac{1}{2}nan + \frac{1}{4}nanan \\ &= n \end{aligned} \tag{2.9}$$

again using $an = -na$ and $n^2 = 0$. Similarly we can show that

$$R\bar{n}\tilde{R} = \bar{n} - 2a - a^2 n \tag{2.10}$$

$$Rx\tilde{R} = x + n(a \cdot x) \tag{2.11}$$

Immediately we see that our interpretation of n being the point at infinity and $\bar{n}$ being the origin is consistent with our claim that R represents the translation $x \mapsto x+a$ since $R(-\bar{n})\tilde{R} = F(a)$ and the point at infinity is unchanged by finite translation.

We can now also see how the rotor acts on $F(x)$

$$\begin{aligned} RF(x)\tilde{R} &= \left(1+\frac{na}{2}\right)\frac{1}{2}(x^2n + 2x - \bar{n})\left(1+\frac{an}{2}\right) \\ &= \frac{1}{2}(x^2n + 2(x + n(a\cdot x)) - (\bar{n} - 2a - a^2n)) \\ &= \frac{1}{2}((x+a)^2n + 2(x+a) - \bar{n}) \\ &= \frac{1}{2}(\hat{x}^2n + 2\hat{x} - \bar{n}) = F(x+a) \end{aligned} \tag{2.12}$$

where $\hat{x} = x+a$ and thus translations in $\mathfrak{A}(p,q)$ can be performed by the rotor $R = T_a$ defined above. To summarise

$$x \mapsto x+a \qquad \Leftrightarrow \qquad F(x) \mapsto T_aF(x)\tilde{T}_a = F(x+a) \tag{2.13}$$

2.4 Dilations

A dilation by a factor of α is the mapping $x \mapsto \alpha x$. In this section we investigate how to form a rotor which has the action of dilating about the origin. We start by considering the rotor $R = D_\alpha = \exp\left(\frac{\alpha}{2}e\bar{e}\right)$ and a number of relations which can easily be verified

$$
\begin{aligned}
-e\bar{e}n = \quad n \quad &= ne\bar{e} \\
-\bar{n}e\bar{e} = \quad \bar{n} \quad &= e\bar{e}\bar{n}
\end{aligned} \tag{2.14}
$$

We can now look at what $RF(x)\tilde{R}$ gives

$$
\begin{aligned}
D_\alpha F(x)\tilde{D}_\alpha &= \exp\left(\frac{\alpha}{2}e\bar{e}\right)\frac{1}{2}\{x^2 n + 2x - \bar{n}\}\exp\left(-\frac{\alpha}{2}e\bar{e}\right) \\
&= \frac{1}{2}(x^2\exp(\alpha e\bar{e})\,n + 2x - \exp(\alpha e\bar{e})\,\bar{n}) \\
&= \frac{1}{2}(x^2\exp(-\alpha)\,n + 2x - \exp(\alpha)\,\bar{n}) \\
&= \exp(\alpha)\,\frac{1}{2}\left\{\exp(-2\alpha)\,x^2 n + 2\exp(-\alpha)\,x - \bar{n}\right\} \\
&= \exp(\alpha)\,\frac{1}{2}\left\{\hat{x}^2 n + 2\hat{x} - \bar{n}\right\}
\end{aligned} \tag{2.15}
$$

where $\hat{x} = \exp(-\alpha)\,x$. The above steps can be verified by considering $\exp\left(-\frac{\alpha}{2}e\bar{e}\right)$ as the expansion $1 - \frac{\alpha}{2}e\bar{e} + \frac{1}{2!}\left(\frac{\alpha}{2}e\bar{e}\right)^2 + \cdots$ and using the relations given in equation 2.14. Again noting that the absolute scale of $F(x)$ doesn't matter we have therefore shown that dilations in $\mathfrak{A}(p,q)$ can be performed by the rotor $R = D_\alpha$

$$
x \mapsto \exp(-\alpha)\,x \qquad \Leftrightarrow \qquad F(x) \mapsto D_\alpha F(x)\tilde{D}_\alpha = \exp(\alpha)\,F(\exp(-\alpha)\,x) \tag{2.16}
$$

Note that the signs are incorrect in the equivalent equations in [9], p.303, equation 3.22. It is worth noting that dilation about any other point may be achieved by concatenating the appropriate rotors to move that point to the origin, dilate and move back.

2.5 Special Conformal Transformations

We have seen above that we are able to express rotations, inversions, translations and dilations in $\mathfrak{A}(p,q)$ by rotations and reflections in $\mathfrak{A}(p+1,q+1)$. This now leads us to consider *special conformal transformations*. These are essentially transformations which preserve angles and are defined by the motion

$$
x \mapsto x\frac{1}{1+ax} \tag{2.17}
$$

A moment's investigation reveals this transform to be a combination of inversion, translation and inversion again

$$\begin{aligned}
x \quad \underset{\text{inversion}}{\longmapsto} \quad & \frac{x}{x^2} \\
\underset{\text{translation}}{\longmapsto} \quad & \frac{x}{x^2} + a \equiv \frac{x}{x^2}(1+xa) \\
\underset{\text{inversion}}{\longmapsto} \quad & \frac{\frac{x}{x^2}+a}{(\frac{x}{x^2}+a)(\frac{x}{x^2}+a)} \\
= \quad & \frac{x+ax^2}{1+2a\cdot x+a^2x^2} = x\frac{1}{1+ax}
\end{aligned} \tag{2.18}$$

since $\frac{1}{1+ax} = \frac{1+xa}{(1+ax)(1+xa)}$. The final line in the above expression shows us that $x\frac{1}{1+ax}$ is indeed a vector since $x + ax^2$ is a vector. As we have built up the special conformal transformation via inversions and translations, we know exactly how to construct the $\mathfrak{A}(p+1,q+1)$ operator that performs such a transformation by simply chaining the rotors for inversion and translation we derived above. The required rotor is therefore given by

$$K_a = eT_ae, \qquad \text{so that} \qquad x \mapsto K_ax\tilde{K}_a \tag{2.19}$$

and

$$K_ax\tilde{K}_a = e\left\{T_a(exe)\tilde{T}_a\right\}e \tag{2.20}$$

Substituting for the rotors above we can write our special conformal rotor as

$$K_a = 1 - \frac{1}{2}\bar{n}a \tag{2.21}$$

We are now in a position to see what happens when we act on $F(x)$ with K_a

$$\begin{aligned}
K_aF(x)\tilde{K}_a &= eT_a(eF(x)e)\tilde{T}_ae \\
&= eT_a(-x^2F(\frac{x}{x^2}))\tilde{T}_ae \\
&= -x^2e\left\{F(\frac{x}{x^2}+a)\right\}e \\
&= -x^2\left\{-\left(\frac{x}{x^2}+a\right)^2F\left(\frac{(\frac{x}{x^2}+a)}{(\frac{x}{x^2}+a)^2}\right)\right\} \\
&= (1+2a\cdot x+a^2x^2)F\left(x\frac{1}{1+ax}\right)
\end{aligned} \tag{2.22}$$

The end result is therefore

$$x \mapsto x\frac{1}{1+ax} \qquad \Leftrightarrow \qquad F(x) \mapsto (1+2a\cdot x+a^2x^2)F\left(x\frac{1}{1+ax}\right) \tag{2.23}$$

2.6 Observations

We can see that, from the above, the following results are true:

$$
\begin{aligned}
Rn\tilde{R} &= n \quad \text{for } R \text{ a rotation, since } \; n\tilde{R} = \tilde{R}n \\
Rn\tilde{R} &= n \quad \text{for } R \text{ a translation (equation 2.9)} \\
n\tilde{R} = \tilde{R}n \; \text{and} \; Rn\tilde{R} &= \exp(-\alpha)n \quad \text{for dilations}
\end{aligned}
$$

Thus rotations, translations and dilations leave n, which we identify with the point at infinity, unchanged up to a scale factor. This is a fact which will be important to us in subsequent sections. Indeed, we find that the underlying geometry described by the rotors is related to the element of the algebra which the rotors hold invariant. We will see later that the 5d conformal setup provides a framework in which we can simply describe non-Euclidean geometries in such terms.

"There is no branch of mathematics, however abstract, which may not some day be applied to phenomena of the real world."

— Nikolai Lobatchevsky

3 Objects and Incidence Relations in the Conformal Space

In this chapter we shall deal with methods of representing lines, planes, circles and spheres in our conformal space via a set of null-spaces. We shall also consider using only the $\lambda(2x-\bar{n})$ part of the conformal representation, $F(x)$, of a point x and show how it enables us to deal with the incidence of planes, lines and so forth as we can in projective geometry [10]. We will observe that it is very similar to the classical method of homogeneous co-ordinates whereby we form $\lambda(x+e_o)$ with x being our original point in Euclidean 3-space and e_o being the extra vector we add on. Further, we shall find find that, in the conformal representation, the signature of the vector we add on is irrelevant for projective geometry and indeed that it is generally better, if dealing only with projective geometry, that we do not have null structures present. This implies we only need add a 4th basis vector which gives a $\mathfrak{A}(p+1,q)$ space. We therefore ask ourselves if there is any advantage in going to a full $\lambda(x^2n+2x-\bar{n})$ representation.

In this chapter we shall also enlarge the representation and, by employing the reflection formula to do inversions, we shall study the incidence relations of spheres and circles in addition to just lines and planes. A second, linked, advantage that will be shown is that in the conformal representation we can represent incidence relations via wedge products just as we can in the GA version of projective geometry and in the Cayley-Grassmann algebra.

3.1 A note on methodology

In this section we develop a useful tool used by a number of proofs later in the chapter. Recall that, in projective geometry, if a line, L, passes through two points a, b, whose (4d) homogeneous representations are A, B, we can represent the line by the bivector $L = A \wedge B$. The representation, X, of any point lying on the line will satisfy

$$X \wedge L = 0$$

In the conformal representation rotations, translations, dilations, inversions are all represented by rotors or reflections, which allows us to infer that any incidence relations remain invariant in form under such operations. This may also be seen explicitly; suppose we have the incidence relation

$$X \wedge Y \wedge \cdots \wedge Z = 0$$

where $X, Y, \cdots, Z \in \mathfrak{A}(p+1, q+1)$. Under reflections in e we have

$$\begin{aligned} X \wedge Y \wedge \cdots \wedge Z &\mapsto (eXe) \wedge (eYe) \wedge \cdots \wedge (eZe) \\ &= e(X \wedge Y \wedge \cdots \wedge Z)e \end{aligned}$$

since $(eXe) \wedge (eYe) = \frac{1}{2}(eXeeYe - eYeeXe) = \frac{1}{2}e(XY - YX)e = e(X \wedge Y)e$, as $e^2 = 1$. Therefore it is true that if $X \wedge Y \wedge \cdots \wedge Z = 0$ then it is also true that $(eXe) \wedge (eYe) \wedge \cdots \wedge (eZe) = 0$.

Similarly, if we consider the relation under some rotor R we have

$$\begin{aligned} X \wedge Y \wedge \cdots \wedge Z &\mapsto (RX\tilde{R}) \wedge (RY\tilde{R}) \wedge \cdots \wedge (RZ\tilde{R}) \\ &= R(X \wedge Y \wedge \cdots \wedge Z)\tilde{R} \end{aligned}$$

again using $(RX\tilde{R}) \wedge (RY\tilde{R}) = \frac{1}{2}(RX\tilde{R}RY\tilde{R} - RY\tilde{R}RX\tilde{R}) = \frac{1}{2}R(XY - YX)\tilde{R} = R(X \wedge Y)\tilde{R}$, as $R\tilde{R} = 1$. Once more we can say that if $X \wedge Y \wedge \cdots \wedge Z = 0$ then it is also true that $(RX\tilde{R}) \wedge (RY\tilde{R}) \wedge \cdots \wedge (RZ\tilde{R}) = 0$.

It is therefore clear that translations, rotations, dilations and inversions can now be brought into the context of projective geometry, giving a significant increase in the usefulness of the representation. It is now possible to build up a set of useful results in this conformal system and to see how lines, planes, circles and spheres are represented. The working above will be the basis for most proofs of constructions in this report; we prove the construction holds for some simple case at the origin and then we can say it must hold for all cases since we can rotate, translate and dilate our setup at the origin to any configuration in space, with no change in the incidence relations.

3.2 *The equation of a line*

As was discussed above, the incidence relations are invariant under rotations and translations in the $\mathfrak{A}(p,q)$ space and hence without loss of generality we can consider the incidence relation for a line in the direction e_1 passing through the origin.

Let three points on this line be x_1, x_2, x_3 with corresponding $\mathfrak{A}(p+1,q+1)$ representations X_1, X_2, X_3. It is clear that $\{X_i\}$ contain only the vectors $n, \bar{n}$ and e_1 as any point x on the line must have the form $x = \lambda e_1$. We therefore claim that, if X is the representation of any other point on the line, we can write the incidence relation

$$X \wedge X_1 \wedge X_2 \wedge X_3 = 0$$

We prove this as follows. Let $L = X_1 \wedge X_2 \wedge X_3$, if we expand this out in terms of the conformal representation we have

$$\begin{aligned} L &= \frac{1}{8}(x_1^2 n + 2x_1 - \bar{n}) \wedge (x_2^2 n + 2x_2 - \bar{n}) \wedge (x_3^2 n + 2x_3 - \bar{n}) \\ &= \alpha(n \wedge e_1 \wedge \bar{n}) \end{aligned} \tag{3.1}$$

where α is a scalar which depends upon x_1, x_2, x_3. If $X = \frac{1}{2}(x^2 n + 2\lambda e_1 - \bar{n})$, then it is easy to see by direct substitution that $X \wedge L = 0$. Since we can rotate and translate our simple line through the origin to any other line in space and still preserve the incidence relations, we know that we may represent all lines in this manner. It is interesting to note that this parallels the projective case and also to note that we would appear to require 3 points in this conformal representation to describe a line instead of the usual 2. We shall return to this later.

3.3 *The equation of a plane*

This section uses a similar method to develop the representation of a plane. Again by translational and rotational invariance we can, without loss of generality, consider initially only the plane spanned by e_1 and e_2 and passing through the origin. If the point x lies in this plane then we can write

$$x = \lambda e_1 + \mu e_2$$

and its conformal representation, X, will only contains the vectors $n, \bar{n}, e_1, e_2$

$$X = \frac{1}{2}(x^2 n + 2(\lambda e_1 + \mu e_2) - \bar{n})$$

Let $\Phi = X_1 \wedge X_2 \wedge X_3 \wedge X_4$, where the points represented by $\{X_i\}$ all lie in the plane. By expanding we can show that Φ must take the form

$$\Phi = \beta(n \wedge \bar{n} \wedge e_1 \wedge e_2)$$

and so, for any representation, X, of a point on the plane we have $X \wedge \Phi = 0$ and therefore

$$X \wedge X_1 \wedge X_2 \wedge X_3 \wedge X_4 = 0 \tag{3.2}$$

is the incidence relation for a plane passing through points represented by X_i, $i = 1, \ldots, 4$. Once again we note the apparent requirement for 4 points to specify the plane as opposed to the usual 3.

We may easily extend this to higher dimensions and specify an r-d hyperplane (where a line is $r = 1$, a plane is $r = 2$ and so forth) via the relation

$$X \wedge X_1 \wedge X_2 \wedge \ldots X_{r+1} \wedge X_{r+2} = 0 \tag{3.3}$$

where $\{X_i\}$ are conformal representations of the $r+2$ points $\{x_i\}$ lying on the hyperplane.

3.4 *The role of inversion: lines and circles*

It may initially appear incongruous to specify $r+2$ points in order to determine an r-d hyperplane. For example, 2 points clearly suffice to determine a line, 3 points for a plane and so on. This section discusses the rôle of these extra points.

To understand the requirement for these extra points and the part inversion plays we shall use a simple example. We shall consider the space $\mathfrak{A}(2,0)$, that is, the ordinary Euclidean plane with basis $\{e_1, e_2\}$, $e_1^2 = 1,\ e_2^2 = 1$.

Let the line L be $x = 1$, or equivalently, $(1,y) : -\infty \leq y \leq +\infty$ and let a be the point (x,y). Suppose we wish to invert points on this line. Doing so gives us the set of points

$$a \mapsto \frac{a}{a^2} \qquad \Longrightarrow \qquad L \mapsto \left(\frac{1}{1+y^2}, \frac{y}{1+y^2}\right) \tag{3.4}$$

Parameterising the original line as $x = 1, y = t;\ -\infty \leq t \leq +\infty$, the inversion produces $(x', y') = \left(\frac{1}{1+t^2}, \frac{t}{1+t^2}\right)$ and it is then easy to show that

$$\left[x' - \frac{1}{2}\right]^2 + y'^2 = \left(\frac{1}{2}\right)^2$$

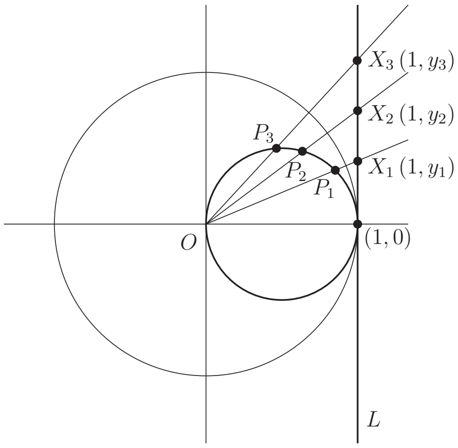

Figure 3.1: *An illustration of the inversion of points on the line $x = 1$ (L) in the unit circle centred on the origin. It produces a circle centred on $(\frac{1}{2}, 0)$ and with radius $\frac{1}{2}$. The points at infinity on the line L map to the origin.*

The inversion, therefore, produces a circle through the origin, centre $(\frac{1}{2}, 0)$ radius $\frac{1}{2}$, see figure 3.1. We may therefore make the connexion

$$\text{straight line} \quad \underset{\text{inversion}}{\longmapsto} \quad \text{circle}$$

We can now state that three points, $x_{1\cdots 3}$ on this line with the representations $X_{1\cdots 3}$ must invert to give three points, $X'_{1\cdots 3}$ on this circle. Let the general point on the line be represented by X; we know, from the above incidence relations, that

$$X \wedge X_1 \wedge X_2 \wedge X_3 = 0$$

and thus, if X' is a general point on the circle, we know that

$$X' \wedge X'_1 \wedge X'_2 \wedge X'_3 = 0$$

We can see this by performing an inversion upon the line via a reflection in e

$$e(X \wedge X_1 \wedge X_2 \wedge X_3)e = eXe \wedge eX_1e \wedge eX_2e \wedge eX_3e$$

This gives a very useful form for the equation of a circle. We derived it for a special case but since we know that we can dilate and translate as we wish, it must in fact be true for a completely general circle. Thus if $X_{1\cdots 3}$ are *any* three points, the equation of the circle passing through these points is

$$X \wedge C = X \wedge X_1 \wedge X_2 \wedge X_3 = 0 \tag{3.5}$$

Where we identify the trivector C as the representation of the circle itself. C is formed by wedging any three points on the circle together. We will see later that all such trivectors are equivalent up to scale.

