1. New Algebraic Tools for Classical Geometry†

DAVID HESTENES, HONGBO LI
Department of Physics and Astronomy
Arizona State University
Tempe, AZ 85287-1504, USA

ALYN ROCKWOOD
Power Take Off Software, Inc.
18375 Highland Estates Dr.
Colorado Springs, CO 80908, USA

1.1 Introduction

Classical geometry has emerged from efforts to codify perception of space and motion. With roots in ancient times, the great flowering of classical geometry was in the 19th century, when Euclidean, non-Euclidean and projective geometries were given precise mathematical formulations and the rich properties of geometric objects were explored. Though fundamental ideas of classical geometry are permanently imbedded and broadly applied in mathematics and physics, the subject itself has practically disappeared from the modern mathematics curriculum. Many of its results have been consigned to the seldom-visited museum of mathematics history, in part, because they are expressed in splintered and arcane language. To make them readily accessible and useful, they need to be reexamined and integrated into a coherent mathematical system.

Classical geometry has been making a comeback recently because it is useful in such fields as Computer-Aided Geometric Design (CAGD), CAD/CAM, computer graphics, computer vision and robotics. In all these fields there is a premium on computational efficiency in designing and manipulating geometric objects. Our purpose here is to introduce powerful new mathematical tools for meeting that objective and developing new insights within a unified algebraic framework. In this and subsequent chapters we show how classical geometry fits neatly into the broader mathematical system of *Geometric Algebra* (GA) and its extension to a complete *Geometric Calculus* (GC) that includes differential forms and much more.

Over the last four decades GC has been developed as a universal geometric language for mathematics and physics. This can be regarded as the culmination of an R & D program innaugurated by Hermann Grassmann in 1844 [G44, H96]. Literature on the evolution of GC with extensive applications to math and physics can be accessed from the *GC web site*

<http://ModelingNTS.la.asu.edu/GC_R&D.html>.

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Here, we draw on this rich source of concepts, tools and methods to enrich classical geometry by integrating it more fully into the whole system.

This chapter provides a synopsis of basic tools in Geometric Algebra to set the stage for further elaboration and applications in subsequent chapters. To make the synopsis compact, proofs are omitted. Geometric interpretation is emphasized, as it is essential for practical applications.

In classical geometry the primitive elements are points, and *geometric objects* are point sets with properties. The properties are of two main types: structural and transformational. Objects are characterized by structural relations and compared by transformations. In his Erlanger program, Felix Klein [K08] classified geometries by the transformations used to compare objects (for example, similarities, projectivities, affinities, etc). Geometric Algebra provides a unified algebraic framework for both kinds of properties and any kind of geometry.

1.2 Geometric Algebra of a Vector Space

The terms "vector space" and "linear space" are ordinarily regarded as synonymous. While retaining the usual concept of linear space, we enrich the concept of vector space by defining a special product among vectors that characterizes their relative directions and magnitudes. The resulting geometric algebra suffices for all the purposes of linear and multilinear algebra. We refer the reader to the extensive treatment of geometric algebra in [HS84] and to the GC Web site, so that we can give a more concise treatment here.

Basic Definitions

As a rule, we use lower case letters to denote vectors, lower case Greek letters to denote scalars and calligraphic capital letters to denote sets.

First, we define geometric algebra. Let $\mathcal{V}^n$ be an n-dimensional vector space over real numbers $\mathcal{R}$. The *geometric algebra* $\mathcal{G}_n = \mathcal{G}(\mathcal{V}^n)$ is generated from $\mathcal{V}^n$ by defining the *geometric product* as a multilinear, associative product satisfying the *contraction rule*:

$$a^2 = \epsilon_a |a|^2, \text{ for } a \in \mathcal{V}^n, \tag{1.1}$$

where ϵ_a is $1, 0$ or -1, $|a| \geq 0$, and $|a| = 0$ if $a = 0$. The integer ϵ_a is called the *signature* of a; the scalar $|a|$ is its *magnitude*. When $a \neq 0$ but $|a| = 0$, a is said to be a *null vector*.

In the above definition, "multi-linearity" means

$$a_1 \cdots (b_1 + \cdots + b_r) \cdots a_s = (a_1 \cdots b_1 \cdots a_s) + \cdots + (a_1 \cdots b_r \cdots a_s), \tag{1.2}$$

for any vectors $a_1, \cdots, a_s, b_1, \cdots, b_r$ and any position of $b_1 + \cdots + b_r$ in the geometric product. Associativity means

$$a(bc) = (ab)c, \text{ for } a, b, c \in \mathcal{V}^n. \tag{1.3}$$

An element M in $\mathcal{G}_n$ is *invertible* if there exists an element N in $\mathcal{G}_n$ such that $MN = NM = 1$. The element N, if it exists, is unique. It is called the *inverse* of M, and is denoted by M^{-1}. For example, null vectors in $\mathcal{V}^n$ are not invertible, but any non-null vector a is invertible, with $a^{-1} = 1/a = a/a^2$. This capability of GA for division by vectors greatly facilitates computations.

From the geometric product, two new kinds of product can be defined. For $a, b \in \mathcal{V}^n$, the scalar-valued quantity

$$a \cdot b = \frac{1}{2}(ab + ba) = b \cdot a \tag{1.4}$$

is called the *inner product*; the (nonscalar) quantity

$$a \wedge b = \frac{1}{2}(ab - ba) = -b \wedge a \tag{1.5}$$

is called the *outer product*. Therefore, the geometric product

$$ab = a \cdot b + a \wedge b \tag{1.6}$$

decomposes into symmetric and anti-symmetric parts.

The outer product of r vectors can be defined as the anti-symmetric part of the geometric product of the r vectors. It is called an *r-blade*, or a blade of *grade* r. A linear combination of r-blades is called an *r-vector*. The set of r-vectors is an $\binom{n}{r}$-dimensional subspace of $\mathcal{G}_n$, denoted by $\mathcal{G}_n^r$. The whole of $\mathcal{G}_n$ is given by the subspace sum

$$\mathcal{G}_n = \sum_{i=0}^{n} \mathcal{G}_n^i. \tag{1.7}$$

A generic element in $\mathcal{G}_n$ is called a *multivector*. According to (1.7), every multivector M can be written in the expanded form

$$M = \sum_{i=0}^{n} \langle M \rangle_i \,, \tag{1.8}$$

where $\langle M \rangle_i$ denotes the i-vector part. The dimension of $\mathcal{G}_n$ is $\sum_{i=0}^{n} \binom{n}{i} = 2^n$.