Treating this as a general incidence relation for a circle and by sandwiching it between e it is clear that we will, in general, obtain another circle under inversion since each $X_i' = eX_ie$ will be another general point in the plane. This only fails if X_1', X_2', X_3' happen to be collinear and, further, co-linearity will only occur if the original circle passes through the origin (as in the case we started with here).

We can see this if we consider what happens to the representation of the origin, $\bar{n}$ under inversion. If we sandwich $\bar{n}$ in between e we obtain $e\bar{n}e = n$ and hence we justify our association of n with the point at infinity (the usual result of inverting the origin). Therefore the incidence relation for a line may always be written in the form

$$X \wedge n \wedge X_1 \wedge X_2 = 0 \tag{3.6}$$

where X_1 and X_2 are the representations of any two finite points on the line. This explains the extra point we appeared to need in describing a line earlier; we can now see that

$$X \wedge X_1 \wedge X_2 \wedge X_3 = 0$$

actually describes a *circle* and, therefore, genuinely requires 3 points whilst a line is just a special case of a circle which passes through the point at infinity. To summarise, any line, L, is represented by a trivector of the form

$$L = X_1 \wedge X_2 \wedge n \tag{3.7}$$

and similarly a circle, C, is represented by a trivector of the form:

$$C = X_1 \wedge X_2 \wedge X_3 \tag{3.8}$$

where no X_i is a multiple of n. This close relationship between circles and lines and the interpretation of a line as a circle passing through the point at infinity is not new. The whole field of *inversive geometry* [1] in the plane has this as its basis. In the above example when we took a line to a circle through the origin we were effectively inverting in the unit circle centred on the origin, as illustrated in figure 3.1. The whole of inversive geometry in the plane can be encapsulated by the operation of reflecting points (X) and lines (L) in general circles (C), ie.

$$X' = CXC$$
$$L' = CLC$$

The huge advantage that we have in our conformal framework is that precisely the same equations can be used in 3d. Conventionally, much of inversive geometry is described by complex numbers and so the jump from the plane to higher-dimension spaces is rarely made.

"A topologist is one who doesn't know the difference between a doughnut and a coffee cup."
— John Kelley

4 Blades as Objects in the Conformal Representation

In this chapter we will summarise our findings of the previous chapters, look at the rôle played by bivectors in this algebra and give some useful alternative representations of lines, planes, circles and spheres based upon those developed previously.

4.1 Vectors and 2-blades

We have seen that we may use null vectors, X, in our 5d space to represent points in Euclidean 3-space. In particular we identify n and $\bar{n}$ with the point at infinity and the origin respectively. In our 5d space there clearly exist vectors which do not square to zero. The interpretation of such vectors will be discussed later.

The term *blade* in GA is used to refer to quantities which can be written as the wedge product of vectors. For example a r-blade can always be written as $A_1 \wedge A_2 \wedge \cdots \wedge A_r$. It is important to distinguish this from an r-vector which may be any linear combination of r-blades. This is important since, in 5d, not all bivectors can be written as 2-blades; for example $e_1 e_2 + e_3 e_4$ cannot be written in the form $a \wedge b$.

In previous sections we have considered the geometric meaning of 3-blades and 4-blades. We shall now consider the geometric meaning of the 2-blade $A \wedge B$ given that A and B are null vectors in $\mathfrak{A}(4,1)$. We start by recalling that null vectors represent points, 3-blades represent generalised circles[1] and 4-blades represent generalised spheres. It will be seen later, when discussing intersections, that the representation of the intersection of a line and a sphere is actually a 2-blade. We also know that the intersection of a sphere and a non-tangential line occurs at two points on the sphere. We postulate, therefore, that $A \wedge B$ represents the pair of points represented by A and

[1]Here we use 'generalised' to indicate that we include the special case involving the point at infinity, i.e. lines and planes

B. This is clear when considering the incidence relation

$$X \wedge A \wedge B = 0$$

which is only true in general for $X = A$ or $X = B$. If we accept that $A \wedge B$ represents two points then, for completeness, we should develop a method of extracting the points A and B.

4.1.1 *Extracting* A *and* B *from* $A \wedge B$

In this section we develop a method of extracting A and B from $A \wedge B$ using a method of *projectors*. Throughout we shall assume that A and B have been normalised so that, for example, $A \cdot n = -1$. We start by considering the 2-blade $T = A \wedge B$ and forming

$$F = \frac{1}{\beta} A \wedge B \tag{4.1}$$

where $\beta > 0$ and $\beta^2 = T^2$, so that $F^2 = 1$ if $\beta^2 \neq 0$. We now use F to define two *projector* operators

$$\begin{aligned} P &= \frac{1}{2}(1+F) \\ \tilde{P} &= \frac{1}{2}(1-F) \end{aligned} \tag{4.2}$$

where $\tilde{P}$ denotes the normal reversion operation applied to P. Note that $P^2 = P$, which can be easily be verified

$$\begin{aligned} PP &= \frac{1}{4}(1+F)(1+F) \\ &= \frac{1}{4}(1+2F+1) = \frac{1}{2}(1+F) \end{aligned} \tag{4.3}$$

Similarly, we can easily show that $\tilde{P}\tilde{P} = \tilde{P}$. These properties justify calling these operators *projectors*, borrowing the term from Physics. An equally important property is that $P\tilde{P} = \tilde{P}P = 0$ which, again, is easy to prove

$$P\tilde{P} = \frac{1}{4}(1+F)(1-F) = \frac{1}{4}(1-1) = 0 \tag{4.4}$$

and similarly for $\tilde{P}P$. We may now see what effect these projectors have on A and B

$$\begin{aligned} PA &= \frac{1}{2}\left[1 + \frac{1}{\beta} A \wedge B\right] A \\ &= \frac{1}{2}\left[A + \frac{1}{\beta}(A \wedge B)A\right] \\ &= \frac{1}{2}\left[A + \frac{1}{\beta}(A \cdot B)A\right] \\ &= \frac{1}{2}(A - A) = 0 \end{aligned} \tag{4.5}$$

since $(A\wedge B)A = (A\wedge B)\cdot A = -A^2B + (A\cdot B)A = (A\cdot B)A$ $(A^2 = 0)$ and $A\cdot B = -\beta$. This is easily seen from $\beta^2 = (A\wedge B)\cdot(A\wedge B) = -A^2B^2 + (A\cdot B)^2$ and the facts that $A^2 = B^2 = 0$ and $A\cdot B$ must be negative as seen in equation 1.7.

Using similar working we can also show the results of P and $\tilde{P}$ acting on A and B

$$PA = 0 \tag{4.6}$$

$$PB = B \tag{4.7}$$

$$\tilde{P}A = A \tag{4.8}$$

$$\tilde{P}B = 0 \tag{4.9}$$

The next step is to consider the vector obtained by dotting $A\wedge B$ with n.

$$(A\wedge B)\cdot n = -n\cdot(A\wedge B) = -(n\cdot A)B + (n\cdot B)A = (B-A) \tag{4.10}$$

using the fact that A and B are normalised points such that $A\cdot n = B\cdot n = -1$. It therefore follows that we have

$$P[(A\wedge B)\cdot n] = P(B-A) = B \tag{4.11}$$

$$-\tilde{P}[(A\wedge B)\cdot n] = -\tilde{P}(B-A) = A \tag{4.12}$$

We note also that since $AP = \tilde{P}A = A$ it follows that $\tilde{P}AP = \tilde{P}\tilde{P}A = \tilde{P}A$. Similar relations hold for BP etc., so that we have

$$\begin{aligned} \tilde{P}AP &= \tilde{P}A \\ PA\tilde{P} &= 0 \\ PB\tilde{P} &= PB \\ \tilde{P}BP &= 0 \end{aligned}$$

which means that we can also write the projections as two-sided operations. Thus from a 2-blade $A\wedge B$ we can extract the two points, A and B that it represents via

$$\begin{aligned} A &= -\tilde{P}[(A\wedge B)\cdot n] \equiv -\tilde{P}[(A\wedge B)\cdot n]P \\ B &= P[(A\wedge B)\cdot n] \equiv P[(A\wedge B)\cdot n]\tilde{P} \end{aligned} \tag{4.13}$$

We will see later that when we perform intersection operations that yield two points the two points in question can then be found using the formulæ in equation 4.13. Usefully we do not have to solve a quadratic equation as we would do using conventional approaches.

4.2 Trivectors

We have already seen that there are two classes of object represented by trivectors. If P, Q, R are null vectors in our 5d space representing points in 3d space then trivectors of the form

$$C = P \wedge Q \wedge R$$

represent circles. Recall that it is specifically a circle passing through points represented by P, Q and R. Trivectors of the form

$$L = P \wedge Q \wedge n$$

represent lines, specifically that line passing through the points represented by P and Q.

In GA it is often found that the operation of taking the dual, that is multiplication by the pseudoscalar I, is usually useful and often has physical or geometric significance. The dual of an element is always just another element of the algebra and does not live in a separate 'dual' or 'tangent' space. Below we shall consider the dual operation with respect to the representations of circles and lines.

4.2.1 Circles as trivectors

Here we will show that taking the dual of the trivector representing a circle gives rise to a useful alternative representation which naturally encodes both the centre and radius. Let us first of all work in the plane so that our conformal space is 4-dimensional, and is $\mathfrak{A}(3,1)$, having basis vectors e_1, e_2 and $e, \bar{e}$. We may easily extend the method to 3d space or higher dimensions as we have done in previous chapters.

We start with the unit circle in the plane and take as three points on it those shown in figure 4.1. For any unit length vector, $\hat{x}$, we know that $F(\hat{x}) = \frac{1}{2}(n + 2\hat{x} - \bar{n}) = (\hat{x} + \bar{e})$. In particular we have

$$F(e_1) \wedge F(e_2) \wedge F(-e_1) = 2e_1e_2\bar{e}$$

and hence the trivector $C = 2e_1e_2\bar{e}$ represents the unit circle. In the plane the pseudoscalar, which we shall write as I_4, is given by $I_4 = e_1e_2e\bar{e}$ and so the dual of C, which we write as C^*, can easily be shown to be given by

$$C^* = CI_4 = 2e = (n + \bar{n}) \tag{4.14}$$

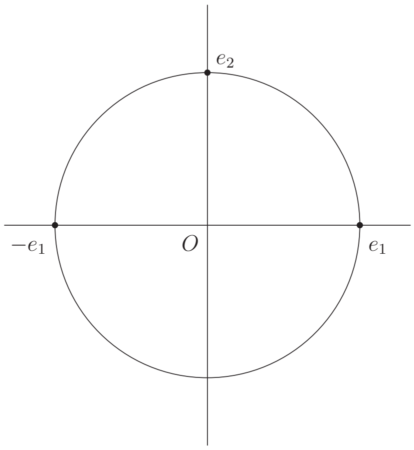

Figure 4.1: *Unit circle with three key points marked*

We know that $X \wedge C = 0$ is the incidence relation for the circle and that $X \wedge C = 0$ can be rewritten as

$$X \cdot (CI_4) = 0 \quad \Leftrightarrow \quad X \cdot C^* = 0$$

Note that C^* is the dual of a trivector in a 4d space and is, therefore, a vector. This suggests a very useful alternative representation for a circle (or a sphere when generalised to higher dimensions) which we now discuss.

We know from equation 1.7 that for any two normalised point representations A and B

$$A \cdot B = -\frac{1}{2}(a-b)^2$$

and thus, if X represents a point on a circle and B represents its centre, we know that we can write

$$X \cdot B = -\frac{1}{2}(x-b)^2 \equiv -\frac{1}{2}\rho^2$$

where ρ is the radius of the circle. For a normalised point representation X this implies that

$$X \cdot (B - \frac{1}{2}\rho^2 n) = 0$$

since $X \cdot n = -1$. Comparing this with $X \cdot C^*$ we see that provided we normalise C^* after taking the dual (so that $C^* \cdot n = -1$), then we find

$$C^* = B - \frac{1}{2}\rho^2 n \tag{4.15}$$

The vector C^*, therefore, encodes in a neat fashion the centre and radius of the circle in the plane.

Also note that if we take our $(n+\bar{n})$ as the representation of the dual of the unit circle centred on the origin and rotate, translate and dilate the expression, we get precisely the same result as in equation 4.15. We can see this by first taking our unit circle at the origin, $C_o^* = n+\bar{n}$ and rotating it. We have seen that rotation rotors leave n and $\bar{n}$ invariant, so C_o^* remains unchanged. Now dilate this with a dilation rotor $D_\alpha = e^{\frac{\alpha}{2}e\bar{e}}$ where $\rho = e^{-\alpha}$ is the dilation factor. We have seen previously that $D_\alpha n\tilde{D}_\alpha = e^{-\alpha}n$ and $D_\alpha \bar{n}\tilde{D}_\alpha = e^{\alpha}\bar{n}$, so that

$$C_o^* \rightarrow W = e^{-\alpha}n + e^{\alpha}\bar{n} \tag{4.16}$$

Now, translate by a using a translation rotor, $T_\alpha = 1+\frac{na}{2}$. Again, using previous results, we know that n is left invariant and $-\frac{1}{2}\bar{n}$ is taken to $F(a)$, giving us

$$W \rightarrow Z = e^{-\alpha}n - 2e^{\alpha}F(a) \tag{4.17}$$

Normalising this to give us our new C^*, where $C^* \cdot n = -1$, means dividing Z above by $-2e^{\alpha}$ (since $A = F(a)$ satisfies $A \cdot n = -1$) leaving us with

$$C^* = -\frac{1}{2}e^{-2\alpha}n + A \tag{4.18}$$

and we can now see that this is precisely equation 4.15 with A representing the centre and $\rho = e^{-\alpha}$ as the radius.

So far we have talked about circles in the plane. We shall now look at the treatment of circles at general positions and orientations in space. Firstly we note that since we can think of C as the wedge product of the representations of three points on the circle, it follows that the *plane* which the circle lies in is $C \wedge n$.

In deriving equation 4.15 for the circle in the plane we used the pseudoscalar for the 4d space. It therefore seems plausible that when we move to general circles in 3d the rôle of this pseudoscalar will be taken by the plane in which the circle lies.

We firstly define the *unit* plane I_c by

$$I_c = \frac{n \wedge C}{\sqrt{(-[n \wedge C]^2)}} \tag{4.19}$$

since $(n \wedge C)$ always squares to give a negative scalar. In future we will find that it is convenient to always take the line, plane, circle, sphere etc. of unit magnitude. The dual of the circle C is then given by the analogous equation to equation 4.15 where we assume that given C, which can be 'unit', we form C^* and then normalise such that $C^* \cdot n = -1$. That is

$$C^* \equiv CI_c = B - \frac{1}{2}\rho^2 n \tag{4.20}$$

where B is again the centre of the circle, ρ is the radius. Note that the 'dual' in this case is with respect the 'unit' plane in which the circle lies. The proof of this is in most respects identical to the previous proof for the plane but, where before we used I_4, we now use the unit plane. This plane is proportional to $e_1 \wedge e_2 \wedge \bar{n} \wedge n$, and is indeed identical to I_4. We can then rotate, dilate and translate as before and note that $R(C_o I_c)\tilde{R} = RC_o\tilde{R}RI_c\tilde{R}$ and hence our new dual has the form given in equation 4.20. The dual is formed by taking the product of the transformed circle with the transformed plane it lies in. We also note that we can find the radius of this general circle very simply by squaring C^*

$$\begin{aligned}(C^*)^2 &= (B - \frac{1}{2}\rho^2 n)^2 \\ &= -\rho^2 B \cdot n = \rho^2 \end{aligned} \tag{4.21}$$

using the facts that $B^2 = 0$, $n^2 = 0$ and $B \cdot n = -1$. From this it then follows that $B = C^* + \frac{1}{2}(C^*)^2 n$. To summarise, from the vector form of any general circle, we can easily obtain the centre and radius as follows

$$(C^*)^2 = \rho^2 \tag{4.22}$$

$$B = C^* \left[1 + \frac{1}{2}C^* n\right] \tag{4.23}$$

Note here that the above relations assume C^* is normalised such that $C^* \cdot n = -1$ since $C^* \cdot n = B \cdot n = -1$, as we assume B is a normalised null vector.

While the above formulation is indeed useful, we will now see that there is a far more elegant way of finding the centre of a circle in 3d. The centre of a circle, C, is also given by simply reflecting the point at infinity, n, in the circle, i.e.

$$CnC \tag{4.24}$$

To prove that CnC gives the centre we can return to our circle centre the origin, radius one. We saw earlier in this section that we can write this circle as $C = 2e_1e_2\bar{e}$

and, again, we can define a *unit circle*, $\hat{C}$, as $\hat{C} = e_1 e_2 \bar{e}$. Thus we see that

$$\begin{aligned} \hat{C} n \hat{C} &= e_1 e_2 \bar{e} n e_1 e_2 \bar{e} \\ &= -e_2 \bar{e} n e_2 \bar{e} \\ &= -\bar{e} n \bar{e} \\ &= \bar{e}(e + \bar{e})\bar{e} = e - \bar{e} \\ &= \bar{n} \end{aligned} \tag{4.25}$$

which is indeed the origin and therefore the centre of the circle. Having proved that the result holds for this simple case, we can now rotate, dilate and translate our circle to give any other circle and the result will still hold. Suppose we apply a rotor, R, which is a composition of rotors which rotate, dilate and translate, to take our circle, C, to any other general circle, $C' = RC\tilde{R}$. Then we see that

$$C' n C' \propto RC\tilde{R}(Rn\tilde{R})RC\tilde{R} \propto R(CnC)\tilde{R} \tag{4.26}$$

since we have seen previously that $Rn\tilde{R} \propto n$ for R composed of rotations, translations and dilations. Equation 4.26 thus tells us that the rotated origin, $R(CnC)\tilde{R}$, is indeed given by $C'nC'$. This type of simple proof, ie proving a result for some simple case at the origin and generalising via rotations, translations and dilations, is a nice feature of the conformal framework. There will many other examples of the XaX formulation producing something interesting in subsequent sections.

In many physical systems we find that GA is very useful in that it exposes how *observables* usually arise by sandwiching a 'fiducial' object between a multivector and its inverse or reverse. For example, in quantum mechanics, we get the spin current in 3d from a Pauli spinor, ϕ, via $s = \phi e_3 \tilde{\phi}$. We see then that this example of sandwiching n between a circle and its reverse (the reverse of a circle is itself up to a scale factor) is an example of this principle at work in the conformal setting.

4.2.2 *Lines as trivectors*

Next we consider the circles passing through the point at infinity which we have already seen are lines. Again we begin by asking ourselves what the dual of a line, L, actually encodes. We suspect the answer to this questions will provide relevant information. As with circles, let us begin by considering lines in the plane. As a starting point we take the line L parallel to the x-axis and distance d from the x-axis.

This line is given by wedging together any two points on the line, say $A = F(de_2)$, $B = F(e_1 + de_2)$, and n;

$$
\begin{aligned}
L &= F(de_2) \wedge F(e_1 + de_2) \wedge n \\
&= \frac{1}{4}(d^2 n + 2de_2 - \bar{n}) \wedge \left(\sqrt{(1+d^2)}n + 2(e_1 + de_2 - \bar{n}\right) \wedge n \\
&= -\frac{1}{2}e_1 \wedge n \wedge \bar{n} - de_1 \wedge e_2 \wedge n
\end{aligned} \tag{4.27}
$$

If we then take the dual of L, again with respect to the pseudoscalar, I_4, for our 2d conformal space, we see that

$$
\begin{aligned}
L^* &= LI_4 = -(\frac{1}{2}e_1 E + de_1 e_2 n)e_1 e_2 e\bar{e} \\
&= e_2 + dn = e_1 I_2 + dn
\end{aligned} \tag{4.28}
$$

using the identities, $e_1 E = Ee_1$, $e_2 E = Ee_2$, $ne\bar{e} = n$ and $Ee\bar{e} = -2$. Thus, we see that the dual of the line in the plane gives the dual, with respect to the pseudoscalar in Euclidean space, $I_2 = e_1 e_2$, of the Euclidean unit vector in the direction of the line plus dn where d is the perpendicular distance of the line from the origin. This holds for any d, and if we apply rotors to rotate our line (noting again that $RnR = n$) we see that we can extend our proof to any line in the plane, therefore giving

$$L^* = \hat{m}I_2 + dn \tag{4.29}$$

where $\hat{m}$ is the unit vector in the direction of the line and d is the perpendicular distance of the line from the origin.

Recalling that $n \cdot \bar{n} = 2$, we can extract d easily from L^* as follows

$$d = \frac{1}{2}L^* \cdot \bar{n} \tag{4.30}$$

Similarly we can then extract $\hat{m}$ as

$$\hat{m} = -[L^* - \frac{1}{2}(L^* \cdot \bar{n})n]I_2 \tag{4.31}$$

As with circles, it should be straightforward to extend this work on lines from the plane into space. However, unlike the circle case, we do not have an obviously specified plane with which to define a 4d pseudoscalar, so instead we look at the dual of the line with respect to the pseudoscalar for the whole 5d space. Clearly we multiply a 3-vector with a 5-vector and get a 2-vector. Once again, consider the same line that we considered above. We look at the dual of this line with $I \equiv I_5$

$$\begin{aligned} L^* &= -(\frac{1}{2}e_1 \wedge n \wedge \bar{n})I - d(e_1 e_2 n)I \\ &= e_2 e_3 - d(e_1 \wedge e_2)In \\ &= e_1 I_3 + [((de_2) \wedge e_1))I]n \end{aligned} \tag{4.32}$$

since $In = nI$. Clearly the first term gives us the unit vector in the direction of the line and the second term gives us its moment. As this holds for any d, if we rotate our line we can produce any line in space and we see that the above can therefore be generalised to hold for any such line. More specifically we write the dual of the line L as

$$L^* = \hat{m}I_3 + [(a \wedge \hat{m})I_3]n$$

where $\hat{m}$ denotes the unit vector (in 3d) in the direction of the line and a is any 3d point on the line. This is analogous to writing the line in terms of Plücker coordinates, where 3 of the coordinates give the line's direction and the other 3 give its moment about the origin.

4.3 4-Vectors

We have already seen that 4-vectors in the conformal setting can represent both spheres and planes. Having seen how we interpret the duals for circles and lines it will now be easy to extend these arguments to spheres and planes.

4.3.1 Spheres as 4-vectors

Given any 4 points whose 5d representations are P, Q, R, S, the sphere through those points is given by the 4-vector Σ

$$\Sigma = P \wedge Q \wedge R \wedge S$$

We know that $X \wedge \Sigma = 0$ for any X lying on the sphere. This can be rewritten as

$$X \cdot (\Sigma I) = 0 \quad \Longrightarrow \quad X \cdot \Sigma^* = 0$$

where $\Sigma^* = \Sigma I$ is the dual to Σ and, hence, is a vector. We can now follow precisely the same workings given for the circle to show that the dual representation of the sphere naturally encodes the centre and the radius.

If X is a point on a sphere and C is its centre we know that we can write

$$X \cdot C = -\frac{1}{2}(x-c)^2 \equiv -\frac{1}{2}\rho^2$$

where ρ is the radius of the sphere. For a normalised point X $(X \cdot n = -1)$ this therefore implies that

$$X \cdot (C - \frac{1}{2}\rho^2 n) = 0$$

Comparing this with $X \cdot \Sigma^*$ we see that provided we normalise Σ^* after taking the dual (such that $\Sigma^* \cdot n = -1$), then we will find

$$\Sigma^* = C - \frac{1}{2}\rho^2 n \tag{4.33}$$

As before, the dual vector Σ^* encodes the centre and radius of the sphere. Whether one wishes to use Σ or Σ^* depends on whether it is most useful to specify the sphere by 4 points lying on it or by its centre and radius. Given a Σ^* (via taking the normalised form of the dual of $\Sigma = P \wedge Q \wedge R \wedge S$) we can immediately get the radius and centre in the manner outlined earlier for the circle

$$(\Sigma^*)^2 = \rho^2 \tag{4.34}$$

$$C = \Sigma^* \left[1 + \frac{1}{2}\Sigma^* n\right] \tag{4.35}$$

As we saw with the case of circles, there is also a more elegant means of extracting the centre of a sphere given Σ or $S = \Sigma^*$. The method is to reflect the point at infinity, n, in the sphere, so that the centre, C, is given by

$$C = SnS = \Sigma n \Sigma \tag{4.36}$$

To show this we consider the sphere of radius 1 centred on the origin, $\Sigma_o = F(e_1) \wedge F(e_2) \wedge F(e_3) \wedge F(-e_1)$ which we can expand to give $2e_1e_2e_3\bar{e}$, so that the dual, S, is given by the particularly concise expression $2e$; we therefore take the 'unit' sphere to be e. We therefore see that

$$SnS = ene = \bar{n} \tag{4.37}$$

which is indeed the origin, which is the centre of the sphere in this case. Now if we rotate, dilate, and translate our sphere at the origin to any other sphere in space, S' say, then, reflecting n in our new sphere gives

$$\begin{aligned} S'nS' &= (RS\tilde{R})n(RS\tilde{R}) = (RS\tilde{R})Rn\tilde{R}(RS\tilde{R}) \\ &= R[SnS]\tilde{R} \end{aligned} \tag{4.38}$$

since $R[SnS]\tilde{R}$ is the transformed origin, i.e. the new centre of the sphere, we see that the formula $S'nS'$ does indeed give the centre.

4.3.2 *Planes as 4-vectors*

A plane, Φ, passing through the 3 points whose 5d representations are P,Q,R, is given by

$$\Phi = P\wedge Q\wedge R\wedge n$$

The physical quantities that we might want to extract from such a 4-vector are clearly the normal to the plane and the perpendicular distance of the plane from the origin. We now investigate how the dual form of the plane helps us to do this. Once again we start by considering the plane $z=d$ which is parallel to the x-y plane and distance d from it. We can represent Φ by

$$\begin{aligned}\Phi &= F(de_3)\wedge F(e_1+de_3)\wedge F(e_2+de_3)\wedge n \\ &= \frac{1}{8}\{(2de_3-\bar{n})\wedge(2[e_1+de_3]-\bar{n})\wedge(2[e_2+de_3]-\bar{n})\wedge n\} \\ &= de_1\wedge e_2\wedge e_3\wedge n-\frac{1}{2}e_1\wedge e_2\wedge\bar{n}\wedge n \\ &= de_1\wedge e_2\wedge e_3\wedge n-e_1\wedge e_2\wedge e\wedge\bar{e} \end{aligned} \tag{4.39}$$

where the second line in equation 4.39 follows because wedging with n removes any term containing n. Then it is simple to show that the dual of Φ is given by

$$\Phi^* = \Phi I = dn+e_3 \tag{4.40}$$

This holds for any d. If we rotate this plane via a rotor we know that the proof must hold and see therefore that since $Rn\tilde{R}=n$ and $Re_3\tilde{R}=\hat{n}$, where $\hat{n}$ is the new rotated normal to the plane, the general equation for the dual of the plane is given by

$$\Phi^* = dn+\hat{n} \tag{4.41}$$

Note that the above assumes that we have normalised the dual such that $[\Phi^*]^2=1$; we can ensure this by normalising the plane such that $\Phi^2=1$. Therefore, given 3 points on the plane, P,Q,R, we form the normalised plane Φ and its dual Φ^*, and we can then extract $\hat{n}$ and d as follows

$$d = \frac{1}{2}\Phi^*\cdot\bar{n} \tag{4.42}$$

$$\hat{n} = \Phi^*-\frac{1}{2}(\Phi^*\cdot\bar{n})n \tag{4.43}$$

Whether we use the 4-vector form or the *normal-distance* form of the plane will depend upon the problem we are solving.

4.3.3 *Summary*

To summarise, we have shown that lines, planes, circles and spheres are represented by blades in the 5d conformal algebra. We can transform them very simply with rotors (which rotate, translate, dilate and invert). This fact also leads to simple proofs of complicated constructions; i.e. we prove for a simple case at the origin and then rotate, translate and dilate to give us any general case in 3-space. The duals of these objects have also been investigated and have been shown to represent physically meaningful properties of the objects. Thus, given 3 points on a circle or its centre, plane and radius, or 4 points on a sphere or its centre and radius, we can very simply construct the object which represents that circle (3-vector) or sphere (4-vector). Similarly with planes. We will see some of the power of this representation in the next chapter when we deal with intersections of these objects.

4.4 *Extension to Higher Dimensions*

A careful reader will have noted that the discussion in the preceding sections need not have restricted itself to 3d. In fact all the methods and results discussed so far transfer immediately to higher dimensions and even different signatures. This section discusses the issues arising from such a generalisation and considers the example of moving from $\mathfrak{A}(2,0)$ to $\mathfrak{A}(3,0)$.

Again we start be considering a special case from which more general cases may be derived by rotation, translation and dilation. Specifically we shall consider the $x=1$ plane. Inverting this plane gives

$$(1,y,z) \mapsto (x',y',z') = \left(\frac{1}{1+y^2+z^2}, \frac{y}{1+y^2+z^2}, \frac{z}{1+y^2+z^2}\right)$$

It is then somewhat trivial to show that x',y',z' satisfy

$$\left(x'-\frac{1}{2}\right)^2 + y'^2 + z'^2 = \left(\frac{1}{2}\right)^2$$

which is the equation of a sphere, radius $\frac{1}{2}$ and centre $(\frac{1}{2},0,0)$. Just as in the 2d case we inverted a line to give a circle through the origin, in 3d we invert a plane to give a sphere through the origin.

We have seen in the previous section that the incidence relation for a plane is

$$X \wedge X_1 \wedge X_2 \wedge X_3 \wedge X_4 = 0 \tag{4.44}$$

where $\{X_{1\cdots 4}\}$ represent any 4 points on the plane. Using a similar method to that discussed in the previous section for circles it is clear that, since we may always rotate and translate a plane to be coincident with the $x = 1$ plane and that inversion of this plane gives a sphere, the above incidence relation should also hold true for spheres. This is, of course, to be expected since it does indeed require 4 points to uniquely specify a sphere in 3d.

A plane is, therefore, now considered a special case of a sphere; one of the points represented in the incidence relation is the point at infinity. Its inverse is a sphere passing through the origin. Again using the arguments of the previous section we see that the incidence relation for a plane now becomes

$$X \wedge n \wedge X_1 \wedge X_2 \wedge X_3 = 0$$

Finally we consider a plane passing through the origin. We may write this as

$$X \wedge n \wedge \bar{n} \wedge X_1 \wedge X_2 = 0$$

with X_1, X_2 representing points lying on the plane. Now invert this to get

$$X \wedge \bar{n} \wedge n \wedge X_1' \wedge X_2' = 0$$

since $ene = \bar{n}$, $e\bar{n}e = n$. Under inversion we know that $x_1 \mapsto \frac{x_1}{x_1^2}$ and similarly for x_2, i.e. the points change only their distance to the origin rather than their direction from it. We therefore see that the plane is mapped onto itself under this inversion operation. We may therefore conclude that a plane passing through the origin is its own inverse since the null vectors $n, \bar{n}$ are just swapped under inversion. These results are all well known using a conventional approach of course, [1]. The novelty here is to show how easy they are to derive in the conformal approach using the key idea of reflection to perform inversion. Further it shows how easy it is to derive similar results in higher dimension and different signature spaces.

"God does not care about our mathematical difficulties. He integrates empirically."

— Albert Einstein

5 Intersections of objects in the 5d algebra

It is often found that many problems in computer graphics, robotics, inverse kinematics and many other fields require finding the intersections of lines, planes, circles and spheres. In this chapter we discuss how to use the techniques and theorems developed in the preceding chapters to find intersections and thereby show that GA provides a highly elegant framework for such problems.

Before looking at particular examples we will look briefly at representations of linear combinations of point representations. When we represent projective geometry via 4d homogeneous coordinates we are still able to consider linear combinations of points if we are willing to make the appropriate normalisation operations; we would also like to be able to consider linear combinations of points in the $\mathfrak{A}(p+1,q+1)$ space. We know that a linear combination of two points a,b, in Euclidean space, of the form

$$\lambda a+\mu b \quad \text{where} \quad \lambda+\mu=1$$

gives another point on the line joining a and b. Similarly, a linear combination of 3 points, a,b,c of the form

$$\lambda a+\mu b+\nu c \quad \text{where} \quad \lambda+\mu+\nu=1$$

gives another point on the plane containing a,b and c. The usual projective representation, where we go up just one dimension, has the advantage of still being linear in the representations of the points e.g. if $x=\lambda a+\mu b+\nu c$ (with $\lambda+\mu+\nu=1$) then its 4d projective representation, $X=x+e$, where e is the extra basis vector which takes 3d to 4d, can also be written in the form

$$X=\lambda A+\mu B+\nu C$$

since

$$\begin{aligned} X &= x+e = \lambda a + \mu b + \nu c + (\lambda + \mu + \nu)e \\ &= \lambda A + \mu B + \nu C \end{aligned}$$

Here we note that we have insisted that the point representation X is 'normalised', i.e. that $X = x+e$ rather than some multiple of this. With the conformal representation, working in $\mathfrak{A}(p+1,q+1)$, we appear to have lost this advantage of linearity. For example, if A and B are the $\mathfrak{A}(p+1,q+1)$ representatives of a and b, then in general

$$\lambda A + (1-\lambda)B \neq \text{ a multiple of } F(\lambda a + (1-\lambda)b)$$

This is due to the presence of the $x^2 n$ term in the representation, which removes linearity. However, the following *is* true and is easy to show from the definition of $F(x)$:

$$F(\lambda a + (1-\lambda)b) = \lambda A + (1-\lambda)B + \lambda(1-\lambda)(A\cdot B)n \tag{5.1}$$

We therefore see that the departure from linear behaviour is given by the addition of a multiple of the representation of the point at infinity. This is relatively benign behaviour and means that many of the techniques we use in the GA version of projective geometry will still work here. For example, this gives us another way of seeing that the equation for a line passing through points a and b is $X \wedge n \wedge A \wedge B = 0$ – the wedging with n knocks out the non-linear term $\lambda(1-\lambda)A\cdot Bn$ and we are left with the usual GA projective geometry result.

Precisely the same sort of thing goes through for a plane. Let a, b, c, in 3d, define a plane and let

$$x = \alpha a + \beta b + \gamma c \quad \text{where} \quad \alpha + \beta + \gamma = 1$$

be a general point on the plane. Then it is easy to show that the representative of x, $X = F(x)$ satisfies

$$X = \alpha A + \beta B + \gamma C + \delta n \;\text{ where }\; \delta = (\alpha\beta A\cdot B + \alpha\gamma A\cdot C + \beta\gamma B\cdot C) \tag{5.2}$$

again making it clear why the equation of the plane can be written as

$$X \wedge n \wedge A \wedge B \wedge C = 0$$

Note that in this chapter, and subsequent chapters where we use the same multiples (α, β, γ etc) in $\mathfrak{A}(p+1, q+1)$ space as in $\mathfrak{A}(p,q)$ space, it is important that the representatives are taken as $F(a)$, $F(b)$ etc and not arbitrary multiples of these. What it amounts to is that we have to enforce our usual normalisation constraint

$$X \cdot n = -1 \tag{5.3}$$

We have already seen that when looking at the dual representation of lines, planes, circles and spheres, it is useful to work with these normalised representations. So far we have avoided using the term homogeneous (a variety of other approaches to the conformal model use the term *homogeneous* e.g. [8]): although we are associating all points λX with a given 3d point x, by insisting upon a normalisation we increase the range of procedures we can employ. If one were truly to use homogeneous coordinates then for many operations there would be a variety of factors to introduce, which would mimic our normalisation. One argument against using normalised coordinates is that we lose the usefulness of being able to avoid carrying around large numbers when dealing with points far from the origin. In practise, we have not met with any difficulties, but everything in this report can be reproduced adopting a homogeneous viewpoint and simply introducing various factors if certain operations are desired. The ease with which the use of normalised points enables one to think about the addition of 5d points and its meaning persuades us that we should maintain this viewpoint for the theoretical analysis we are presenting. In practise, the points need not be stored as normalised points.