By associativity and multi-linearity, the outer product extended to any finite number of multivectors and to scalars, with the special proviso

$$\lambda \wedge M = M \wedge \lambda = \lambda M, \text{ for } \lambda \in \mathcal{R}, M \in \mathcal{G}_n. \tag{1.9}$$

The inner product of an r-blade $a_1 \wedge \cdots \wedge a_r$ with an s-blade $b_1 \wedge \cdots \wedge b_s$ can be defined recursively by

$$\begin{aligned}&(a_1 \wedge \cdots \wedge a_r) \cdot (b_1 \wedge \cdots \wedge b_s) \\ &\quad = \begin{cases} ((a_1 \wedge \cdots \wedge a_r) \cdot b_1) \cdot (b_2 \wedge \cdots \wedge b_s) & \text{if} \quad r \geq s \\ (a_1 \wedge \cdots \wedge a_{r-1}) \cdot (a_r \cdot (b_1 \wedge \cdots \wedge b_s)) & \text{if} \quad r < s \end{cases},\end{aligned} \tag{1.10a}$$

and

$$\begin{aligned}&(a_1 \wedge \cdots \wedge a_r) \cdot b_1 \\ &\quad = \sum_{i=1}^{r} (-1)^{r-i} a_1 \wedge \cdots \wedge a_{i-1} \wedge (a_i \cdot b_1) \wedge a_{i+1} \wedge \cdots \wedge a_r, \\ &a_r \cdot (b_1 \wedge \cdots \wedge b_s) \\ &\quad = \sum_{i=1}^{s} (-1)^{i-1} b_1 \wedge \cdots \wedge b_{i-1} \wedge (a_r \cdot b_i) \wedge b_{i+1} \wedge \cdots \wedge b_s.\end{aligned} \tag{1.10b}$$

By bilinearity, the inner product is extended to any two multivectors, if

$$\lambda \cdot M = M \cdot \lambda = 0, \text{ for } \lambda \in \mathcal{R},\ M \in \mathcal{G}_n. \tag{1.11}$$

For any blades A and B with nonzero grades r and s we have

$$A \cdot B = \langle AB \rangle_{|r-s|}, \tag{1.12}$$
$$A \wedge B = \langle AB \rangle_{r+s}. \tag{1.13}$$

These relations can be adopted as alternative definitions of inner and outer products or derived as theorems.

An *automorphism* f of $\mathcal{G}_n$ is an invertible linear mapping that preserves the geometric product:

$$f(M_1M_2) = f(M_1)f(M_2), \text{ for } M_1, M_2 \in \mathcal{G}_n. \tag{1.14}$$

More generally, this defines an *isomorphism* f from one geometric algebra to another.

An *anti-automorphism* g is a linear mapping that reverses the order of geometric products:

$$g(M_1M_2) = g(M_2)g(M_1), \text{ for } M_1, M_2 \in \mathcal{G}_n. \tag{1.15}$$

The *main anti-automorphism* of $\mathcal{G}_n$, also called *reversion*, is denoted by "$\dagger$", and defined by

$$\langle M^\dagger \rangle_i = (-1)^{\frac{i(i-1)}{2}} \langle M \rangle_i, \text{ for } M \in \mathcal{G}_n,\ 0 \le i \le n. \tag{1.16}$$

An *involution* h is an invertible linear mapping whose composition with itself is the identity map:

$$h(h(M)) = M, \text{ for } M \in \mathcal{G}_n. \tag{1.17}$$

The *main involution* of $\mathcal{G}_n$, also called *grade involution* or *parity conjugation*, is denoted by "$*$", and defined by

$$\langle M^* \rangle_i = (-1)^i \langle M \rangle_i, \text{ for } M \in \mathcal{G}_n,\ 0 \le i \le n. \tag{1.18}$$

For example, for vectors $a_1, \cdots, a_r$, we have

$$(a_1 \cdots a_r)^\dagger = a_r \cdots a_1,\ \ (a_1 \cdots a_r)^* = (-1)^r a_1 \cdots a_r. \tag{1.19}$$

A multivector M is said to be *even*, or have *even parity*, if $M^* = M$; it is *odd*, or has *odd parity*, if $M^* = -M$.

The concept of magnitude is extended from vectors to any multivector M by

$$|M| = \sqrt{\sum_{i=0}^{n} |\langle M \rangle_i|^2}, \tag{1.20a}$$

where

$$|\langle M\rangle_i| = \sqrt{|\langle M\rangle_i \cdot \langle M\rangle_i|}. \tag{1.20b}$$

A natural *scalar product* on the whole of $\mathcal{G}_n$ is defined by

$$\langle MN^\dagger\rangle = \langle NM^\dagger\rangle = \sum_{i=0}^{n}\langle\langle M\rangle_i\langle N\rangle_i^\dagger\rangle, \tag{1.21}$$

where $\langle\cdots\rangle = \langle\cdots\rangle_0$ denotes the "scalar part." When scalar parts are used frequently it is convenient to drop the subscript zero.

In $\mathcal{G}_n$, the maximal grade of a blade is n, and any blade of grade n is called a *pseudoscalar*. The space $\mathcal{V}^n$ is said to be *non-degenerate* if it has a pseudoscalar with nonzero magnitude. In that case the notion of duality can be defined algebraically. Let I_n be a pseudoscalar with magnitude 1, designated as the *unit pseudoscalar*. The *dual* of a multivector M in $\mathcal{G}_n$ is then defined by

$$\widetilde{M} = M^\sim = MI_n^{-1}, \tag{1.22}$$

where I_n^{-1} differs from I_n by at most a sign. The dual of an r-blade is an $(n-r)$-blade; in particular, the dual of an n-blade is a scalar, which is why an n-blade is called a pseudoscalar.

Inner and outer products are *dual* to one another, as expressed by the following identities that hold for any vector a and multivector M:

$$(a\cdot M)I_n = a\wedge(MI_n)\,, \tag{1.23a}$$

$$(a\wedge M)I_n = a\cdot(MI_n)\,. \tag{1.23b}$$

Duality enables us to define the *meet* $M\vee N$ of multivectors M, N with (grade M) + (grade N) $\geq n$ by

$$M\vee N = \widetilde{M}\cdot N\,. \tag{1.24}$$

The meet satisfies the "deMorgan rule"

$$(M\vee N)^\sim = \widetilde{M}\wedge\widetilde{N}. \tag{1.25}$$

As shown below, the meet can be interpreted as an algebraic representation for the intersection of vector subspaces. More generally, it can be used to describe "incidence relations" in geometry [HZ91].

Many other products can be defined in terms of the geometric product. The *commutator product* $A\times B$ is defined for any A and B by

$$A\times B \equiv \tfrac{1}{2}(AB - BA) = -B\times A\,. \tag{1.26}$$

Mathematicians classify this product as a "derivation" with respect to the geometric product, because it has the "distributive property"

$$A\times(BC) = (A\times B)C + B(A\times C)\,. \tag{1.27}$$

This implies the *Jacobi identity*

$$A \times (B \times C) = (A \times B) \times C + B \times (A \times C)\,, \tag{1.28}$$

which is a derivation on the commutator product. The relation of the commutator product to the inner and outer products is grade dependent; thus, for a vector a,

$$a \times \langle\, M \,\rangle_k = a \wedge \langle\, M \,\rangle_k \qquad \text{if } k \text{ is odd}\,, \tag{1.29a}$$

$$a \times \langle\, M \,\rangle_k = a \cdot \langle\, M \,\rangle_k \qquad \text{if } k \text{ is even}\,. \tag{1.29b}$$

The commutator product is especially useful in computations with bivectors. With any bivector A this product is grade preserving:

$$A \times \langle\, M \,\rangle_k = \langle\, A \times M \,\rangle_k\,. \tag{1.30}$$

In particular, this implies that the space of bivectors is closed under the commutator product. Consequently, it forms a Lie algebra. The geometric product of bivector A with M has the expanded form

$$AM = A \cdot M + A \times M + A \wedge M \qquad \text{for grade } M \geq 2\,. \tag{1.31}$$

Compare this with the corresponding expansion (1.6) for the product of vectors.