This formalism for dealing with linear combinations of 5d points has some interesting properties. Given two null vectors A and B representing 3d points a and b, we know from the above that taking $X = \frac{1}{2}(A+B)$ will give us $C + \alpha n$ where C is the 5d representative of the midpoint of the line joining a and b, i.e. $c = \frac{1}{2}(a+b)$. Extracting C from X is now a special case of the more general problem of extracting P, where $P^2 = 0$, from an expression of the form

$$X = \alpha P + \beta n$$

To do this we first note that $XnX = \alpha^2 PnP$ since sandwiching n gets rid of the βn term. Secondly, we note that in taking PnP we are effectively *reflecting* n in the point representation P. It is to be expected that this can produce only one thing, which is a multiple of the point itself. Indeed this is the case, and it is not difficult to prove that

$$PnP = -2P$$

Thus using the fact that $X \cdot n = \alpha P \cdot n = -\alpha$, we see that $P = -(XnX)/2\alpha^2$ leading to

$$P = \frac{-(XnX)}{2(X \cdot n)^2} \tag{5.4}$$

We will use the above extraction procedure a number of times in subsequent constructions.

We now go on to look at some explicit examples of intersecting objects in this framework. Firstly we look at the general *meet* operation, familiar from projective geometry, and how it appears in geometric algebra. Suppose we are intersecting two objects W_r and W_s (an r-blade and an s-blade representing r and s-grade objects), if X lies on the intersection of W_r and W_s then we know that

$$X \wedge W_r = 0 \quad \text{and} \quad X \wedge W_s = 0$$

It can then be shown that

$$X \cdot \langle W_r W_s \rangle_{2n-r-s} = 0 \quad \text{or equivalently } X \wedge \{ \left[\langle W_r W_s \rangle_{2n-r-s} \right] I_n \} = 0 \tag{5.5}$$

where n is the dimension of the space, in this case, $n = 5$. Thus the intersection of the two objects W_r and W_s is given by taking the $2n - r - s$ grade part of the product of W_r and W_s. This is equivalent to the form of the meet given in [10, 9],

$$W_r \vee W_s = \{W_r^* \wedge W_s^*\}^* \tag{5.6}$$

where the operator $\vee$ denotes the meet or intersection operation and the * refers to taking the dual with respect to the *join* of the two objects (e.g. in the case of the intersection of 2 lines, the join is not the whole space and we would not therefore be able to use our pseudoscalar I_5 for evaluating the dual in the above expression for the meet). In the case where the join is indeed the whole space and we can use the pseudoscalar I_5, we can also write the meet as

$$W_r \vee W_s = W_r \cdot (W_s)^*$$

Clearly this need for evaluating the join is cumbersome, and the formula given in equation 5.5 is far preferable as it can be computed with ease. We will now go on to see how useful this formalism is in intersecting objects – it is a generic formula regardless of the nature of our objects (from within the set of spheres, circles, planes and lines), and can thus be programmed up very easily.

5.1 *Intersecting spheres with spheres or planes*

Let us consider the intersection of two spheres, Σ_1 and Σ_2 where they may intersect in a circle, at a point or not at all. Suppose we take the formula for the meet (where now we use $*$ to indicate multiplication by the pseudoscalar I_n):

$$C = \Sigma_1 \vee \Sigma_2 = \left[\langle \Sigma_1 \Sigma_2 \rangle_{2n-r-s}\right]^* \tag{5.7}$$

$2n - r - s = 10 - 4 - 4 = 2$, so that the dual quantity will have grade $5 - 2 = 3$ – generally, this will give the trivector representing the circle of intersection. We can tell whether we have a circle, a point intersection or no intersection according to whether

$$C^2 > 0 \quad C^2 = 0 \;\text{ or }\; C^2 < 0 \tag{5.8}$$

In the case of $C^2 > 0$ we can extract the centre and radius according to equation 4.3.1. If we also attempt to extract the centre and radius via these same formulæ from C where $C^2 = 0$, we will find that the circle will have zero radius and its centre will be the point of tangency of the two spheres. Similarly, attempting to extract the radius and centre from C in the case $C^2 < 0$ (i.e. no intersection) leads to an imaginary radius and a centre which lies on the shortest line joining the surfaces of the spheres (i.e. that joining the centres). If the two spheres have the same radii, it is the midway point on this line.

The above all follows through if instead of having a second sphere, Σ_2, we have a plane, Φ_2 – we again get a trivector for our intersection object via the meet and the sign of the square of this trivector tells us whether the two objects are tangent, intersect in a circle or do not intersect at all.

Although we will not go into this here, it is interesting to note that in the case of the non-intersecting objects, we can also extract information such as the perpendicular vector between the surfaces from the product $W_r W_s$.

5.2 *Intersecting spheres with circles or lines*

Let us now intersect a sphere Σ_1 (4-blade) with a circle C_2 (3-blade). According to our meet formulæour intersection is a 2-blade, B, given by

$$B = \Sigma_1 \vee C_2 = \left[\langle \Sigma_1 C_2 \rangle_{2n-r-s}\right]^* \tag{5.9}$$

where $2n-r-s=10-4-3=3$, so that the dual object has grade 2. We have already seen that these 2-blades represent 2 points – precisely as we would expect, since an intersecting sphere and circle will do so at 2 points. Now, again, we look at the sign of the resulting 2-blade, B, and we will find that there are two, zero or one point of intersection according to

$$B^2 > 0 \quad B^2 = 0 \quad \text{or} \quad B^2 < 0$$

We have seen earlier that given a bivector B, such that $B^2 > 0$, of the above form, we can extract the two points of intersection via the projectors given in equation 4.13. Now, if $B^2 = 0$ we cannot form the projector, but it is trivial to find the representation of the point of intersection, X, in this case using the following

$$X = BnB$$

i.e. for a 2-blade of the form $W = P \wedge Q$, reflecting n in W, WnW, would give us the midpoint of the line joining P and Q – for our case where $B^2 = 0$, the construction BnB will therefore give us a representation of the point of intersection. These results can easily be shown by considering simple cases at the origin and then extending the proof via rotors as previously.

Precisely the same working holds if we replace our circles above with lines – the meet again gives a 2-vector whose square tells us whether there are 2, 1 or no intersections, and from which the intersection points can be obtained easily. We will return to the intersections of lines with spheres when we later consider reflections of lines in spheres.

Again, in the non-intersecting case the product $W_r W_s$ can provide us with information about the vectors between the surfaces.

5.3 *Intersecting planes with planes, circles and lines*

Consider two planes Φ_1 and Φ_2; taking the meet gives

$$L = \Phi_1 \vee \Phi_2 = \left[\langle \Phi_1 \Phi_2 \rangle_{2n-r-s} \right]^* \tag{5.10}$$

where $2n-r-s=10-4-4=2$, so that the dual object has grade 3 – as we would expect, if the planes intersect to give a line. We are able to tell whether the planes intersect by looking at the sign of L^2 – if $L^2 = 0$ we know that the planes are parallel and do not intersect, if $L^2 > 0$, the planes intersect in the line L.

Now consider a plane Φ_1 and a circle C_2; we take the meet of these two objects to give

$$B = \Phi_1 \vee C_2 = \left[\langle \Phi_1 C_2 \rangle_{2n-r-s}\right]^* \tag{5.11}$$

where $2n-r-s = 10-4-3 = 3$, so that the dual object has grade 2 – the plane and the circle intersect in a maximum of two points, and the 2-blade, B encodes these two points as with the sphere-circle intersection. Once again we can assert that there are 2,1 or 0 intersections according to whether $B^2 > 0, B^2 = 0, B^2 < 0$. In the case of two intersections, the points are extracted from B by projectors as before, and in the case of tangency, the one point of contact is obtained by taking BnB.

It is worth thinking about what happens when the circle C_2 lies in the plane Φ_1 so that the intersection is C_2 itself. As one might expect, in this case there *is no* grade 3 part of $\Phi_1 C_2$. In the case of the $z = 0$ plane and the unit circle lying in the plane and centred on the origin this can easily be confirmed:

$$\Phi_1 = F(e_1) \wedge F(e_2) \wedge F(-e1) \wedge n \propto e_1 e_2 e\bar{e}$$

$$C_2 = F(e_1) \wedge F(e_2) \wedge F(-e1) \propto e_1 e_2 \bar{e}$$

thus

$$\Phi_1 C_2 \propto e_1 e_2 e\bar{e} e_1 e_2 \bar{e} = e \text{ hence } \langle \Phi_1 C_2 \rangle_3 = 0$$

We can also note that, in this case, the dual of $\Phi_1 C_2$ with respect to $e_1 e_2 e\bar{e}$ is indeed C_2.

If we now replace our circle by a line, L_2, it is clear that the meet will still give us a 2-blade, B; but we know that the line and the plane intersect in at most one position, so should we not be looking for a vector rather than a 2-blade? The answer is that if the plane and the line intersect, and the meet gives us B, then B is always of the form

$$B = X \wedge n$$

where X is the representation of the point of intersection. This can be proved easily by again considering a simple case at the origin. If $B^2 > 0$ the line and plane intersect in a point, if $B^2 = 0$ the line and plane do not intersect and if $B = 0$ the line lies in the plane. If there is one point of intersection so that B is of the above form, we can extract the 3d point of intersection, $x = x^i e_i,\ i = 1,2,3$ (and hence X), by simply equating x^i to the coefficient of the $e_i \wedge n$ term or by using the following expansion

$$x = (B \wedge \bar{n}) \cdot E \tag{5.12}$$

where $E = n \wedge \bar{n}$ as given earlier.

5.4 *Intersecting circles with circles and lines*

Consider two circles, C_1 and C_2, taking their meet gives

$$X = C_1 \vee C_2 = \left[\langle C_1 C_2 \rangle_{2n-r-s}\right]^* \tag{5.13}$$

where $2n - r - s = 10 - 3 - 3 = 4$, so that the dual object has grade 1. We know, however, that the intersection of two circles has at most 2 intersections (only possible if they lie in the same plane), so how do we get two intersections from our grade 1 object? In fact we find that the following is true

$$\begin{aligned} C_1 \vee C_2 &= X \ \text{ where } X^2 = 0 \ \text{ if circles have one intersection} \\ C_1 \vee C_2 &= X \ \text{ where } X^2 \neq 0 \ \text{ if circles have no intersection} \\ C_1 \vee C_2 &= 0 \quad \text{if circles have two intersections} \end{aligned} \tag{5.14}$$

In the case where the meet gives zero and we know there are two intersections, these can easily be found by intersecting the plane of one of the circles with the other circle, i.e.

$$B = C_1 \vee (C_2 \wedge n) = \left[\langle C_1 (C_2 \wedge n) \rangle_{2n-r-s}\right]^* \tag{5.15}$$

where $2n - r - s = 10 - 3 - 4 = 3$, so that the dual object has grade 2, and the two points of intersection can be extracted from the 2-blade B using equations 4.13.

If we now replace C_2 by a line L_2 we see that we again get a grade 1 object when we take the meet, and the situation above is exactly replicated, i.e.

$$\begin{aligned} C_1 \vee L_2 &= X \ \text{ where } X^2 = 0 \ \text{ if circle and line have one intersection} \\ C_1 \vee L_2 &= X \ \text{ where } X^2 \neq 0 \ \text{ if circle and line have no intersection} \\ C_1 \vee L_2 &= 0 \quad \text{if circle and line have two intersections} \end{aligned} \tag{5.16}$$

As before, in the case where the meet gives zero the two intersections can easily be found by intersecting the plane of one of the circles with the line. It is also interesting to note here that in the case where the circle and the line do not intersect, with the meet giving a vector, X, which is not null, the sign of X^2 tells us whether the line passes through the circle ($X^2 < 0$) or does not pass through the circle ($X^2 > 0$) – such simple checks can often be useful in graphics applications.

Given that we have had a little difficulty with circles intersecting circles, we might expect some slight difficulties with lines and lines. It turns out that many interesting constructions emerge when we start to consider the intersections between two lines, these will be discussed in the following section.

5.5 *Intersecting lines with lines*

Let us consider two lines, L_1 and L_2. Taking the meet of these two lines gives

$$X = L_1 \vee L_2 = \left[\langle L_1 L_2 \rangle_{2n-r-s}\right]^* \tag{5.17}$$

where $2n-r-s = 10-3-3 = 4$, so that the dual object has grade 1. We might expect that if the lines intersect at a point, the meet, X, will give this intersection point – however, this is not the case. We find that the following is true

$$\begin{aligned} L_1 \vee L_2 &= 0 \quad \text{if the lines intersect} \\ L_1 \vee L_2 &\propto n \quad \text{if the lines do not intersect} \end{aligned} \tag{5.18}$$

We therefore have a simple of way for checking for intersecting lines, but if the meet gives us zero so that we know there is an intersection point, we can no longer find this point in a fully covariant way by intersecting one line with the a plane defined by the other line, since such a plane is not uniquely defined. We could, in practise, intersect one line with the plane formed by the other line and the origin, but then if the other line passes through the origin, this will not work, and we are left with a non-covariant procedure and one which entails us forming conditionals for a number of cases. We would instead like to look for a method which works co-variantly – such a method exists, and in the process of describing it, we see a number of other useful constructions.

Again take our arbitrary lines, L_1 and L_2 (assume they are normalised, such that $L_1^2 = L_2^2 = 1$). Suppose we reflect line L_1 in line L_2 – this statement is not well-defined in

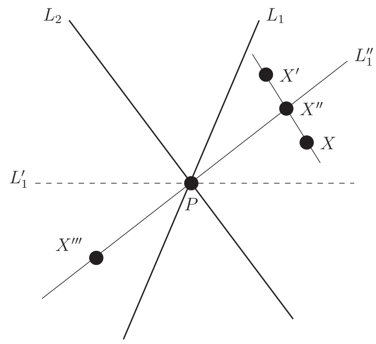

Figure 5.1: *The rotation of intersecting lines to produce two lines intersecting at right angles, via the construction* $L_1 - L_2L_1L_2$.

a conventional sense, but in GA, we have seen that reflection of an object in another object is indeed well defined and is brought about by sandwiching the object to be reflected between the object that it is being reflected in. So, our reflected line, L_1' is given by

$$L_1' = L_2L_1L_2$$

The operation of reflection is grade preserving if we are dealing with blades, and therefore we know that we get another line. This construction will work for any two lines, but let us now suppose that our lines intersect at a point, we can then expect that the reflected line L_1' will be the line formed by the intersection point, represented by P, and any point on L_1 reflected in L_2. This is indeed what L_1' is. Now, however, it becomes possible very easily to form the line which is perpendicular to L_2, passing through the intersection point, P, and in the plane defined by L_1 and L_2 via the following

$$L_1'' = L_1 - L_2L_1L_2 \tag{5.19}$$

Clearly what we are doing here is rotating one line in the plane defined by the two lines, to be perpendicular to the other line. We will return to this rotor description later. Now we have two perpendicular lines which intersect, we can find the point of intersection relatively easily. Take any arbitrary point representation X and reflect it in L_1'' (assuming again that we have normalised L_1'') via $X' = L_1''XL_1''$, then take the midpoint of X and X', we know that this must lie on the line $L1'$

$$X' = L_1''XL_1'' \quad X'' = \frac{1}{2}(X + X')$$

(recall that X'' will be our real midpoint plus some multiple of n). Now we reflect X''

in L_2 to give X''' and again take the midpoint – the midpoint must now give us the intersection point P plus some multiple of n;

$$X''' = L_2 X'' L_2 \quad P' = \frac{1}{2}(X'' + X''')$$

we then extract the null vector corresponding to the representation of our intersection point P via

$$P = \frac{-(P' n P')}{2(P' \cdot n)^2}$$

This construction is independent of X and is a beautiful illustration of the ability to manipulate objects using geometric algebra. Although it appears involved, it can be programmed up very easily ($\bar{n}$ can be used as the point X) and is an entirely covariant way of intersecting two lines. Of course, in practise, one can also check to see if there is at least one line that does not pass through the origin ($\bar{n} \wedge L = 0$ if L passes through the origin), if there is, say this is L_1, we can then form the plane $\bar{n} \wedge L_1$ and intersect this with L_2, if both lines pass through the origin then the intersection point is the origin. Note that we can use precisely the same type of argument to extract the plane formed by two intersecting lines. For example, take any arbitrary point X, reflect it in the line L_1 via $L_1 X L_1$, so that the midpoint of this line is given by P where P is given (up to some additional multiple of n) by

$$P = \frac{1}{2}(X + L_1 X L_1)$$

P must clearly lie on L_1, thus the plane formed by the two lines must be given by

$$L_2 \wedge P = L_2 \wedge \frac{1}{2}(X + L_1 X L_1) \tag{5.20}$$

Again, $\bar{n}$ can be used for our point X in real computations. The derivations given here are entirely covariant and the same constructions will intersect 'lines' in different geometries, we will illustrate this towards the end of the report by considering 3d hyperbolic geometry.

Recall that previously we found the reflection of L_1 in L_2 via $L_2 L_1 L_2$, now note that we can rewrite this as

$$L_2 L_1 L_2 = (L_2 L_1) L_1 \widetilde{(L_2 L_1)} = R L_1 \tilde{R} \tag{5.21}$$

since $\widetilde{(L_2 L_1)} = \tilde{L}_1 \tilde{L}_2 = L_1 L_2$. Thus we see that the quantity $L_2 L_1$ acts as a rotor (if the lines are normalised) which rotates through twice the angle between the lines about an axis through the intersection point and perpendicular to the plane containing the lines. In fact, when we write this reflection as a rotation, there is nothing to insist that our lines must intersect. If L_1 and L_2 do not intersect, then $L_2L_1L_2$ will still perform a *reflection of* L_1 *in* L_2, but we are now able to interpret exactly what this reflection means for non-intersecting lines if we regard the operator as a rotor. It turns out that the rotor L_2L_1 is the product of a rotation rotor and a translation rotor – the rotation is in the plane normal to the common perpendicular of the lines and the translation is along the common perpendicular so that one line is taken onto and then through the other. We can show that it is possible to write the product L_2L_1 as

$$L_2L_1 = (\cos\theta + \hat{B}\sin\theta)(1 + dn) \tag{5.22}$$

where d is the 3d vector representing the length and direction of the common perpendicular, $\hat{B} = \hat{d}I_3$ (with $\hat{d} = d/\sqrt{(d^2)}$) and θ is the angle between the lines as measured when translated to lie in the same plane along d. We see that the above is a combination of a rotation rotor and a translation rotor, we will see later that rotors which take one object (of the same grade) into another object are often formed in the way we have outlined here for lines.

"Mathematics consists of proving the most obvious thing in the least obvious way."

— George Polye

6 Reflections in the Conformal Space

We have already used the recipe for reflecting one object in another a number of times in previous chapters. In this chapter we will look at reflections in more detail, in particular at those constructions which will be particularly useful in computer graphics. We will start with the most basic operation of reflecting lines in planes and go on to look at more sophisticated reflections.