Blades and Subspaces

The elements of $\mathcal{G}_n$ can be assigned a variety of different geometric interpretations appropriate for different applications. For one, geometric algebra characterizes geometric objects composed of points that are represented by vectors in $\mathcal{V}^n$.

To every r-dimensional subspace in $\mathcal{V}^n$, there is an r-blade A_r such that the subspace is the solution set of the equation

$$x \wedge A_r = 0, \text{ for } x \in \mathcal{V}^n. \tag{1.32}$$

According to this equation, A_r is unique to within a nonzero scalar factor. This subspace generates a geometric algebra $\mathcal{G}(A_r)$. Conversely, the subspace itself is uniquely determined by A_r. Therefore, as discussed in [HS84], the blades in $\mathcal{V}^n$ determine an algebra of subspaces for $\mathcal{V}^n$. The blade A_r can be regarded as a directed measure (or r-volume) on the subspace, with magnitude $|A_r|$ and an orientation. Thus, since A_r determines a unique subspace, the two blades A_r and $-A_r$ determine two subspaces of the same vectors but opposite orientation. The blades of grade r form a manifold $G(r, n)$ called a *Grassmannian*. The algebra of blades is therefore an algebraic structure for $G(r, n)$, and the rich literature on Grassmannians [H92] can be incorporated into GC.

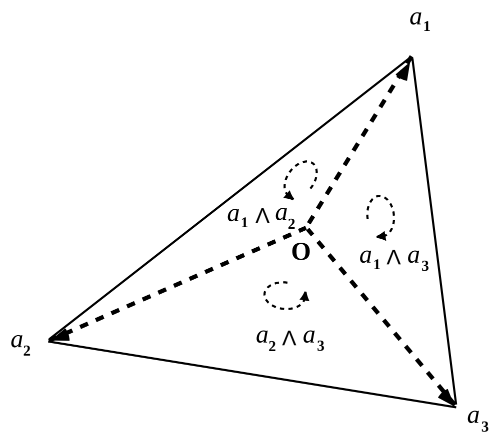

Fig 1.1. Blades in the space of $a_1 \wedge a_2 \wedge a_3$, where the a_i are vectors.

Vectors $a_1, \cdots, a_r$, are linearly dependent if and only if $a_1 \wedge \cdots \wedge a_r = 0$. The r-blade $a_1 \wedge \cdots \wedge a_r$ represents the r-dimensional subspace of $\mathcal{V}^n$ spanned by them, if they are linearly independent. The case $r = 3$ is illustrated in Fig 1.1.

The square of a blade is always a scalar, and a blade is said to be *null* if its square is zero. Null blades represent degenerate subspaces. For a non-degenerate r-subspace, we often use a unit blade, say I_r, to denote it. For any two pseudoscalars A_r, B_r in $\mathcal{G}(I_r)$, since $A_r B_r^{-1} = B_r^{-1} A_r$ is a scalar, we can write it A_r/B_r.

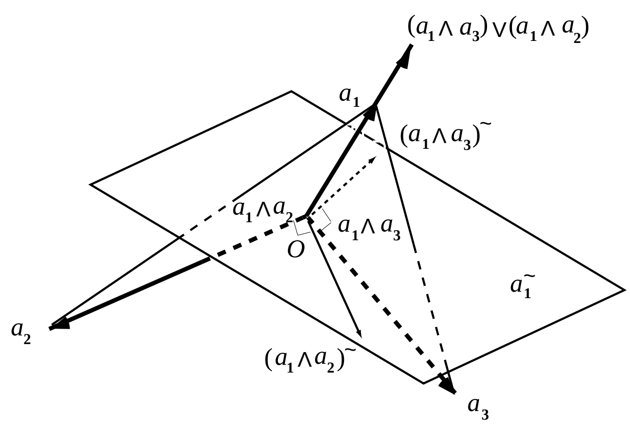

Fig 1.2. Dual and meet in the space of $a_1 \wedge a_2 \wedge a_3$.

Figure 1.2 illustrates duality and meet in $\mathcal{G}_3$ generated by vectors a_1, a_2, a_3. Notice the collinearity of vectors a_1 and $(a_1 \wedge a_3) \vee (a_1 \wedge a_2)$.

In a non-degenerate space, the dual of a blade represents the orthogonal complement of the subspace represented by the blade. The meet of two blades, if nonzero, represents the intersection of the two subspaces represented by the two blades respectively.

Two multivectors are said to be *orthogonal* or *perpendicular* to one another, if their inner product is zero. An r-blade A_r and s-blade B_s are orthogonal if and only if (1) when $r \geq s$ there exists a vector in the space of B_s, that is orthogonal to every vector in the space of A_r, and (2) when $r < s$ there exists a vector in the space of A_r that is orthogonal to every vector in the space of B_s.

Let A_r be a non-degenerate r-blade in $\mathcal{G}_n$. Then any vector $x \in \mathcal{V}^n$ has a *projection* onto the space of A_r defined by

$$P_{A_r}(x) = (x \cdot A_r)\, A_r^{-1}. \tag{1.33}$$

Its *rejection* from the space of A_r is defined by

$$P^{\perp}_{A_r}(x) = (x \wedge A_r)\, A_r^{-1}. \tag{1.34}$$

Therefore,

$$x = P_{A_r}(x) + P^{\perp}_{A_r}(x) \tag{1.35}$$

is the *orthogonal decomposition* of $\mathcal{V}^n$ with respect to the A_r.

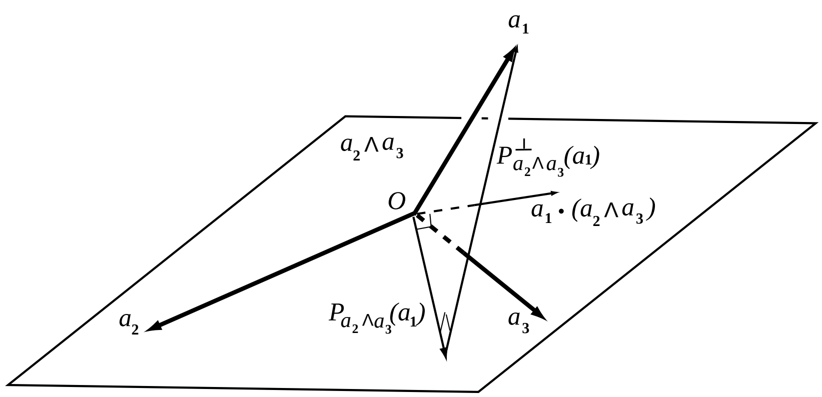

Fig 1.3. Projection, rejection and inner product in the space of $a_1 \wedge a_2 \wedge a_3$.

Figure 1.3 shows the projection and rejection of vector a_1 with respect to the space $a_2 \wedge a_3$, together with the inner product $a_1 \cdot (a_2 \wedge a_3)$. Note that the vector $a_1 \cdot (a_2 \wedge a_3)$ is perpendicular to the vector $P_{a_2 \wedge a_3}(a_1)$.

Frames

If a blade A_r admits the decomposition

$$A_r = a_1 \wedge a_2 \wedge \ldots \wedge a_r\,, \tag{1.36}$$

the set of vectors $\{a_i\,;\, i = 1, \ldots, r\}$ is said to be a *frame* (or *basis*) for A_r and the vector space it determines. Also, A_r is said to be the *pseudoscalar* for the frame. A *dual frame* $\{a^i\}$ is defined by the equations

$$a^i \cdot a_j = \delta^i_j. \tag{1.37}$$

If A_r is invertible, these equations can be solved for the a^i, with the result

$$a^i = (-1)^{i+1}(a_1 \wedge \ldots \wedge \check{a}_i \wedge \ldots \wedge a_r)A_r^{-1}, \tag{1.38}$$

where $\check{a}_i$ indicates that a_i is omitted from the product. Moreover,

$$A_r^{-1} = a^1 \wedge a^2 \wedge \ldots \wedge a^r. \tag{1.39}$$

In this way geometric algebra greatly facilitates manipulations with frames for vector spaces.

Differentiation

Let $a = \sum_i \alpha^i a_i$ be a *vector variable* with coordinates $\alpha^i = a \cdot a_i$ defined on the space of A_r as specified by (1.36). The *vector derivative* with respect to a on A_r can be defined by

$$\partial_a = \sum_i a^i \frac{\partial}{\partial \alpha^i}. \tag{1.40}$$

It can be defined without introducing coordinates, but this approach relates it to standard partial derivatives. The following table of derivatives is easily derived. For any vector b in $\mathcal{V}^n$:

$$\partial_a(a \cdot b) = b \cdot \partial_a a = \textstyle\sum_i a^i(a_i \cdot b) = P_{A_r}(b), \tag{1.41a}$$

$$\partial_a a^2 = 2a, \tag{1.41b}$$

$$\partial_a a = \partial_a \cdot a = r, \tag{1.41c}$$

$$\partial_a \wedge a = 0. \tag{1.41d}$$

Of course A_r could be the pseudoscalar for $\mathcal{V}^n$.

[HS84] generalizes differentiation to any multivector variable defined on $\mathcal{G}_n$ or any of its subspaces. The derivative ∂_X with respect to a multivector variable X is characterized by the basic identity

$$\partial_X \langle XA \rangle = \langle A \rangle_X = \langle A \partial_X \rangle X\,, \tag{1.42}$$

where A is independent of X, $\langle \ldots \rangle$ means scalar part, and $\langle A \rangle_X$ means "select from A only those grades which are contained in X." If A has the same grade as X then $\langle A \rangle_X = A$. It follows that

$$\partial_X \langle \widetilde{X} A \rangle = \langle \widetilde{A} \rangle_X\,. \tag{1.43}$$

The operator $\langle A\partial_X \rangle$ is a kind of generalized directional derivative defined by

$$\langle A\partial_X \rangle F(X) \equiv \frac{d}{d\epsilon} F(X + \epsilon A)\big|_{\epsilon=0}\,, \tag{1.44}$$

where ϵ is a scalar parameter and F is any differentiable function of X. Applied to $F(X) = X$, this yields the right equality in (1.42). If A has the same grade as X, then the left equality in (1.42) gives

$$\partial_X = \partial_A \langle A \partial_X \rangle\,, \tag{1.45}$$

so the general multivector derivative $\partial_X F(X)$ can be obtained from the "directional derivative" (1.44). From (1.44) one derives the sum, product and chain rules for differential operators. Of course, the vector derivative with its properties (1.40) to (1.41d) is a special case, as is the usual scalar derivative.

Signature

So far, a particular signature has not been attributed to $\mathcal{V}^n$ to emphasize the fact that, for many purposes, signature is inconsequential. To account for signature, we introduce the alternative notation $\mathcal{R}^{p,q,r}$ to denote a real vector space of dimension $n = p + q + r$, where p, q and r are, respectively, the dimensions of subspaces spanned by vectors with positive, negative and null signatures. Let $\mathcal{R}_{p,q,r} = \mathcal{G}(\mathcal{R}^{p,q,r})$ denote the geometric algebra of $\mathcal{R}^{p,q,r}$, and let $\mathcal{R}^k_{p,q,r}$ denote the $\binom{n}{r}$-dimensional subspace of k-vectors, so that

$$\mathcal{R}_{p,q,r} = \sum_{k=0}^{n} \mathcal{R}^k_{p,q,r}\,. \tag{1.46}$$

A pseudoscalar I_n for $\mathcal{R}^{p,q,r}$ factors into

$$I_n = A_p B_q C_r\,, \tag{1.47}$$

where the factors are pseudoscalars for the three different kinds of subspaces.

The algebra is said to be *non-degenerate* if I_n is invertible. That is possible only if $r = 0$, so $n = p + q$ and

$$I_n = A_p B_q\,. \tag{1.48}$$

In that case we write $\mathcal{R}^{p,q} = \mathcal{R}^{p,q,0}$ and $\mathcal{R}_{p,q} = \mathcal{R}_{p,q,0}$. The algebra is said to be *Euclidean* (or *anti-Euclidean*) if $n = p$ (or $n = q$). Then it is convenient to use the notations $\mathcal{R}^n = \mathcal{R}^{n,0}$, $\mathcal{R}^{-n} = \mathcal{R}^{0,n}$, etcetera.

Any degenerate algebra can be embedded in a non-degenerate algebra of larger dimension, and it is almost always a good idea to do so. Otherwise, there will be subspaces without a complete basis of dual vectors, which will complicate algebraic manipulations. The n-dimensional vector spaces of every possible signature are subspaces of $\mathcal{R}^{n,n}$. For that reason, $\mathcal{R}_{n,n}$ is called the *mother algebra* of n-space. As explained in [DHSA93], it is the proper arena for the most general approach to linear algebra.

1.3 Linear Transformations

The terms "linear function," "linear mapping" and "linear transformation" are usually regarded as synonymous. To emphasize an important distinction in GA, let us restrict the meaning of the last term to "linear vector-valued functions of a vector variable." Of course, every linear function is isomorphic to a linear transformation. The special importance of the latter derives from the fact that the tools of geometric algebra are available to characterize its structure and facilitate applications. Geometric algebra enables coordinate-free analysis and computations. It also facilitates the use of matrices when they are desired or needed.