6.1 Reflections of lines in planes

Given a set of plane reflectors with normals $n_1, n_2, n_3 ....$, and an incoming ray direction m, we are familiar with the fact that we form the final output direction after multiple bounces via

$$m' = \pm n_i n_{i-1} \dots n_1 m n_1 \dots n_{i-1} n_i \tag{6.1}$$

This formula holds in 3d and works well for direction, but does not keep track of the position of the rays and hence we will not be able to use it to see if, under reflection from particular combinations of planes, a ray will undergo multiple bounces. In order to do this we need to efficiently compute, given the position and direction of a ray in space together with the position and orientation of a plane, the position of intersection and the direction of reflection. By using the 5d conformal geometry we are able to include the positional information in a very simple manner. Let Φ be the plane and L be the line we wish to reflect in the plane, where Φ and L are given by

$$\Phi = I(dn + \hat{n}) \equiv n \wedge A \wedge B \wedge C \tag{6.2}$$

$$L = n \wedge P \wedge Q \tag{6.3}$$

with $\hat{n}$ the unit normal to the plane and d the perpendicular distance of the plane to the origin, A, B, C points on the plane and P, Q points on the line. Now consider the line L' given by

$$L' = \Phi L \Phi \tag{6.4}$$

Note that we have again used this sandwiching technique to carry out the reflection. We now show that the line L' not only has the direction of the ray L reflected in Φ, but also passes through the point of intersection of L and Φ – it is thus giving us exactly what we require, the expression for the actual reflected ray and not merely its direction. Note also how similar equation (6.4) is to the equation giving the direction of the reflected ray for an incoming ray a reflecting in a plane of normal m, namely $-mam$. Suppose we take any point, $x = x_1e_1 + x_2e_2 + x_3e_3$, in space, and let X be the 5d representation of x. To begin with we take Φ to be the x-y plane, our first task is then to find the conformal representation of Φ; take three points, a,b,c on this plane to be 0, e_1 and e_2. The conformal representations of these points are

$$A = F(0) = \frac{1}{2}(0n + 2x0 - \bar{n}) = -\bar{n} \tag{6.5}$$

$$B = F(e_1) = \frac{1}{2}(n + 2e_1 - \bar{n}) = (e_1 + \bar{e}) \tag{6.6}$$

$$C = F(e_2) = \frac{1}{2}(n + 2e_2 - \bar{n}) = (e_2 + \bar{e}) \tag{6.7}$$

(using the fact that $n - \bar{n} = 2\bar{e}$). We can therefore write the plane Φ as

$$\Phi = n \wedge (-\bar{n}) \wedge (e_1 + \bar{e}) \wedge (e_2 + \bar{e}) \tag{6.8}$$

which, writing n and $\bar{n}$ in terms of e and $\bar{e}$, can be written as

$$\Phi = -(e - \bar{e}) \wedge (e + \bar{e}) \wedge (e_1 + \bar{e}) \wedge (e_2 + \bar{e}) \tag{6.9}$$

$$= -\{e \wedge e_1 \wedge e_2 \wedge \bar{e} - \bar{e} \wedge e_1 \wedge e_2 \wedge e\} \tag{6.10}$$

$$= -2e \wedge \bar{e} \wedge e_1 \wedge e_2 \tag{6.11}$$

$$= -2e\bar{e}e_1e_2 \tag{6.12}$$

We can simplify the representation of the plane by taking its dual, Φ^* where

$$\Phi^* = \Phi I = -2e\bar{e}e_1e_2e_1e_2e_3e\bar{e} = -2e_3 \tag{6.13}$$

Thus, up to scale we can take the 5d representation of the e_1-e_2 plane to be given by $\Phi = Ie_3$. Now we return to our arbitrary point in space, X, and see what happens when we form the quantity $\Phi X \Phi$;

$$\Phi X \Phi = Ie_3 X Ie_3 = Ie_3 \frac{1}{2}(x^2 n + 2x - \bar{n}) Ie_3 \tag{6.14}$$

Since $Ie_3 = e_3I$ and $e_3ne_3 = -n$, $e_3xe_3 = -x_1e_1 - x_2e_2 + x_3e_3$ and $e_3\bar{n}e_3 = -\bar{n}$, we can reduce equation (6.14) to

$$\Phi X \Phi == I\frac{1}{2}(\tilde{x}^2 n + 2\tilde{x} - \bar{n})I = \frac{1}{2}(\tilde{x}^2 n + 2\tilde{x} - \bar{n}) = \tilde{X} \tag{6.15}$$

where $\tilde{x} = x_1e_1 + x_2e_2 - x_3e_3$ and $\tilde{X} = \frac{1}{2}(\tilde{x}^2 n + 2\tilde{x} - \bar{n})$. Thus, we see that sandwiching X between Φ has the effect of reflecting the point in the plane (i.e. reversing the sign of the e_3 coordinate). If we take another point in space, whose 5d representation is Y, then similarly $\Phi Y \Phi$ reflects the point Y in the Φ-plane. Now, we know that the 5d representation of the line joining points X and Y, is L where

$$L = n \wedge X \wedge Y \tag{6.16}$$

Since $\Phi(A \wedge B)\Phi$ can be written as $\Phi A \Phi \wedge \Phi B \Phi$ (to see this we simply write $A \wedge B$ as $\frac{1}{2}(AB - BA)$ and use the fact that $\Phi\Phi$ gives a scalar), we can extend this to show that

$$\Phi(A_1 \wedge A_2 \wedge A_3 \wedge \ldots \wedge A_n)\Phi = \Phi A_1 \Phi \wedge \Phi A_2 \Phi \wedge \Phi A_3 \Phi \wedge \ldots \wedge \Phi A_n \Phi \tag{6.17}$$

Therefore, we are able to see that, up to a scale factor, $\Phi L \Phi$ can be written as

$$\Phi L \Phi = \Phi n \Phi \wedge \Phi X \Phi \wedge \Phi Y \Phi = n \wedge \Phi X \Phi \wedge \Phi Y \Phi = L' \tag{6.18}$$

since $\Phi n \Phi = \lambda n$ with $\lambda = \Phi^2$ (one can verify this most easily by using the alternative form of the conformal plane, i.e. $\Phi = I(dn + \hat{n})$ as given in equation (6.3)). In equation (6.18) L' is clearly the reflection of line L in the plane Φ – as it is defined in terms of the reflected points $\Phi X \Phi$ and $\Phi Y \Phi$, it very naturally gives both the direction and position in space of the reflected line. Once again, this result can be generalised to hold for any line and any plane by rotating and translating: in this case, suppose we rotate and translate our plane Φ via a rotor, R, in 5d. Note now that we can apply R to equation (6.15) to give

$$R\tilde{X}\tilde{R} = (R\Phi\tilde{R})(RX\tilde{R})(R\Phi\tilde{R}) \tag{6.19}$$

which can be viewed as

$$\tilde{X}_R = (R\Phi\tilde{R})X_R(R\Phi\tilde{R}) \tag{6.20}$$

since we can express any point in the space as X_R, and since $\tilde{X}_R$ then gives the reflection in the $\tilde{\Phi} = R\Phi\tilde{R}$ plane, it is clear that the results above hold for any arbitrary plane.

The result that $\Phi L \Phi$ gives the reflection of line L in plane Φ is currently used extensively in our ray-tracing programs and in programs which generate the reflections from 2d and 3d surfaces used in studies of surface characterisation and evolution. It provides a very efficient way of evaluating reflections of many incoming rays from a given plane.

Given the general equation of a plane it is not hard to show that if we take Φ^2 where $\Phi = A \wedge B \wedge C \wedge n$, we get a multiple of the area of the plane segment described by the triangle defined by the 3 points, A, B, C – this fact has been used in surface scattering problems. To show this we again consider a simple case at the origin, prove it for this and because we can rotate, translate and dilate, it will be true for all planes/triangles. Take the triangle at the origin whose vertices are given by a, e_1 and the origin. Thus,

$$\begin{aligned} \Phi &= \frac{1}{2}(a^2 n + 2a - \bar{n}) \wedge 2(e_1 + \bar{e}) \wedge n \wedge \bar{n} \\ &= 2a \wedge e_1 \wedge n \wedge \bar{n} \equiv 2a \wedge e_1 \wedge E \end{aligned} \tag{6.21}$$

Thus, since $(B \wedge E)^2 = B^2 E^2 = 4B^2$ (where B is a euclidean bivector and $E = n \wedge \bar{n}$), we see that $\Phi^2 = 16(a \wedge e_1)^2$. But the area of the triangle is clearly given by $-B^2$ where $B = \frac{1}{2}(a \wedge e_1)$, so that the relationship is as follows

$$\text{area of triangle}^2 = -\frac{1}{64}\Phi^2 \tag{6.22}$$

Or, the area of the triangle is given by one eighth of the magnitude of the 4-vector formed from the three points and n. We would now like to say that since we have proved this for the simple case at the origin, we can extend via rotating, dilating and translating to say that it holds for any triangle in space, as in previous proofs. However, in this case, we are interested in an actual number, the magnitude of the area, not just an incidence relation, and we must therefore check that all scale factors behave properly under dilation (rotation and translation are clearly well behaved). To do this consider operating on our triangle at the origin with a dilation rotor, $R_D = \exp\left(\frac{\alpha}{2} e\bar{e}\right)$. Since we have scaled our points a and e_1 by $\exp(-\alpha)$, we expect an overall scaling of $\exp(-2\alpha)$ on the area. Our new Φ' is given by

$$\Phi' = F(\exp(-\alpha)\, a) \wedge F(\exp(-\alpha)\, e_1) \wedge n \wedge \bar{n} \tag{6.23}$$

Multiplying both sides of the above equation and using our earlier results on dilation rotors we have

$$\begin{aligned}
\exp(2\alpha)\Phi' &= \exp(\alpha)F(\exp(-\alpha)a)\wedge\exp(\alpha)F(\exp(-\alpha)e_1)\wedge n\wedge\bar{n} \\
&= R_D A\tilde{R}_D\wedge R_D E_1\tilde{R}_D\wedge R_D n\tilde{R}_D\wedge R_D\bar{n}\tilde{R}_D \\
&= R_D(A\wedge E_1\wedge n\wedge\bar{n})\tilde{R}_D \\
&= R_D\Phi\tilde{R}_D \qquad (6.24)
\end{aligned}$$

where $A = F(\exp(-\alpha)a)$ and $E_1 = F(\exp(-\alpha)e_1)$. The above makes use of the facts that $R_D n\tilde{R}_D = n$ and $R_D\bar{n}\tilde{R}_D = \bar{n}$. Squaring the result in equation 6.24 tells us that $\exp(4\alpha)\Phi'^2 = \Phi^2$ or

$$\Phi'^2 = (\exp(-2\alpha)\Phi)^2 \qquad (6.25)$$

which is precisely the correct scaling. Therefore, we are able to use our simple proof at the origin and extend it via rotors to a proof for any triangle in space.

6.2 *Reflection of lines in triangular facets*

We have seen how nicely we are able to reflect lines in planes, but often our complicated objects are made up of facets. Let us assume that the surface we are reflecting from is made up of triangular facets. Let one of these facets form the plane, Φ, which is defined by three points, p_1, p_2, p_3, whose 5d representations are P_1, P_2, P_3. The plane Φ can therefore be written as

$$\Phi = n\wedge P_1\wedge P_2\wedge P_3 \qquad (6.26)$$

It is then possible to define a *reciprocal frame*, P^1, P^2, P^3, satisfying

$$P^i\cdot P_j = \delta^i_j \quad \text{for} \quad i,j = 1,2,3 \qquad (6.27)$$

via the following formulæ

$$P^i = ||\Phi||^{-2}(n\wedge L_i)\cdot\Phi \quad \text{for} \quad i = 1,2,3 \qquad (6.28)$$

where $L_1 = P_2\wedge P_3$, $L_2 = P_3\wedge P_1$, $L_3 = P_1\wedge P_2$ and $||\Phi|| = \sqrt{(-\Phi^2)}$. The form of the reciprocal vectors given in equation 6.28 makes it easy to see how they satisfy equation 6.27:

$$\begin{aligned}
[(n\wedge L_i)\cdot\Phi]\cdot P_j &\equiv P_j\cdot[(n\wedge L_i)\cdot\Phi] \\
&= (P_j\wedge n\wedge L_i)\cdot\Phi \\
&= -(n\wedge P_j\wedge L_i)\cdot\Phi \qquad (6.29)
\end{aligned}$$

the above rearrangement uses the relationship $A_r \cdot (B_s \cdot C_t) = (A_r \wedge B_s) \cdot C_t$ for $r + s \leq t$ (see [9]). Now, it is easy to see that for $i = j$, $(n \wedge P_i \wedge L_i) = -\Phi$ so that $(n \wedge P_i \wedge L_i) \cdot \Phi = ||\Phi||^2$ and for $i \neq j$, $(n \wedge P_j \wedge L_i) = 0$ since L_i will contain P_j. The conditions for the reciprocal frame given in equation (6.27) are therefore satisfied. Notice also from equation (6.29) it is easy to see that

$$P^i \cdot n = -(n \wedge n \wedge L_i) \cdot \Phi = 0 \quad \text{for all} \quad i \tag{6.30}$$

Now if we have a general point p (3d representation) in the plane defined by p_1, p_2, p_3 we know that we can write $p = \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3$ with $\lambda_1 + \lambda_2 + \lambda_3 = 1$. In Section 5 we have seen that if P, P_i are the 5d representations of p, p_i, then P can be written as

$$P = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3 + \lambda_n n \tag{6.31}$$

Thus, it is very easy to recover the λs by simply dotting with the reciprocal vectors, i.e.

$$\lambda_i = (P^i \cdot P) \quad \text{for} \quad i = 1, 2, 3 \tag{6.32}$$

These results follow because $P^i \cdot n = 0$ as shown above. When we have the λs it is then simple to ascertain if a ray intersects with a particular facet – this is the case if

$$0 \leq \lambda_i \leq 1 \quad \text{for} \quad i = 1, 2, 3 \tag{6.33}$$

if these conditions are not satisfied, the ray does not intersect the facet. Thus, coupled with the multiple reflection formulæ, this can lead to very fast numerics. The results described in these last two sections have been used for surface evolution and scattering and are now being applied in ray-tracing algorithms.

6.3 *Reflecting from spheres*

Suppose we have a ray, represented by a line L, impinging on a sphere Σ or S ($S = \Sigma^*$). We have seen in the previous section that we can take the meet of L and S to form a 2-blade, $B = L \cdot S$. Analysis of B tells us whether the line intersects with the sphere, is tangent to it or does not intersect. If the line intersects, we can extract the points of intersection P and Q from B (see equation 4.13). When the line comes in, hits the sphere, at P say, and is reflected, it is reflected in the tangent plane at P. If we let this tangent plane be Φ_p, then the reflected line, L', is clearly

$$L' = \Phi_p L \Phi_p$$

Now, given any point X on the sphere Σ, we know that $X \wedge \Sigma = 0$, therefore $X\Sigma = X \cdot \Sigma$ and we can think of $X\Sigma$ as taking the dual of the point X in the sphere Σ – thus one might naturally suppose that it somehow encodes the tangent plane at X. $X \cdot \Sigma$ is a trivector and it is not hard to show that the tangent plane at X is actually given by

$$\Phi_x = (X \cdot \Sigma) \wedge n \tag{6.34}$$

Thus, once we have the intersection point of the line, we can form the tangent plane and reflect the line in this plane. It is also interesting to ask ourselves, what does the quantity SLS or $\Sigma L \Sigma$ give, since we are now used to regarding this as the *reflection of the line L in the sphere S*. We know that this operation of reflection is grade-preserving, so that our line L which is a 3-blade, will be taken to another 3-blade; however, this 3-blade can be another line or a circle. To investigate this, suppose the line intersects the sphere at two points, P and Q, which do not form a diameter (i.e. the line does not pass through the origin). The line can then be written as

$$L = P \wedge Q \wedge n$$

Now, if we reflect L in S (or Σ), we get

$$SLS = S(P \wedge Q \wedge n)S = SPS \wedge SQS \wedge SnS$$

When we reflect a point on the sphere in the sphere, we naturally return the point itself, thus $SPS = P$ and $SQS = Q$. Also, we saw earlier, that reflection of n in the sphere gave us the centre which we will call C. Therefore, our reflection gives us

$$SLS = P \wedge Q \wedge C$$

which is clearly a circle passing through the centre of the sphere, and the two intersection points. If the line passes through the centre of the sphere, i.e. P and Q form a diameter, then the line is unchanged by reflection in S since SLS again gives $= P \wedge Q \wedge C$, but P,Q and C are now collinear and this trivector again gives something of the form $A \wedge B \wedge n$ forming a line. In fact, what we are seeing here is an example of *inversive geometry*. Textbooks which discuss inversive geometry (e.g. [1]) will look at the reflection of lines and circles in circles in the plane, finding that they are mapped to lines and circles depending upon the original position of the line or circle relative to the reflecting line or circle. The whole of inversive geometry in the plane can be dealt with easily using our reflection formulæ, C_1LC_1 for reflection of lines in circle C_1, and C_1CC_1 for reflection of circles in circle C_1 – we at no point need to revert to a complex number interpretation to see what is going on. The enormous advantage of this is that whatever we have done in 2d will extend directly to 3d with almost no changes. Thus we are able to reproduce the whole of inversive geometry in space with the simple formulæ S_1LS_1, S_1SS_1, $S_1\Phi S_1$ and S_1CS_1, for reflection of lines, spheres, planes and circles in spheres.

Now that we know the nature of the quantity SLS, can we use it directly for reflecting lines from spheres without going through the intermediate stage of forming the tangent plane? If P and Q are again our points of intersection of L with S, then $(P\cdot(SLS))$ will produce a 2-vector. Again, since $P\wedge(SLS)=0$ because P lies on SLS, we know that $P\cdot(SLS)=P(SLS)$, so that we can once more think about this operation as taking the dual of P in our circle SLS. The dual of P (lying on the circle) with respect to the circle gives a 2-vector which, it seems likely, will lie in the tangent line to that circle. Thus, wedging this 2-vector with n will give the tangent line, which indeed turns out to be the reflected line. Thus, in summary, if a line L hits the sphere S at a point P (point of first hitting) the reflected line, L', is given by

$$L' = (P\cdot(SLS))\wedge n \tag{6.35}$$

"Out of nothing I have created a strange new universe."
— Janos Bolyai

7 A Conformal Approach to non-Euclidean Geometry

The conformal approach to Euclidean geometry discussed above can also be extended to non-Euclidean geometry. This has been discussed previously, in an abstract setting, by Li ([14]), but a concrete realisation which shows how the traditional methods can be encompassed in a GA approach, has not so far been given. This turns out to be a rich and fascinating subject, and we only give short illustrations here.