To relate a linear transformation $\underline{f}$ on $\mathcal{V}^n$ to its matrix representation $f_i^{\,j}$, we introduce a basis $\{e_i\}$ and its dual $\{e^j\}$, so that

$$\underline{f}e_i = \underline{f}(e_i) = \sum_j e_j f_i^{\,j}\,, \tag{1.49a}$$

and

$$f_i^{\,j} = e^j \cdot \underline{f}e_i = (\bar{f}\,e^j) \cdot e_i\,. \tag{1.49b}$$

The last equality defines the *adjoint* $\bar{f}$ of $\underline{f}$, so that

$$\bar{f}\,e^j = \sum_i f_i^{\,j} e^i\,. \tag{1.50}$$

Without reference to a basis the adjoint is defined by

$$b \cdot \underline{f}a = a \cdot \bar{f}\,b\,, \tag{1.51a}$$

whence,

$$\bar{f}\,b = \partial_a(b \cdot \underline{f}a) = \sum_i e^i(b \cdot \underline{f}\,e_i)\,. \tag{1.51b}$$

Within geometric algebra, it seldom helps to introduce matrices unless they have been used in the first place to define the linear transformations of interest, as, for example, in a computer graphics display where coordinates are needed. Some tools to handle linear transformations without matrices are described below.

Outermorphism

Every linear transformation $\underline{f}$ on $\mathcal{V}^n$ extends naturally to a linear function $\underline{f}$ on $\mathcal{G}_n$ with

$$\underline{f}(A \wedge B) = (\underline{f}A) \wedge (\underline{f}B)\,. \tag{1.52}$$

This extension is called an *outermorphism* because it preserves the outer product. Any ambiguity in using the same symbol $\underline{f}$ for the transformation and its extension can be removed by displaying an argument for the function. For any blade with the form (1.36) we have

$$\underline{f}A_r = (\underline{f}a_1) \wedge (\underline{f}a_2) \wedge \cdots \wedge (\underline{f}a_r)\,. \tag{1.53}$$

This shows explicitly how the transformation of vectors induces a transformation of blades. By linearity it extends to any multivector.

The outermorphism of the adjoint $\bar{f}$ is easily specified by generalizing (1.51a) and (1.51b); thus, for any multivectors A and B,

$$\langle B\,\underline{f}A\rangle = \langle A\bar{f}\,B\rangle\,. \tag{1.54}$$

By multivector differentiation,

$$\bar{f}\,B = \bar{f}\,(B) = \partial_A\langle A\bar{f}\,(B)\rangle\,. \tag{1.55}$$

We are now equipped to formulate the fundamental theorem:

$$A\cdot(\,\underline{f}B) = \underline{f}[(\bar{f}\,A)\cdot B] \qquad \text{or} \qquad (\,\underline{f}B)\cdot A = \underline{f}[B\cdot\bar{f}\,A]\,. \tag{1.56}$$

for (grade A) $\le$ (grade B).

This theorem, first proved in [HS84], is awkward to formulate without geometric algebra, so it seldom appears (at least in full generality) in the literature on linear algebra. It is important because it is the most general *transformation law for inner products.*

Outermorphisms generalize and simplify the theory of determinants. Let I be a pseudoscalar for $\mathcal{V}^n$. The determinant of $\underline{f}$ is the eigenvalue of $\underline{f}$ on I, as expressed by

$$\bar{f}\,I = (\det\underline{f})I\,. \tag{1.57}$$

If I is invertible, then

$$I^{-1}\bar{f}\,I = I^{-1}\cdot(\,\underline{f}I) = \det\underline{f} = \det\bar{f} = \det f_i^{\,j}\,. \tag{1.58}$$

To prove the last equality, we can write $I = e_1\wedge e_2\wedge\ldots\wedge e_n$ so that

$$\det\underline{f} = (e^n\wedge\ldots\wedge e^1)\cdot[(\,\underline{f}e_1)\wedge\ldots\wedge\,\underline{f}(e_n)]\,. \tag{1.59}$$

Using the identities (1.10a) and (1.10b), the right side of (1.14) can be expanded to get the standard expression for determinants in terms of matrix elements $f_i^{\,j}$. This exemplifies the fact that the Laplace expansion and all other properties of determinants are easy and nearly automatic consequences of their formulation within geometric algebra.

The law (1.56) has the important special case

$$A\bar{f}\,I = \bar{f}\,[(\,\underline{f}A)I]\,. \tag{1.60}$$

For $\det\underline{f}\neq 0$, this gives an explicit expression for the inverse outermorphism

$$\underline{f}^{-1}A = \frac{\bar{f}\,(AI)I^{-1}}{\det\underline{f}}\,. \tag{1.61}$$

Applying this to the basis $\{e^i\}$ and using (1.38) we obtain

$$\underline{f}^{-1}e^i = \frac{(-1)^{i+1}\bar{f}\,(e_1\wedge\cdots\wedge\check{a}_i\wedge\cdots\wedge e_n)\cdot(e_1\wedge\cdots\wedge e_n)}{\det\underline{f}}\,. \tag{1.62}$$

Again, expansion of the right side with the help of (1.10a) and (1.10b) gives the standard expression for a matrix inverse in terms of matrix elements.

The composition of linear transformations $\underline{g}$ and $\underline{f}$ can be expressed as an operator product:

$$\underline{h} = \underline{g}\,\underline{f}\,. \tag{1.63}$$

This relation extends to their outermorphism as well. Applied to (1.57), it immediately gives the classical result

$$\det \underline{h} = (\det \underline{g}) \det \underline{f}\,, \tag{1.64}$$

from which many more properties of determinants follow easily.

Orthogonal Transformations

A linear transformation $\underline{U}$ is said to be *orthogonal* if it preserves the inner product of vectors, as specified by

$$(\underline{U}a)\cdot(\underline{U}b) = a\cdot(\overline{U}\underline{U}b) = a\cdot b\,. \tag{1.65}$$

Clearly, this is equivalent to the condition $\underline{U}^{-1} = \overline{U}$. For handling orthogonal transformations geometric algebra is decisively superior to matrix algebra, because it is computationally more efficient and simpler to interpret geometrically. To explain how, some new terminology is helpful.

A *versor* is any multivector that can be expressed as the geometric product of invertible vectors. Thus, any versor U can be expressed in the factored form

$$U = u_k \cdots u_2 u_1\,, \tag{1.66}$$

where the choice of vectors u_i is not unique, but there is a minimal number $k \le n$. The parity of U is even (odd) for even (odd) k.

Every versor U determines an orthogonal transformation $\underline{U}$ given by

$$\underline{U}(x) = UxU^{*-1} = U^*xU^{-1}, \qquad \text{for} \qquad x \in \mathcal{V}^n\,. \tag{1.67}$$

Conversely, every orthogonal transformation $\underline{U}$ can be expressed in the *canonical form* (1.67). This has at least two great advantages. First, any orthogonal transformation is representable (and therefore, visualizable) as a set of vectors. Second, the composition of orthogonal transformations is reduced to multiplication of vectors.