Euclidean geometry, the geometry of the plane, is defined by *Euclid's Postulates*:

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as centre.
4. All right angles are congruent.
5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

This last postulate is equivalent to what is known as the *parallel postulate* which loosely states that two parallel lines will never meet, even at infinity. Hyperbolic space is one of the simplest geometries that satisfy all but this last postulate.

Ignoring the last postulate may, initially, seem a purely intellectual exercise although the last postulate has never been proved to exist in real-space by experiment. Who is to say that parallel beams of light emitted from Earth won't eventually con-

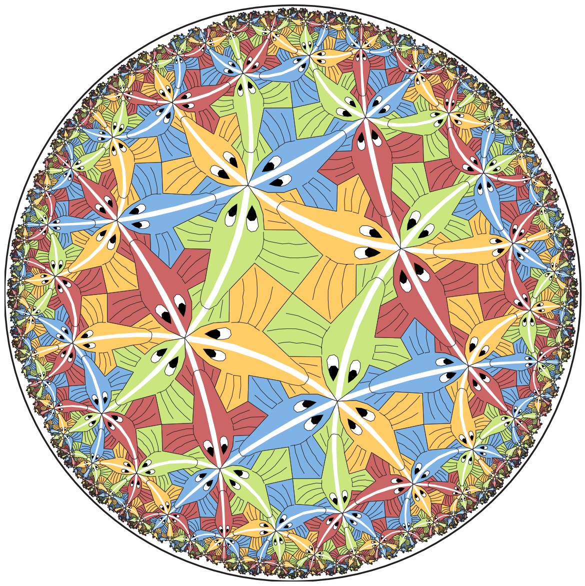

Figure 7.1: *A re-creation of Escher's* Circle Limit III, *a depiction of hyperbolic geometry on the Poincaré disc. Taken from [6].*

verge somewhere out by Andromeda? Indeed, the 'bending' effect upon light by large gravitational fields almost guarantees the convergence of some light beams.

Hyperbolic geometry is usually represented in two dimensions on the *Poincaré disc*. Here the boundary circle of the disc represents the points at infinity. Everywhere outside the disc is inaccessible to the hyperbolic geometry. Straight lines, called d-lines by Brannan *et al.* [1], are represented by circular arcs which erupt normal to the boundary circle. Perhaps the most famous example of this geometry is given in Escher's *Circle Limit* series of wood prints (see figure 7.1). In these, infinite tessellations of hyperbolic space are represented mapped to the Poincaré disc. They clearly show the circular nature of d-lines.

We seek to find a method of representing this geometry within our conformal framework.

7.1 2D non-Euclidean Space

All rigid-body transformation (i.e. rotation and translation) rotors in the Euclidean approach above leave n invariant, i.e. $Rn\tilde{R} = n$ for all rotors R. They also leave $\bar{n}$ invariant. We have already identified n and $\bar{n}$ with the points at infinity and the origin respectively. The question arises: 'What if we restrict rotors to keep other vectors invariant?'. Since a geometry is defined by its congruence transformations, changing those transformations will therefore reflect a different geometry.

Suppose we choose to restrict the rotors such that they keep e invariant. Without loss of generality, we deal with a conformal extension of $\mathbb{R}^2$ and write down a set of four basis vectors

$$E_1 = e_1 \quad E_2 = e_2 \quad E_3 = e \quad E_4 = \bar{e} \tag{7.1}$$

and thus form the rotors $R_{k\ell} = \exp\left(\frac{\alpha}{2} E_k \wedge E_\ell\right)$ with $k, \ell \in \{1,2,3,4\}$. Applying them to e via $R_{k\ell} e \tilde{R}_{k\ell} \equiv R_{k\ell} e R_{\ell k}$ we find the bivector generators of the rotors which preserve e are $\bar{e}e_1$, $\bar{e}e_2$ and e_1e_2. The latter just correspond to rotations in the e_1e_2 plane and hence, the former two must be the generators of translations.

Hence we can say that a rotor which translates the origin to the vector x must be given by

$$T_x = \exp\left(\frac{f(r)}{2}\bar{e}\hat{r}\right) \tag{7.2}$$

where $r = |x|$, $\hat{r} = x/|x|$ and $f(r)$ is some function of r yet to be determined. Noting that $(\bar{e}e_1)^2 = (\bar{e}e_2)^2 = +1$ and therefore $(\bar{e}\hat{r})^2 = +1$ we can take the power series expansion of T_x and collect like-coefficients to obtain

$$T_x = \cosh\left(\frac{f(r)}{2}\right) + \bar{e}\hat{r}\sinh\left(\frac{f(r)}{2}\right) \tag{7.3}$$

The choice of the origin is only restricted in that it must differ from the point at infinity and must not contain components of e_1 or e_2 (to retain isotropy). Either n or $\bar{n}$ is a suitable choice but we choose a multiple of $\bar{n}$ to retain compatibility with the Euclidean case.

Again, as with the Euclidean case, we wish to impose a normalisation condition on the null-vectors, $X = F_e(x)$, such that

$$X \cdot e = -1 \tag{7.4}$$

where we use $F_e(x)$ to represent the mapping defined by the geometry generated by the rotors which preserve e. If this is to hold then the origin must in fact be $-\bar{n}$.

We can now find the representation of the general point x as the translation along

x of the origin. Writing $c = \cosh\left(\frac{f(r)}{2}\right)$ and $s = \sinh\left(\frac{f(r)}{2}\right)$

$$\begin{aligned} F_e(x) &= T_x(-\bar{n})\tilde{T}_x & (7.5)\\ &= [c + \bar{e}\hat{r}s](-\bar{n})[c - \bar{e}\hat{r}s] & (7.6)\\ &= -c^2\bar{n} + 2sc\hat{r} + s^2 n & (7.7)\end{aligned}$$

then letting $C = \cosh(f(r))$ and $S = \sinh(f(r))$ giving $c^2 = (C+1)/2$, $s^2 = (C-1)/2$, $sc = S/2$ and hence

$$\begin{aligned} F_e(x) &= \frac{1}{2}n(C-1) - \frac{1}{2}\bar{n}(C+1) + S\hat{r} & (7.8)\\ &= \frac{1}{2}\left[(C-1)n + 2S\hat{r} - (C+1)\bar{n}\right] & (7.9)\end{aligned}$$

As required $(F_e(x))^2 = 0$ and $F_e(x) \cdot e = -1$.

It remains to choose a sensible form for $f(r)$. We seek to choose $f(r)$ such that the representation $F_e(x)$ is similar to our Euclidean representation $F(x)$ since this will allow us to use many of the same techniques we developed for the Euclidean case. We firstly note that our original representation is actually dimensionally inconsistent. We correct this by adding in a fundamental length scale, λ, which is unity in the Euclidean case. If we also express the vector x as $r\hat{r}$ where $\hat{r}$ is a unit vector parallel to x and r is an approptiate scalar, we can re-write our Euclidean representation as

$$F(x) = \frac{1}{2\lambda^2}(r^2 n + 2\lambda r\hat{r} - \lambda^2\bar{n}) \qquad (7.10)$$

If we wish that $F_e(x)$ be similar to $F(x)$ then we have the conditions

$$\frac{S}{C+1} = \frac{\sinh(f(r))}{\cosh(f(r))+1} = \frac{r}{\lambda} \qquad (7.11)$$

and

$$\frac{C-1}{S} = \frac{\cosh(f(r))-1}{\sinh(f(r))} = \frac{r}{\lambda} \qquad (7.12)$$

so the mapping function becomes

$$\begin{aligned} F_e(x) &= \frac{C+1}{2\lambda^2}\left[x^2 n + 2\lambda x - \lambda^2\bar{n}\right] & (7.13)\\ &= \frac{\cosh(f(r))+1}{2\lambda^2}\left[x^2 n + 2\lambda x - \lambda^2\bar{n}\right] & (7.14)\end{aligned}$$

which has a degree of similarity to the expression for $F(x)$. Further, assuming r and λ are positive, we can see from equation 7.11 that $r < \lambda$ since $\sinh(A) < 1 + \cosh(A)$ for all A.

Given equations 7.11 and 7.12, we can eliminate $\sinh(f(r))$ to give

$$\frac{\cosh(f(r))-1}{\cosh(f(r))+1}=\frac{r^2}{\lambda^2} \tag{7.15}$$

and hence $\cosh(f(r))=(\lambda^2+r^2)/(\lambda^2-r^2)$. Substituting into either 7.11 or 7.12 gives

$$f(r)=\sinh^{-1}\left(\frac{2\lambda r}{\lambda^2-r^2}\right) \tag{7.16}$$

and hence we can form the following expressions for $\sinh(f(r))$ and $\cosh(f(r))$

$$\sinh(f(r))=\frac{2\lambda r}{\lambda^2-r^2} \quad \text{and} \quad \cosh(f(r))=\frac{2\lambda^2}{\lambda^2-r^2}-1 \tag{7.17}$$

Inserting these into equation 7.10 gives the final form of the non-Euclidean mapping function

$$F_e(x)=\frac{1}{\lambda^2-x^2}(x^2+2\lambda x-\lambda^2\bar{n}) \tag{7.18}$$

We can also show, by substituting the results in equation 7.17, that the form of the translation rotor given in equation 7.3 can also be written as

$$T_x=\frac{1}{\sqrt{\lambda^2-x^2}}(\lambda+\bar{e}x) \tag{7.19}$$

Some discussion of the relevance of λ is worthwhile here. Notice that in order for the translator to remain real-valued, $x^2\leq\lambda^2$. We can never therefore translate the origin outside of a circle radius λ centred upon it. The value of λ effectively defines a region of inaccessible space from the origin, effectively a boundary to the geometry.

This circle corresponds directly to the unit-circle boundary in the Poincaré disc representation if $\lambda=1$ and simple dilations of the Poincaré representation if $\lambda\neq 1$. To maintain compatibility with the Poincaré representation, we usually set λ to be unity.

The rotor in equation 7.19 is, by construction, the rotor which takes us from the origin to position x. A difference from the Euclidean case is that now T_x and T_y do not commute for two different positions x and y. This means that T_{y-x} is *not* the rotor taking us from position x to position y. However, this is not a problem, since we can always achieve this motion via going back through the origin, and forming

$$T_{x\mapsto y}=T_yT_{-x} \tag{7.20}$$

Since composition of rotors always produces another rotor, this means that we have the same freedom as in the Euclidean case to prove a relation we are interested in, at some special position and orientation, and then use the covariant rotor structure to generalise the result to general positions and orientations. Spatial rotations are of course achieved as before with rotors of the form $R = \exp\left(\frac{\theta}{2}e_1e_2\right)$.

The final motion we should consider is the analogue of inversion. Unlike in the Euclidean case, inversion using reflection in e is now a fully covariant operation. Specifically, if R represents any combination of rotation and translation, and A is some object in the space, we have

$$eAe \mapsto Re\tilde{R}RA\tilde{R}Re\tilde{R} = eRA\tilde{R}e = R(eAe)\tilde{R} \qquad (7.21)$$

The last equality means that transforming A first and then reflecting is the same as reflecting and then transforming, which is what is required of a covariant operation. The availability of this reflection operation is very useful. The translation rotors discussed above clearly only allow us to move around within the interior of the disc $r < \lambda$. By reflection in e, as we shall see below, we are able to jump into a 'dual world' outside the disc.

7.2 *The Poincaré Disc*

Having achieved a representation function, and discussed the set of motions, we should examine this new space in relation to the Poincaré disc, which we have said it is equivalent to. The first thing is to examine the distance function, i.e. how we assign a non-Euclidean distance function between points in the space.

If we consider a simple rotation, $\exp\left(\frac{\theta}{2}\hat{B}\right)$, where $\hat{B}$ is some unit spatial bivector, then we are used to the idea that θ is the correct measure of distance (here angular) to describe the transformation. Thus if we consider again the translation rotor in equation 7.3, $T_x = \exp\left(\frac{f(r)}{2}\bar{e}\hat{r}\right)$, we would expect the correct distance measure to associate with it would be $f(r)$. This would be a viable option, except that for points close to the origin of the disc ($r \ll \lambda$) we would like ordinary Euclidean notions of distance to apply, at least approximately. Now $\bar{e} = (1/2)(n - \bar{n})$ and the $\bar{n}$ part of this, when exponentiated and applied as in equation 7.9, has no effect to first order on the origin point $-\bar{n}$, whereas the n part does; e.g.

$$\begin{aligned} T_x(-\bar{n})\tilde{T}_x &\approx \left(1+\frac{f(r)}{2}\frac{(n-\bar{n})\hat{r}}{2}\right)(-\bar{n})\left(1-\frac{f(r)}{2}\frac{(n-\bar{n})\hat{r}}{2}\right) \\ &= \left(1+\frac{f(r)}{2}\frac{n\hat{r}}{2}\right)(-\bar{n})\left(1-\frac{f(r)}{2}\frac{n\hat{r}}{2}\right) \end{aligned} \tag{7.22}$$

This means that, to first order near the origin, the T_x rotor approximates in its actions the Euclidean translation rotor corresponding to distance $f(r)/2$ rather than $f(r)$, i.e.

$$T_x \approx 1+\frac{f(r)}{2}\frac{n\hat{r}}{2} \tag{7.23}$$

For this reason, we take the non-Euclidean distance between a point and the origin to be given by $f(r)/2$ rather than $f(r)$. Calling this non-Euclidean distance function $d(r)$, and using equation 7.17 along with the identity

$$\sinh\left(\frac{z}{2}\right) = \left[\frac{\cosh z - 1}{2}\right]^{\frac{1}{2}},$$

gives

$$d(r) = \sinh^{-1}\left(\frac{r}{\sqrt{\lambda^2 - r^2}}\right) \tag{7.24}$$

We note this approximates to r/λ for $r \ll \lambda$, i.e. we recover the Euclidean distance measured in units of λ.

This function gives us the distance of a point from the origin, but what about the distance between two general points, neither of which is at the origin? One of the major advantages of the conformal approach to Euclidean geometry is that it gives us an inner product formula for computing the distance between any two points, and we would hope that the same would be possible here. This is indeed the case. Let X be the null vector corresponding to the point $x = r\hat{r}$ using our representation, equation (7.10). Then, since $n\cdot\bar{n} = 2$, we can rewrite eqn (7.24) as

$$d(r) = \sinh^{-1}\left(\sqrt{-\frac{1}{2}X\cdot(-\bar{n})}\right) \tag{7.25}$$

Note, $X\cdot(-\bar{n})$ is the inner product of X with the null vector representing the origin. Two points in any general positions can be wound back using a common translation rotor so that one of them ends up at the origin. For example, let T_y be the translation rotor that takes Y back to the origin, then

$$X' = T_y X \tilde{T}_y \qquad Y' = T_y Y \tilde{T}_y = -\bar{n}$$

so that $X' \cdot Y' = (T_y X \tilde{T}_y) \cdot T_y Y \tilde{T}_y = X \cdot Y = X' \cdot (-\bar{n})$. Since we know that a function of $-\frac{1}{2}X' \cdot (-\bar{n})$ gives the distance between the two points from equation (7.25), we can now write this distance in terms of $X \cdot Y$. Note that at no stage in the process is the inner product between the null vectors changed (the inner product is rotor invariant), and thus we have succeeded in defining a distance between general points in terms of inner products. If the general points are x and y with representatives X and Y, the expression for the distance function is thus

$$d(x,y) = \sinh^{-1}\left(\sqrt{-\frac{1}{2}X \cdot Y}\right) \tag{7.26}$$

which is a satisfyingly simple relationship. As in the Euclidean case, it is a monotonic function of $X \cdot Y$. If we take the inner product of X and Y we obtain, using equation 7.10,

$$\begin{aligned} X \cdot Y &= \frac{1}{(\lambda^2 - x^2)(\lambda^2 - y^2)}\left[-2\lambda^2 x^2 - 2\lambda^2 y^2 + 4\lambda^2 xy\right] \\ &= \frac{-2\lambda^2}{(\lambda^2 - x^2)(\lambda^2 - y^2)}(x-y)^2 \end{aligned} \tag{7.27}$$

Written in terms of the points themselves the distance between the points is therefore

$$d(x,y) = \sinh^{-1}\left(\lambda\sqrt{\frac{(x-y)^2}{(\lambda^2 - x^2)(\lambda^2 - y^2)}}\right) \tag{7.28}$$

Armed with this distance function, we can now investigate geodesics in the disc. These are the lines that are ‘straightest’ in the geometry defined by the distance function (more precisely, the arc length along them is extremal.) We will not give the details, but show some numerically computed examples in Fig. 7.2. In calculating these we have taken $\lambda = 1$, so the disc is now the unit disc ordinarily considered in the Poicaré model. One finds that each geodesic is an arc of a circle, and that each of them asymptotically approaches the bounding unit circle at right angles. At this point we can start making contact with the description of the classical approach to the Poincaré disc given in Brannan *et al.* The geodesics there are called *d-lines*. They are not justified in terms of being geodesics, but simply defined as being arcs of circles which cut the boundary at right angles. However, Brannan *et al* do give a distance function, which in terms of distance to the origin is

$$d(x,0) = \tanh^{-1}(|x|) \tag{7.29}$$

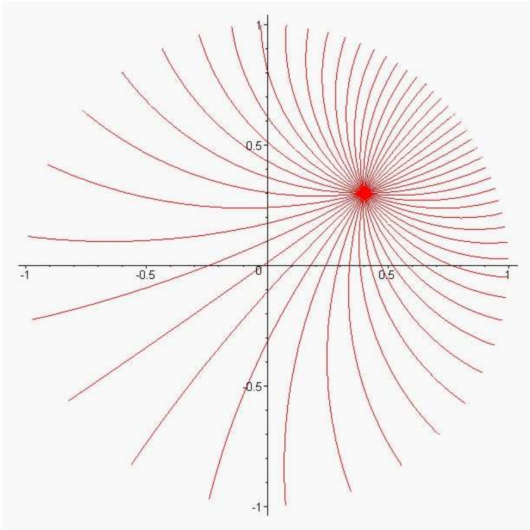

Figure 7.2: *Geodesics emanating from a point in hyperbolic space. They all intersect the unit circle at right angles and each is in fact the arc of a circle. ($\lambda = 1$ has been taken here.)*

and therefore agrees with our identification of $d(r) = f(r)/2$, using equation (7.17), and taking $\lambda = 1$.