The outermorphism of (1.67) is

$$\underline{U}(M) = UMU^{-1} \qquad \text{for} \qquad U^* = U\,, \tag{1.68a}$$

or

$$\underline{U}(M) = UM^*U^{-1} \qquad \text{for} \qquad U^* = -U\,, \tag{1.68b}$$

where M is a generic multivector.

An even versor $R = R^*$ is called a *spinor* or *rotor* if

$$RR^\dagger = |\, R\,|^2 \,, \tag{1.69a}$$

so that

$$R^{-1} = \frac{1}{R} = \frac{R^\dagger}{RR^\dagger} = \frac{R^\dagger}{|\, R\,|^2} \,. \tag{1.69b}$$

Alternative, but closely related, definitions of "spinor" and "rotor" are common. Often the term rotor presumes the normalization $|\, R\,|^2 = 1$. In that case, (1.67) takes the simpler form

$$\underline{R}x = RxR^\dagger \tag{1.70}$$

and $\underline{R}$ is called a *rotation*. Actually, the form with R^{-1} is preferable to the one with $R^\dagger$, because $\underline{R}$ is independent of $|\, R\,|$, and normalizing may be inconvenient.

Note that for $U = u_2u_1$, the requirement $|\, U\,|^2 = u_2^2u_1^2$ for a rotation implies that the vectors u_1 and u_2 have the same signature. Therefore, when they have opposite signature U is the prime example of an even versor which is not a spinor, and the corresponding linear operator $\underline{U}$ in (1.67) is not a rotation.

In the simplest case where the versor is a single vector $u_1 = -u_1^*$, we can analyze (1.67) with the help of (1.35) as follows:

$$\begin{aligned} \underline{u}_1(x) = u_1^* x u_1^{-1} &= -u_1(P_{u_1}(x) + P_{u_1}^\perp(x))u_1^{-1} \\ &= -P_{u_1}(x) + P_{u_1}^\perp(x) \,. \end{aligned} \tag{1.71}$$

The net effect of the transformation is to re-*verse* direction of the component of x along u_1, whence the name *versor* (which dates back to Hamilton). Since every invertible vector is normal to a hyperplane in $\mathcal{V}^n$, the transformation (1.71) can be described as *reflection in a hyperplane*. In view of (1.66), *every orthogonal transformation can be expressed as the composite of at most n reflections in hyperplanes*. This is widely known as the Cartan-Dieudonné Theorem.

The reflection (1.71) is illustrated in Fig. 1.4 along with the next simplest example, a rotation $\underline{u_3u_2}$ induced by two successive reflections. The figure presumes that the rotation is elliptic (or Euclidean), though it could be hyperbolic (known as a Lorentz transformation in physics), depending on the signature of the plane.

Given that a rotation takes a given vector a into $b = \underline{R}a$, it is often of interest to find the simplest spinor R that generates it. It is readily verified that the solution is

$$R = (a + b)a = b(a + b) \,. \tag{1.72}$$

Without loss of generality, we can assume that a and b are normalized to $a^2 = b^2 = \pm 1$, so that

$$|\, R\,|^2 = a^2(a + b)^2 = 2\,|\, a \cdot b \pm 1\,| \,. \tag{1.73}$$

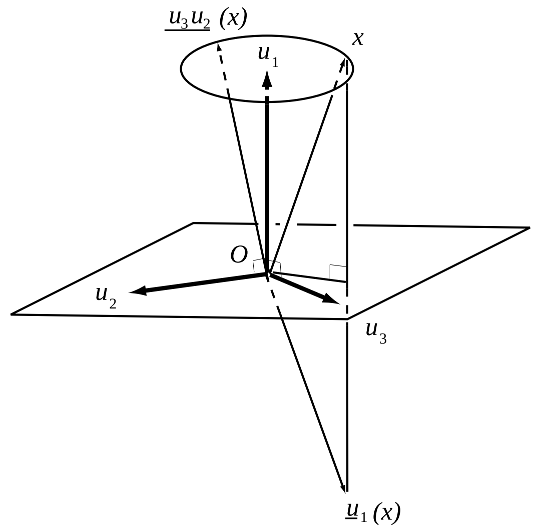

Fig 1.4. Versor (vector and rotor) actions. Here u_1 is orthogonal to both u_2, u_3.

This is a case where normalization is inconvenient. Besides destroying the simplicity of the unnormalized form (1.72), it would require taking the square root of (1.73), an unnecessary computation because it does not affect $\underline{R}$. Note that $|R| = 0$ and R^{-1} is undefined when a and b are oppositely directed. In that case a and b do not define a unique plane of rotation.

Although it is helpful to know that rotors can be "parameterized" by vectors as in (1.66) and (1.70), there are many other parameterizations, such as Euler angles, which are more suitable in certain applications. A detailed treatment of alternatives for 3-D rotations is given in [H98].

The versors in $\mathcal{G}_n$ form a group under the geometric product, called the *versor group*. The versors of unit magnitude form a subgroup, called the *pin group*. Since $\underline{U}$ in (1.67) is independent of the sign and magnitude of U, the two groups are equivalent *double coverings* of the *orthogonal group* $\mathrm{O}(p, q)$, where the signature of $\mathcal{V}^n = \mathcal{R}^{p,q}$ is displayed to enable crucial distinctions. At first sight the pin group seems simpler than the versor group, but we have already noted that it is sometimes more efficient to work with unnormalized versors.

For any $\underline{U}$

$$\det \underline{U} = \pm 1 \,, \tag{1.74}$$

where the sign corresponds to the parity of the versor U which generates it. Those with positive determinant compose the *special orthogonal group* $\mathrm{SO}(p, r)$. It is doubly covered by the subgroup of even versors. The subgroup of elements in $\mathrm{SO}(p, r)$ which are continuously connected to the identity is called the *rotation group* $\mathrm{SO}^+(p,r)$. The versor group covering $\mathrm{SO}(p,r)$ is called the *spin group* $\mathrm{Spin}(p,r)$. Let us write $\mathrm{Spin}^+(p,r)$ for the group of rotors covering $\mathrm{SO}^+(p,r)$. This group is distingiushed by the condition (1.68a) on its elements, and that ensures that the rotations are continuously connected to the identity. The distinction between SO and SO^+ or between Spin and Spin^+ is not always recognized in the literature, but that seldom creates problems. For Euclidean or anti-Euclidean signature there is no distinction.

The Spin groups are more general than anyone suspected for a long time. It has been proved in [DHSA93] that *every Lie group* can be represented as a spin group in some Geometric Algebra of suitable dimension and signature. The corresponding *Lie algebra* is then represented as an algebra of bivectors under the commutator product (1.26). All this has great theoretical and practical advantages, as it is computationally more efficient than matrix representations. Engineers who compute 3-D rotations for a living are well aware that quaternions (the rotors in $\mathcal{R}_3$) are computationally superior to matrices.