Thus far, apart from the formula for non-Euclidean distance in terms of $X \cdot Y$, we have only mirrored what is already known. We now start to show the power of the conformal GA approach to this geometry, by showing how the operations and objects defined previously in conformal Euclidean geometry have immediate analogues here, representing a considerable unification, and saving of effort. It may also be worth noting here that the power of this approach extends immediately to 3 dimensions. All the above formulation, and what we do next, transfers seamlessly to 3d, where the Poincaré disc becomes a ball. All the formulae are correct already for this transition. On the other hand, to provide a computational scheme which is somewhat akin to our rotor formulation, Brannan *et al* introduce complex coordinates to work in the Poincaré disc, and use Mobius transformations in place of the rotors. These are effective computationally in 2d, but of course the complex variable apparatus does not extend at all to 3d (unless one uses the GA approach!). Also of course, the conformal setup enabling points, lines, circles, spheres and planes to be integrated into one algebraic system does not exist in the complex variable approach, even in 2d.

7.3 *New approaches to computation in non-Euclidean geometry*

Having dealt with distances between points, let us start with the next most fundamental objects — lines. In Euclidean space we have seen that these are given by $L = n \wedge A \wedge B$, where A and B are (the representatives of) any two points on the line. Rotor transformations are able to move lines around successfully because for either a rotation or translation, $Rn\tilde{R} = n$. Thus when we perform $RL\tilde{R}$ we end up with $n \wedge (RA\tilde{R}) \wedge (RB\tilde{R})$ which is a line through the transformed points. (Dilations also fit into this, since although they introduce a scale factor when acting on both n and general points, this still produces the intended line up to a scale factor in the Euclidean case.)

This discussion makes it clear what a line must be in the conformal approach to non-Euclidean geometry. Instead of using n, we must use the invariant object e. Thus we define a non-Euclidean line as

$$L = e \wedge A \wedge B \tag{7.30}$$

where A and B are the two points through which we wish it to pass. This construction will guarantee covariance of the definition of a line, for the same reasons as in the Euclidean case (namely here that $Re\tilde{R} = e$ for any allowable rotor R).

The immediate, very important question, of course, is what precisely *is* the object we have constructed. Ideally it should correspond to the d-line geodesics we have just discussed. To determine whether a position X lies on this line, we need to solve $X \wedge L = 0$. Let us take $x = x_1e_1 + x_2e_2$, $a = a_1e_1 + a_2e_2$ and $b = b_1e_1 + b_2e_2$. Then, taking $\lambda = 1$ for convenience, it is easy to show the resulting equation for x is of the form

$$x_1^2 + x_2^2 - 2px_1 - 2qx_2 + 1 = 0 \tag{7.31}$$

where $p^2 + q^2 > 1$. Specifically, one finds

$$p = -\frac{1}{2}\,\frac{\left\{(a_1{}^2 + a_2{}^2 + 1)\,b_2 - (b_1{}^2 + b_2{}^2 + 1)\,a_2\right\}}{a_1\,b_2 - a_2\,b_1}$$
$$q = -\frac{1}{2}\,\frac{\left\{-(a_1{}^2 + a_2{}^2 + 1)\,b_1 + (b_1{}^2 + b_2{}^2 + 1)\,a_1\right\}}{a_1\,b_2 - a_2\,b_1}$$

Equation (7.31) is precisely the form of equation Brannan *et al* give for d-lines (their p.283) and shows that indeed our recipe in terms of wedging with e has worked!

We can combine the notions of lines and distance, by asking for the 'non-Euclidean midpoint' of the line segment joining two positions a and b. This is that point lying on the d-line between a and b which is an equal non-Euclidean distance from each. Brannan *et al* deal with this on p.288, but consider only the easiest case where the two points lie along a diameter of the unit disc.

We know in the Euclidean case, that forming $A+B$ will give a non-null point consisting of a multiple of the null vector representing the desired midpoint, plus a multiple of the point at infinity n. We hypothesise that we will get the same behaviour here, but with e playing the role of n. We can use two different methods to obtain these results. The second is faster than the first, but we outline both, since they both typify the techniques available for use, and serve as good illustrations of the power of the covariant method for non-Euclidean geometry.

Firstly, since we can move objects around at will, let us establish the result in the simplest case, where the two points are symmetrically disposed about the origin, e.g. let $a = \alpha e_1$ and $b = -\alpha e_1$. Clearly, by symmetry the non-Euclidean midpoint must be the origin itself, so we write

$$A+B = \beta(-\bar{n}) + \delta e \tag{7.32}$$

where δ and β are scalar multiples to be determined. Since $A+B = \frac{2}{\lambda^2-\alpha^2}(\alpha^2 n - \lambda^2 \bar{n})$ it is easy to see that we must have

$$\delta = \frac{4\alpha^2}{\lambda^2-\alpha^2} \quad \text{and} \quad \beta = \frac{2(\alpha^2+\lambda^2)}{\lambda^2-\alpha^2} \tag{7.33}$$

Meanwhile, we note that

$$A \cdot B = -\frac{8\alpha^2\lambda^2}{(\lambda^2-\alpha^2)^2} \tag{7.34}$$

and some straightforward manipulation then tells us that

$$\delta = \sqrt{4-2A\cdot B} - 2 \quad \text{and} \quad \beta = \sqrt{4-2A\cdot B} \tag{7.35}$$

We have by this means actually solved the general problem! To see this, note that we can now rearrange equation (7.32) to get

$$-\bar{n} = \frac{1}{\sqrt{1-\frac{1}{2}A\cdot B}} \left\{ \frac{1}{2}(A+B) - e\left(\sqrt{1-\frac{1}{2}A\cdot B} - 1\right) \right\} \tag{7.36}$$

Everything on the r.h.s. is covariant, and the separation between A and B was controlled by a variable which was kept general (α), so by employing translation and rotation rotors on the r.h.s. we can make A and B line up with any two desired points. Meanwhile the l.h.s. will keep track, and must still remain the midpoint. (To see this, note that its 'dot' with the new points that A and B transform into, will remain constant during this process.) The e term just remains invariant. For two completely general points A and B therefore, their non-Euclidean midpoint is given by

$$X_{\text{mid}} = \frac{1}{\sqrt{1-\frac{1}{2}A\cdot B}}\left\{\frac{1}{2}(A+B) - e\left(\sqrt{1-\frac{1}{2}A\cdot B} - 1\right)\right\} \tag{7.37}$$

which is a fully covariant expression.

The alternative method, which is quicker computationally, is as follows. Let us write equation (7.32) again but this time with X_{mid} in place of $-\bar{n}$ and with a general A and B in place from the beginning. So

$$A+B = \beta X_{\text{mid}} + \delta e \tag{7.38}$$

Rearranging, we have

$$X_{\text{mid}} = \frac{1}{\beta}(A+B-\delta e) \tag{7.39}$$

which shows that our assumption about the form of the midpoint is in fact valid. This is because dotting the r.h.s. with A and B in turn, we obtain

$$X_{\text{mid}}\cdot A = X_{\text{mid}}\cdot B = \frac{1}{\beta}(A\cdot B+\delta) \tag{7.40}$$

This means that X_{mid} is indeed equidistant, in a non-Euclidean sense, from A and B. Moreover,

$$X_{\text{mid}} \wedge (e\wedge A\wedge B) = 0 \tag{7.41}$$

so it correctly lies on the d-line joining them.

It just remains to fix δ and β via requiring that X_{mid} is null and is correctly normalised. The null requirement gives

$$2A\cdot B + 4\delta + \delta^2 = 0 \tag{7.42}$$

and requiring $X_{\text{mid}}\cdot e = -1$ yields

$$\beta = \delta + 2 \tag{7.43}$$

Solving these yields (7.35) as before, and we recover (7.37).

7.4 *Non-Euclidean circles*

A great deal of non-Euclidean geometry is concerned with non-Euclidean circles, i.e. the locus of points that are at a constant non-Euclidean distance from a given centre. We can immediately hypothesise that such a circle, passing through the points A, B, D should be given by the trivector.

$$C = A \wedge B \wedge D \tag{7.44}$$

Thus the set of points X which satisfy $X \wedge C = 0$ can be found from the Euclidean case, since we have the same null points being wedged together here, up to scale, and this will not affect the outcome (e.g. in the Euclidean case we know that $X = \frac{1}{2\lambda^2}[x^2 n + 2\lambda x - \lambda^2 \bar{n}]$ and in the non-Euclidean case $X = \frac{1}{\lambda^2 - x^2}[x^2 n + 2\lambda x - \lambda^2 \bar{n}]$). If the above hypothesis is true, we can immediately deduce the somewhat surprising, though in fact true, conclusion that non-Euclidean circles are also Euclidean circles. It turns out that it is just their centres that are in general different.

To establish that C is a non-Euclidean circle (as well as clearly a Euclidean one) we can take the special case of a circle centred at the origin. By symmetry, this must be both a Euclidean and non-Euclidean circle. Let this have Euclidean radius ρ. If we recall our earlier work, then for the scaled version of the null point representative, the inner product between two points A and B is given by

$$A \cdot B = -\frac{1}{2\lambda^2}(a-b)^2 \tag{7.45}$$

Then, since $X \cdot B = -\frac{1}{2\lambda^2}(x-b)^2 = -\frac{1}{2\lambda^2}(\rho)^2$ for a circle centre B and radius ρ, we have that

$$X \cdot (B - \frac{1}{2\lambda^2}(\rho)^2 n) = 0 \tag{7.46}$$

giving $C^* = B - \frac{1}{2\lambda^2}(\rho)^2 n$, where C is the trivector describing the circle. Thus if B is the origin, so that $B = -\frac{1}{2}\bar{n}$, the dual of C is given by $C^* = -\frac{1}{2\lambda^2}\{\rho^2 n + \lambda^2 \bar{n}\}$. Now, since $n + \bar{n} = 2e$ we can write $\rho^2 n + \lambda^2 \bar{n}$ as

$$\rho^2 n + \lambda^2 \bar{n} = 2\rho^2 e + (\lambda^2 - \rho^2)\bar{n} \tag{7.47}$$

We can normalise this to $C^2 = (C^*)^2 = 1$ via the scaling

$$C^* \mapsto \alpha(\rho^2 n + \lambda^2 \bar{n}) \quad \text{such that} \quad (C^*)^2 = 1$$

and hence $\alpha = \frac{1}{2\rho\lambda}$, thus

$$C^* = \frac{1}{2\rho\lambda}\left(\rho^2 n + \lambda^2 \bar{n}\right) = \frac{1}{2\rho\lambda}[2\rho^2 e + (\lambda^2 - \rho^2)\bar{n}]. \tag{7.48}$$

This displays the dual to C as a linear combination of the vector representing the origin, which is here the centre of the circle, and a multiple of e. This is the covariant form we require for generalisation. Note

$$X \cdot (IC) = \frac{1}{2\rho\lambda}\left\{X \cdot (2\rho^2 e + (\lambda^2 - \rho^2)\bar{n})\right\} = \frac{1}{2\rho\lambda}\left\{-2\rho^2 - (\lambda^2 - \rho^2)X \cdot (-\bar{n})\right\} \tag{7.49}$$

shows us how X maintaining a constant non-Euclidean distance from the centre $(-\bar{n})$ is guaranteed by $X \cdot (IC) = 0$, in which case we have

$$X \cdot (-\bar{n}) = \frac{-2\rho^2}{(\lambda^2 - \rho^2)} \tag{7.50}$$

Let us act on $IC = I(A \wedge B \wedge D)$ with non-Euclidean rotors, to move the 3 points around as we wish. These same rotors acting on the right hand side of (7.47) mean that we will continue to get the required behaviour of constant non-Euclidean distance from the transformed centre, since the rotors will take the origin to the new centre of the circle, say P. Thus indeed, the wedge of 3 points generates a non-Euclidean circle, and moreover the above enables us to extract its (non-Euclidean) radius and centre from the dual object. Again we see that the rôle of the n appearing in the Euclidean expressions is replaced by e here. Specifically we find the following: the (normalised dual to the) non-Euclidean circle with non-Euclidean centre P, Euclidean radius ρ and non-Euclidean radius d, is given by

$$IC = \frac{1}{2\rho\lambda}\left(2\rho^2 e + (\rho^2 - \lambda^2)P\right) \tag{7.51}$$

with $d = \sinh^{-1}(\rho/\sqrt{\lambda^2 - \rho^2})$, from equation 7.26.

7.5 Non-Euclidean reflection

A very useful operation in non-Euclidean geometry is the notion of non-Euclidean reflection, as defined for example in [1]. It is extremely easy to calculate the non-Euclidean reflection of a point X in the d-line L in the GA approach. Assuming L is normalised to satisfy $L^2 = 1$, we just form LXL, the standard form of reflecting one object in another. Since $LeL = e$ for any (normalised) line, we see that $X' = LXL$ is both null and satisfies $X' \cdot e = -1$ thus qualifying it to represent a point. Moreover it is co-variantly constructed; under a rotor transformation it just rotates to $RX'\tilde{R}$ and thus must represent something physical. It is not hard to show that the point it represents is that found by moving along a d-line intersecting L at right angles and passing through X, by an equal non-Euclidean distance on the other side of the line as X is on the original side. This is indeed the definition of reflection in this case.

7.6 *Extension to higher dimensions and other geometries*

All the above transfers seamlessly to 3 and higher dimensions, in most cases with no changes at all to the formulae. This is in contrast to e.g. the methods introduced in Brannan et al, which rely on standard complex analysis and so only work in the 2d plane. Furthermore, we can extend all the above analysis to the case of *spherical geometry* as well. This involves replacing the role of e with that of $\bar{e}$, which changes some trigonometric functions and ranges of applicability, but otherwise most of the above discussion goes through unchanged.

> *"As soon as we started programming, we found to our surprise that it wasn't as easy to get programs right as we had thought. Debugging had to be discovered. I can remember the exact instant when I realized that a large part of my life from then on was going to be spent in finding mistakes in my own programs."*
>
> — Maurice Wilkes

8 Software Based Implementation and Visualisation

So far we have seen that the 5d conformal geometric algebra enables us to define objects (lines, circles, planes and spheres) as blades and then manipulate these blades as elements of the algebra to perform intersections, reflections and transformations. Once we have these objects defined, we can rotate, translate, dilate and invert these objects with remarkable ease. The question now is how do we efficiently implement this in software and how does the computational load compare with conventional linear algebra approaches to these problems. At the outset, we note that in our 5d algebra each object can be regarded as a sparse length 32 vector (there are 32 elements in the 5d algebra: one scalar, 5 vectors, 10 bivectors, 10 trivectors, 5 4-vectors and one pseudoscalar), which might seem to indicate that there will be a lot of redundant computation. However, we also note that in our formula for intersections, reflections etc, we do not have to solve any sets of equations, problems which are quadratic in nature are solved in a linear fashion. Moreover, much of the forming of conditionals or consideration of special cases has been removed; the covariant nature of the approach also means that we are be able to extend our code easily to deal with the non-Euclidean geometries discussed in earlier chapters.

8.1 Implementation

Although the CGA appears notationally far more efficient, how efficient does it become when actually implemented on current hardware? A software implementation was written in the C programming language and used the OpenGL visualisation library for output. Note that the OpenGL library is based in classical linear algebra and so various algorithms were used to extract information about objects for the purposes of visualisation. These algorithms will be briefly discussed later. The oper-

$$A^G = \begin{bmatrix} +A_0 & +A_1 & +A_2 & +A_3 & -A_{12} & -A_{13} & -A_{23} & -A_{123} \\ +A_1 & +A_0 & +A_{12} & +A_{13} & -A_2 & -A_3 & -A_{123} & -A_{23} \\ +A_2 & -A_{12} & +A_0 & +A_{23} & +A_1 & +A_{123} & -A_3 & +A_{13} \\ +A_3 & -A_{13} & -A_{23} & +A_0 & -A_{123} & +A_1 & +A_2 & -A_{12} \\ +A_{12} & -A_2 & +A_1 & +A_{123} & +A_0 & +A_{23} & -A_{13} & +A_3 \\ +A_{13} & -A_3 & -A_{123} & +A_1 & -A_{23} & +A_0 & +A_{12} & -A_2 \\ +A_{23} & +A_{123} & -A_3 & +A_2 & +A_{13} & -A_{12} & +A_0 & +A_1 \\ +A_{123} & +A_{23} & -A_{13} & +A_{12} & +A_3 & -A_2 & +A_1 & +A_0 \end{bmatrix}$$

Figure 8.1: *Example product matrix for the geometric product in* $\mathscr{A}(3,0)$. $A_{ij...k}$ *is the element of* A *proportional to* $e_i e_j ... e_k$.

ations and computational load required to find this information are included in the presented results since it is anticipated that any real system would also have to perform these operations and so their inclusion gave a more representative benchmark.

The software library was designed using an object-orientated methodology where multivectors are represented to the programmer as an opaque data-type and all access to the multivector is through 'method functions'. Figure 8.2 shows a typical program where the intersection of two spheres is found and the resulting circle plotted.

Any multivector M in $\mathscr{A}(p,q)$ can be formed from a linear combination of all possible basis vector products up to grade $p+q$. For $\mathscr{A}(4,1)$, the highest grade object is $e_1e_2e_3e\bar{e}$ so

$$M = a_1 + a_2e_1 + a_3e_2 + ... + a_7e_1e_2 + ... + a_{32}e_1e_2e_3e\bar{e} \quad (8.1)$$

Initially the software library stored a multivector as the vector $[a_1, a_2, ...a_{32}]'$ implemented as an array in C as used in many other implementations such as the CLU project [16] by Christian Perwass and the Gaigen project [7] by Fontijne *et al.*. Later optimisations added the ability to keep track of which grades were present in the multivector but discussion of this will be left to the optimisation section. Such a change in underlying implementation without change to the application programmers' interface (API) was only possible due to the object-orientated nature of the library.

```
multiv_t *a, *b, *c;
/* Create multivectors */
cga_new_multiv(a); cga_new_multiv(b); cga_new_multiv(c);
/* Form two spheres */
cga_make_sphere(0,1,0, 5, a); cga_make_sphere(1,1,0, 3, a);
/* Find intersection */
cga_meet(a,b, c);
/* Plot result */
cga_draw_circle_outer(c);
/* Tidy up */
cga_free_multiv(a); cga_free_multiv(b); cga_free_multiv(c);
```

Figure 8.2: *Finding the intersection of two spheres.*

8.1.1 Optimisations

Many existing implementations such as CLU and Gaigen represent the geometric product AB as a linear map, dependent on A, applied to the components of B. The matrix generally used to represent the mapping, A^G, depends only on the components of A. An example matrix for the geometric product in $\mathscr{A}(3,0)$ generated by Gaigen is shown in Figure 8.1. The vector-representation of the product, C, is found in the following manner,

$$\mathbf{C} = A^G \mathbf{B} \tag{8.2}$$

Initial experiments were performed to investigate the computational load of CGA with reference to classical linear algebra. In a typical case, extracting the centre, radius and normal vector of a circle passing through three points was found to require 109 floating-point operations (flops) using linear algebra and 3519 flops using the CGA approach (measured using a naive Matlab implementation and its built-in `flops()` function).

Despite the intuitive nature of the CGA approach (for example finding intersections is dramatically easier when you have many different classes of object) it was almost 30 times slower than the classical approach.

It was found that a significant reduction in the operation count could be obtained by writing specialised product routines. For example the full geometric product calculation for $\mathscr{A}(4,1)$, with its 32 orthonormal basis-elements, requires a 32×32 matrix multiplication which requires 1024 floating-point multiplies. When you calculate the result of a bivector–bivector product, however, you need only perform 100 multiplies.

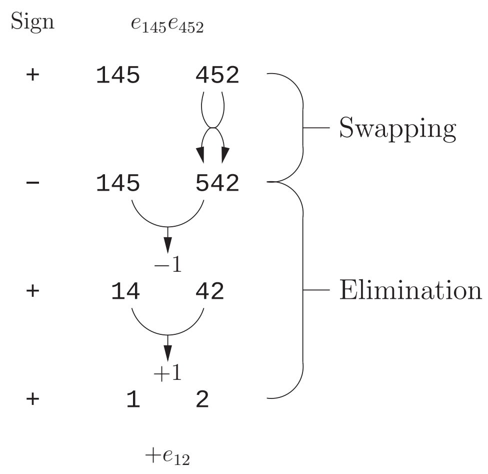

Figure 8.3: *Example of finding that* $e_{145}e_{452} = e_{12}$ *with* $e_5^2 = -1$, $e_4^2 = 1$.

The problem with this approach was that by having to use specialised routines, the generality CGA provides was lost. A solution that was found was to keep track of which grades are present in the multivector.

8.1.2 *Implementation details*

Product tables were generated with a Perl[1] script. n-vector components were represented by a string of digits. For example, the trivector $e_1e_4e_5$ was represented as '`145`'. The resulting geometric product was calculated using the algorithm represented in figure 8.3. The algorithm can be outlined as followed:

1. Sort first component numerically by exchanging neighbours and alternate sign once for each swap.
2. Reverse sort second component numerically by exchanging neighbours and alternate sign once for each swap.
3. If the last digit of the first component and first digit of the second component match, change sign as appropriate to the square of the components and remove.
4. If there are more pairs to match, goto step 3.
5. Concatenate components and output.

[1]Practical Extraction and Report Language—see http://www.perl.com/

Other product tables were computed from the geometric product. For example $a \cdot b = 0.5(ab + ba)$.

These product tables were then used to generate optimal n-vector, m-vector product routines in C which operated directly on the multivector components.

8.2 *Grade tracking*

The representation was changed so that a general multivector M was represented as a sum of single-grade objects

$$M = \langle M \rangle_0 + \langle M \rangle_1 + \ldots + \langle M \rangle_n \tag{8.3}$$

and so the product of multivectors A and B is

$$AB = \langle A \rangle_0 \langle B \rangle_0 + \langle A \rangle_0 \langle B \rangle_1 + \ldots \tag{8.4}$$

$$+ \langle A \rangle_1 \langle B \rangle_0 + \langle A \rangle_1 \langle B \rangle_1 + \ldots \tag{8.5}$$

$$+ \langle A \rangle_n \langle B \rangle_{n-1} + \langle A \rangle_n \langle B \rangle_n \tag{8.6}$$

If G_A is the set of grades present in A and G_B is the set of grades present in B then

$$\langle A \rangle_i = 0 \quad \text{if } i \notin G_A \tag{8.7}$$

$$\langle B \rangle_i = 0 \quad \text{if } i \notin G_B \tag{8.8}$$

hence

$$\langle A \rangle_i \langle B \rangle_j = 0 \quad \text{if } i \notin G_A \text{ or } j \notin G_B \tag{8.9}$$

and need not be computed. If G_A and G_B are sufficiently small with respect to the dimension of the space, then significant advantage may be obtained. Since we always know a bivector–scalar product will return another bivector, the set of grades produced by the product may be computed directly from G_A and G_B. Hence, if the set G_M is kept with the representation of M we can always know which single-grade products will always return zero and thus need not be computed.

Figure 8.4 shows this algorithm in block diagram form. Multivectors are broken down into single-grade components and those components which are zero are discarded. Each pair of components from each input multivector is passed to a specialised product routine and the results from each routine are concatenated. Since

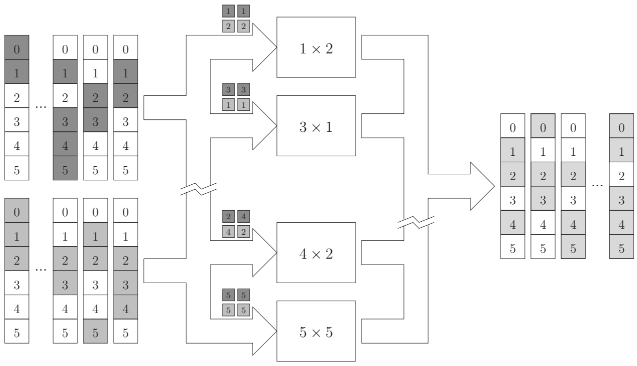

Figure 8.4: *Block diagram showing how the product function may be parallelised.*

we discard some components prior to finding the products, some of the specialised routines will have no inputs and hence need not be calculated.

Such a diagram also suggests a possible hardware implementation. Each single-grade product function can be replaced by an independent hardware module and then the time taken to form a product of two multivectors is only limited by the slowest optimised product block used.

With efficient queueing, stalling and re-ordering algorithms (such as those found in modern processor pipelines) it may be possible to increase the speed of computation for lists of multivectors. Figure 8.4 can now be interpreted as a hardware block diagram. Such a device could be used in a SIMD (Single Instruction, Multiple Data) manner to find the products of many multivector pairs using a single machine instruction.

8.3 Visualisation

8.3.1 3D Euclidean Space

In order to visualise the operations performed using CGA, a method was required to find appropriate parameters of spheres, lines and planes in order to plot them using OpenGL. Finding the Euclidean representation of a point is trivial (simply apply the inverse-mapping).

The visualisation of a line was performed by intersecting (via the meet operator) the line with some suitably large sphere such that if the line did not intersect the sphere, it would probably not intersect the viewing volume. The intersection was of

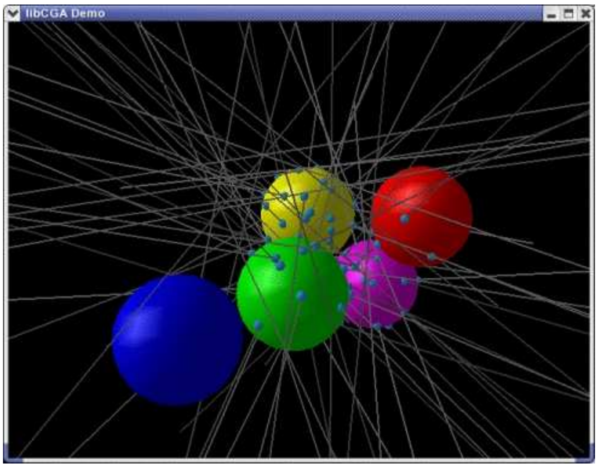

Figure 8.5: *Sample output from a program which intersects elements of an array containing lines and spheres.*

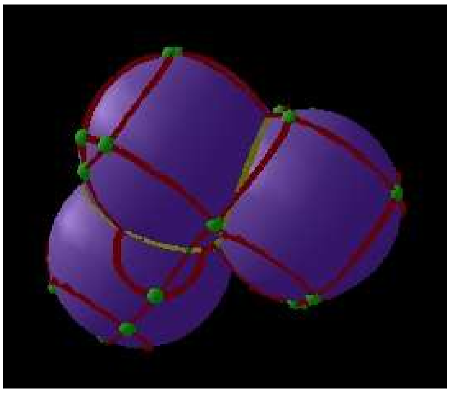

Figure 8.6: *The same program as in figure 8.5 but with the array filled with planes and spheres. The colours used to plot the objects were changed to increase contrast.*

the form $A \wedge B$ where A and B were the null-vector representations of the points of intersection of the line and sphere. Once found, a line could be drawn through these points.

To extract A and B the method of projectors, introduced earlier in this report, may be used.

Similarly the methods and algorithms above may be used to find centres and radii of spheres and circles for visualisation. These methods are already used in alternative implementations such as the CLU project.

The use of the meet operator makes finding intersections simple and hence im-

Sphere–line intersection	
Without tracking	42,037 flops/intersection
With tracking	3,269 flops/intersection
	∼ 13 times faster
Sphere–plane intersection	
Without tracking	175,901 flops/intersection
With tracking	10,709 flops/intersection
	∼ 16 times faster
Facet–line intersection & reflection	
Without tracking	4,253 flops/facet
With tracking	181 flops/facet
	∼ 23 times faster

Table 8.1: *Speedup obtained when using Grade Tracking.*

plementing complex algorithms becomes easier. For example, a program was written which filled an array of multivector objects with a collection of spheres and lines. The resulting array was iterated over, intersecting every element with each other and plotting the result. The result is shown in figure 8.5. The advantage of this approach was that, while computing the intersections, we did not have to worry about the nature of the objects. Indeed, a one-line change to the program which filled the array with a set of planes and spheres instead gave rise to figure 8.6. It is this generality of application which we believe makes this approach powerful.

Finally, an implementation of the facet-ray intersection algorithm introduced above was performed and the number of operations per facet to perform the intersection and reflection were found. Sample output from this program is shown in figure 8.7.

Measurements of the operation count for each of the example programs were made with and without the use of the grade tracking optimisation. The differences are shown in table 8.1.

Although there is a significant improvement, the algorithm is still high in operation count when compared to typical linear algebra-based algorithms. The CGA approach does, however, have a significant degree of generality when one considers what can be intersected and in what dimensions. The extension of standard intersection routines to higher dimensions is not always a trivial matter.

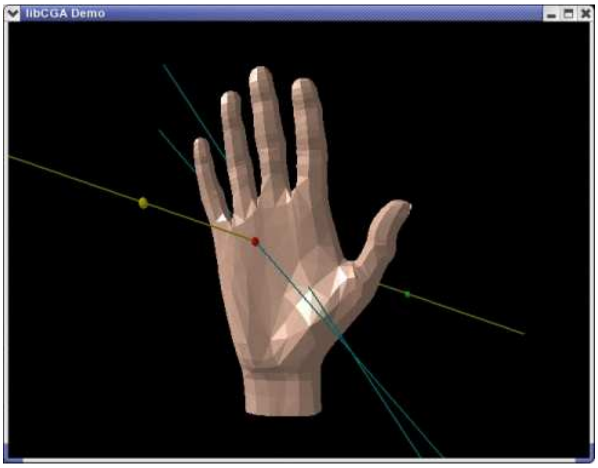

Figure 8.7: *A complex example program showing facet-ray intersection performed entirely within the conformal model.*

8.3.2 2D Hyperbolic Space

A similar technique was used to plot lines in hyperbolic space. The term *d-line* is used [1] when describing lines in hyperbolic space. They also show that *d*-lines, when mapped to the Poincaré disc, are circular arcs which erupt at right-angles from the unit circle. We can approximate the boundary points (the points where they intersect the unit disc) by forming a circle corresponding to the unit-circle and finding the meet of the line with this circle. This gives the null-vector representation of the boundary points A and B as the bivector $A \wedge B$ as before.

It would be advantageous to find a NURBS (Non-Uniform Rational B-Spline) based representation for the *d*-line since NURBS curve rendering is a standard part of the OpenGL specification and could conceivably be hardware-assisted. Figure 8.8 shows the control points used, they are the origin O and the boundary points themselves A and B.

It is a property of NURBS that the curve will be tangential to OA at A and OB at B. Since OA and OB are radii of the unit-circle, they clearly intersect it at right-angles and hence the curve also intersects at right-angles.

It remains to choose the control point weights so that the curve is actually a circular arc. It has been shown [2] that a suitable weight for O is $\cos\gamma$ and 1 for A and B where γ is the angle either of the radii make with the chord AB.

[2] See http://www.ddt.pwp.blueyonder.co.uk/evgeny/Intro/NURBS.htm for a demonstration of this.

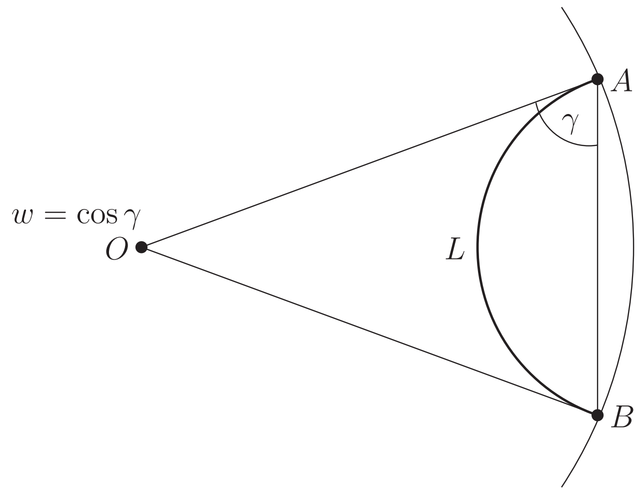

Figure 8.8: *NURBS-based rendering of d-lines. Here O is the origin and A and B are the boundary points of the line L.*

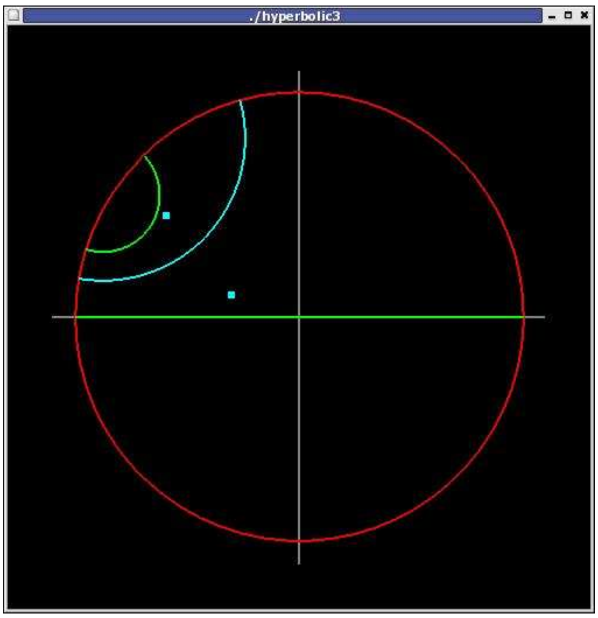

Figure 8.9: *Example output from the code when using the Poincaré disc visualisation. Here L_0 is the perpendicular bisector of A and B and L_1' is the reflection in L_0 of the x-axis L_1.*

Figure 8.9 shows typical output from a sample program. Here the positions of A and B are set with the mouse. The line $L_0 = (A-B)^*$ is computed (which is the perpendicular bisector of A and B) and finally the x-axis (represented by L_1) is reflected in L_0 to form $L'_1 = L_0 L_1 L_0$. Note the ease with which these operations are performed and that they also work with the Euclidean mapping.

Bibliography

[1] Brannan, D. A., Esplen, M. F., and Gray, J. J. *Geometry*. Cambridge University Press, 1999. Ch. 6.

[2] Doran, C., and Lasenby, A. Physical Applications of Geometric Algebra, 2002. Available at http://www.mrao.cam.ac.uk/~clifford/ptIIIcourse/.

[3] Doran, C., and Lasenby, A. *Geometric Algebra for Physicists*. Cambridge University Press, 2003.

[4] Dorst, L., Mann, S., and Bouma, T. GABLE: a Geoemtric AlgeBra Learning Environment, 2001. Available for download at http://www.science.uva.nl/~leo/GABLE/.

[5] Dress, A., and Havel, T. Distance geometry and geometric algebra. *Foundations of Physics 23*, 10 (Oct. 1991), 1357–1374.

[6] Dunham, D. Transformation of Hyperbolic Escher Patterns. *Visual Mathematics 1*, 1 (1999).

[7] Fontijne, S., Dorst, L., and Bouma, T. GAIGEN: a Geoemtric Algebra GENerator, 2001. Available for download at http://carol.wins.uva.nl/~fontijne/gaigen/about.html.

[8] Hestenes, D. Old wine in new bottles: a new algebraic framework for computational geometry. In *Geometric Algebra with Applications in Science and Engineering* (Boston, 2001), E. Bayro-Corrochano and G. Sobcyzk, Eds., Birkhauser, pp. 1–16.

[9] Hestenes, D., and Sobczyk, G. *Clifford Algebra to Geometric Calculus: A unified language for mathematics and physics*. D. Reidel, Dordrecht, 1984.

[10] HESTENES, D., AND ZIEGLER, R. Projective geometry with clifford algebra. *Acta Applicandæ Mathematicæ 23* (1991), 25–63.

[11] LASENBY, A., AND LASENBY, J. Surface Evolution and Representation using Geometric Algebra. In *The Mathematics of Surfaces IX: Proceedings of the ninth IMA conference on the mathematics of surfaces* (London, 2000), R. Cipolla and R. Martin, Eds., Springer, pp. 144–168.

[12] LASENBY, J., AND BAYRO-CORROCHANO, E. Analysis and Computation of Projective Invariants from Multiple Views in the Geometric Algebra Framework. In *Invariants for Pattern Recognition and Classification* (2000), M. Rodrigues, Ed., vol. 42, World Scientific. Series in Machine Perception and Artificial Intelligence, 233pp, ISBN 981-02-4278-6.

[13] LASENBY, J., FITZGERALD, W., LASENBY, A., AND DORAN, C. New geometric methods for computer vision: An application to structure and motion estimation. *International Journal of Computer Vision 26*, 3 (1998), 191–213.

[14] LI, H. Hyperbolic Geometry. In *Geometric Algebra with Applications in Science and Engineering* (Boston, 2001), E. Bayro-Corrochano and G. Sobcyzk, Eds., Birkhauser, pp. 61–85.

[15] LI, H., HESTENES, D., AND ROCKWOOD, A. Generalized homogeneous coordinates for computational geometry. In *Geometric Computing with Clifford Algebra* (2001), G. Sommer, Ed., Springer, pp. 25—58.

[16] PERWASS, C. The CLU Project. Available for download at `http://www.perwass.de/cbup/clu.html`.

[17] ROCKWOOD, A., HESTENES, D., DORAN, C., LASENBY, J., DORST, L., AND MANN, S. Geometric Algebra. Course Notes. Course 31, Siggraph 2001, Los Angeles.