1.4 Vectors as geometrical points

The elements of $\mathcal{G}_n$ can be assigned a variety of geometric interpretations appropriate for different applications. The most common practice is to identify vectors with geometric points, so that geometric objects composed of points are represented by point sets in a vector space. In this section we show how geometric algebra can be used to characterize some of the most basic geometric objects. This leads to abstract representations of objects by their properties without reference to the points that compose them. Thus we arrive at a kind of "algebra of geometric properties" which greatly facilitates geometric analysis and applications. We have already taken a large step in this direction by constructing an "algebra of vector subspaces" in Section 1.2. Here we take two more steps. First, we displace the subspaces from the origin to give us as algebra of k-planes. Second, we break the k-planes into pieces to get a "simplicial algebra." In many applications, such as finite element analysis, simplexes are basic building blocks for complex structures. To that end, our objective here is to sharpen the tools for manipulating simplexes. Applications to Geometric Calculus are described in [S92].

In physics and engineering the vector space $\mathcal{R}^3$ is widely used to represent Euclidean 3-space $\mathcal{E}^3$ as a model of "physical space." More advanced applications use $\mathcal{R}^{1,3}$ as a model for spacetime with the *"spacetime algebra"* $\mathcal{R}_{1,3}$ to characterize its properties. Both of these important cases are included in the present treatment, which applies to spaces of any dimension and signature.

r-planes

An r-dimensional plane parallel to the subspace of A_r and through the point a is the solution set of

$$(x-a)\wedge A_r = 0, \text{ for } x \in \mathcal{V}^n\,. \tag{1.75}$$

It is often called an r-*plane*, or r-*flat*. It is called a *hyperplane* when $r = n - 1$.

As detailed for $\mathcal{R}^3$ in [H98], A_r is the *tangent* of the r-plane, and $M_{r+1} = a \wedge A_r$ is the *moment*. When A_r is invertible, $d = M_{r+1}A_r^{-1}$ is the *directance*. Let n be a unit vector collinear with d, then $d \cdot n$ is the *signed distance* of the r-plane from the origin in the direction of n, or briefly, the n-*distance* of the r-plane.

An r-plane can be represented by

$$A_r + M_{r+1} = (1 + d)A_r \,, \tag{1.76}$$

where A_r is the tangent, M_{r+1} is the moment, and d is the directance. A point x is on the r-plane if and only if $x \wedge M_{r+1} = 0$ and $x \wedge A_r = M_{r+1}$. This representation, illustrated in Fig 1.4, has applications in rigid body mechanics [H98]. The representation (1.76) is equivalent to the *Plücker coordinates* for an r-plane [P1865].

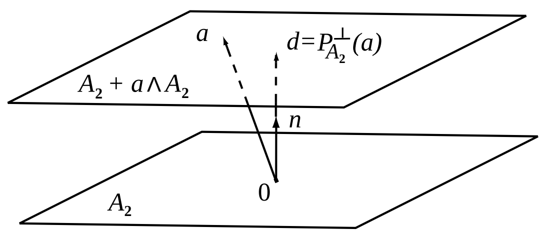

Fig 1.5. A 2-plane in the space of $a \wedge A_2$.

A linear transformation $\underline{f}$ of points in $\mathcal{V}^n$ induces a transformation of (1.75) via its outermorphism: thus,

$$\underline{f}[(x - a) \wedge A_r] = (\underline{f}x - \underline{f}a) \wedge (\underline{f}A_r) = (x' - a') \wedge A'_r = 0 \,. \tag{1.77}$$

This proves that every nonsingular linear transformation maps straight lines into straight lines and, more generally, k-planes into k-planes. This generalizes trivially to affine transformations.

Simplexes

An r-dimensional *simplex* (r-*simplex*) in $\mathcal{V}^n$ is the convex hull of $r+1$ points, of which at least r are linearly independent. A set $\{a_0, a_1, a_2, \ldots a_r\}$ of defining points is said to be a *frame* for the simplex. One of the points, say a_0, is distinguished and called the *base point* or *place* of the simplex. It will be convenient to introduce the notations

$$A_r \equiv a_0 \wedge a_1 \wedge a_2 \wedge \cdots \wedge a_r = a_0 \wedge \bar{A}_r \,, \tag{1.78a}$$

$$\begin{aligned}\bar{A}_r &\equiv (a_1 - a_0) \wedge (a_2 - a_0) \wedge \cdots \wedge (a_r - a_0) \\ &= \bar{a}_1 \wedge \bar{a}_2 \wedge \cdots \wedge \bar{a}_r \,,\end{aligned} \tag{1.78b}$$

$$\bar{a}_i \equiv a_i - a_0 \qquad \text{for} \qquad i = 1, \ldots, r \,. \tag{1.78c}$$

$\overline{A}_r$ is called the *tangent* of the simplex, because it is tangent for the r-plane in which the simplex lies (See Fig 1.6.). It must be nonzero for the simplex to have a convex hull. We also assume that it is non-degenerate, so we don't deal with complications of null vectors. The tangent assigns a natural *directed measure* $\overline{A}_r/r!$ to the simplex. As shown by Sobczyk [S92], this is the appropriate measure for defining integration over chains of simplexes and producing a generalized Stokes Theorem. The scalar *content* (or volume) of the simplex is given by $(r!)^{-1}|\,\overline{A}_r\,|$. For the simplex in Fig 1.6 this is the area of the triangle (3-hedron) with sides $\overline{a}_1$ and $\overline{a}_2$. In general, it is the volume of an $(r+1)$-hedron. The tangent $\overline{A}_r$ assigns a definite *orientation* to the simplex, and interchanging any two of the vectors $\overline{a}_i$ in (1.78b) reverses the orientation. Also, $\overline{A}_r$ is independent of the choice of origin, though A_r is not.

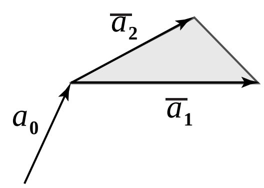

Fig 1.6. Simplex at a_0 with tangent $\overline{A}_2 = \overline{a}_1 \wedge \overline{a}_2$.

In accordance with (1.75), the equation for the plane of the simplex is

$$x \wedge \overline{A}_r = a_0 \wedge \overline{A}_r = A_r\,. \tag{1.79}$$

Thus A_r is the *moment* of the simplex. It will be convenient to use A_r as a name for the simplex, since the expanded form (1.78a) displays all the defining points of the simplex. We also assume $A_r \neq 0$, since it greatly facilitates analysis of simplex properties. However, when the simplex plane (1.79) passes through the origin, its moment $a_0 \wedge \overline{A}_r$ vanishes. There are two ways to deal with this problem. One is to treat it as a degenerate special case. A better way is to remove the origin from $\mathcal{V}^n$ by embedding $\mathcal{V}^n$ as an n-plane in a space of higher dimension. Then all points will be treated on equal footing and the moment $a_0 \wedge \overline{A}_r$ never vanishes. This is tantamount to introducing homogeneous coordinates, an approach which is developed in Chapter 2. Note that the simplex A_r is oriented, since interchanging any pair of vectors in (1.78a) will change its sign.

Since (1.78a) expresses A_r as the pseudoscalar for the simplex frame $\{a_i\}$ it determines a dual frame $\{a^i\}$ given by (1.36). The *face opposite* a_i in simplex A_r is represented by its moment

$$\mathcal{F}_i^r A_r \equiv A_r^i \equiv a^i \cdot A_r = (-1)^{i+1} a_0 \wedge \cdots \wedge \check{a}_i \wedge \cdots \wedge a_r\,. \tag{1.80}$$

This defines a *face operator* $\mathcal{F}_i$ (as illustrated in Fig 1.7). Of course, the face A_r^i is an $(r-1)$-dimensional simplex. Also

$$A_r = a_i \wedge A_r^i\,, \qquad \text{for any} \qquad 0 \leq i \leq r\,. \tag{1.81}$$

The *boundary* of simplex A_r can be represented formally by the multivector sum

$$\partial\!\!\!/ A_r = \sum_{i=0}^{r} A_r^i = \sum_{i=0}^{r} \mathcal{F}_i A_r \,. \tag{1.82}$$

This defines a boundary operator $\partial\!\!\!/$. Obviously,

$$A_r = a_i \wedge \partial\!\!\!/ A_r \,, \qquad \text{for any} \qquad 0 \le i \le r \,. \tag{1.83}$$

Comparing the identity

$$(a_1 - a_0) \wedge (a_2 - a_0) \wedge \cdots \wedge (a_r - a_0) = \sum_{i=0}^{r} (-1)^i a_0 \wedge \cdots \wedge \check{a}_i \wedge \cdots \wedge a_r \tag{1.84}$$

with (1.78b) and (1.80) we see that

$$\overline{A}_r = \partial\!\!\!/ A_r \,. \tag{1.85}$$

Also, note the symmetry of (1.84) with respect to the choice of base point.

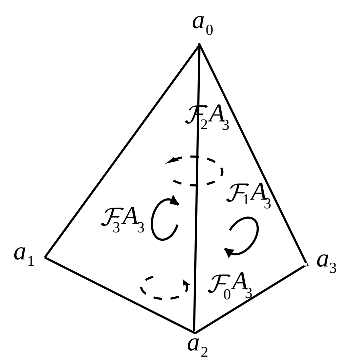

Fig 1.7. Simplex $a_0 \wedge a_1 \wedge a_2 \wedge a_3$.

For the two operators $\mathcal{F}_i$ and $\partial\!\!\!/$ we have

$$\mathcal{F}_i \mathcal{F}_i = 0 \,, \tag{1.86a}$$
$$\mathcal{F}_i \mathcal{F}_j = -\mathcal{F}_j \mathcal{F}_i \,, \quad \text{for} \quad i \ne j \,, \tag{1.86b}$$
$$\mathcal{F}_i \partial\!\!\!/ = -\partial\!\!\!/ \mathcal{F}_i \,, \tag{1.86c}$$
$$\partial\!\!\!/ \partial\!\!\!/ = 0 \,. \tag{1.86d}$$

These operator relations are strongly analogous to relations in algebraic topology.

If a point x lies in the r-plane of the simplex, we can write

$$x = \sum_{i=0}^{r} \alpha^i a_i \,, \tag{1.87}$$

where $\alpha^i = x \cdot a^i$. The point lies within the simplex if and only if

$$\sum_{i=0}^{r} \alpha^i = 1 \qquad \text{and} \qquad 0 \le \alpha^i \le 1\,. \tag{1.88}$$

Subject to these conditions the α^i are known as *barycentric coordinates* of the point.

From (1.80) it follows that

$$a^i \cdot A_r^i = (a^i \wedge a^i) \cdot A_r = 0 \tag{1.89a}$$

and

$$a^i A_r^i = a^i \wedge A_r^i = A_r\,. \tag{1.89b}$$

Thus, a^i is *normal* to the (moment of) the *i th face* A_r^i and is contained in A_r. In view of (1.88), a^i can be regarded as an *outward normal* to the face.

Precisely analogous relations hold for the tangent $\overline{A}_r$ and its faces. By virtue of (1.78b), the frame $\{\overline{a}_i\}$ has a dual frame $\{\overline{a}^i\}$ of *outward normals* to the faces of $\overline{A}_r$. These normals play a crucial role in an invariant formulation of Stokes Theorem that has the same form for all dimensions ([HS84], [S92]).

1.5 Linearizing the Euclidean group

The Euclidean group is the group of rigid displacements on $\mathcal{E}^n$. With $\mathcal{E}^n$ represented as $\mathcal{R}^n$ in $\mathcal{R}_n$, any rigid displacement can be expressed in the canonical form

$$\widehat{D}_c : x \to x' = \widehat{D}_c x = \widehat{T}_c \underline{R} x = RxR^{-1} + c\,, \tag{1.90}$$

where, in accordance with (1.67), R is a rotor determining a rotation $\underline{R}$ about the origin and $\widehat{T}_c$ is a translation by vector c. The composition of displacements is complicated by the fact that rotations are multiplicative but translations are additive. We alleviate this difficulty with a device that makes translations multiplicative as well.

We augment $\mathcal{R}^n$ with a null vector e orthogonal to $\mathcal{R}^n$. In other words, we embed $\mathcal{R}^n$ in a vector space $\mathcal{V}^{n+1} = \mathcal{R}^{n,0,1}$, so the degenerate geometric algebra $\mathcal{R}_{n,0,1}$ is at our disposal. In this algebra we can represent the translation $\widehat{T}_c$ as a spinor

$$T_c = e^{\frac{1}{2}ec} = 1 + \tfrac{1}{2}ec\,, \tag{1.91a}$$

with inverse

$$T_c^{-1} = 1 - \tfrac{1}{2}ec = T_c^*\,, \tag{1.91b}$$

where "$*$" is the main involution in $\mathcal{R}_n$, so $c^* = -c$ but $e^* = e$. If we represent the point x by

$$X = 1 + ex\,, \tag{1.92}$$

the translation $\widehat{T}_c$ can be replaced by the equivalent linear operator

$$\underline{T}_c : X \to X' = T_c X T_c^{*-1} = T_c(1 + ex)T_c = 1 + e(x + c)\,. \tag{1.93}$$

Consequently, any rigid displacement $\widehat{D}_c$ has the spinor representation

$$D_c = T_c R\,, \tag{1.94a}$$

and the linear representation

$$\underline{D}_c(X) = D_c X D_c^{*-1} \tag{1.94b}$$

in conformity with (1.67).

We call the representation of points by (1.92) the *affine model for Euclidean space*, because it supports *affine transformations*, which are composites of linear transformations (Section 1.3) with translations. It has the advantage of linearizing the Euclidean group through (1.94b). More important, it gives us the spinor representation (1.94a) for the group elements. This has the great advantage of reducing the group composition to the geometric product. For example, let us construct the spinor R_c for rotation about an arbitrary point c. We can achieve such a rotation by translating c to the origin, rotating about the origin, and then translating back. Thus

$$R_c = T_c R T_c^{-1} = R + e\,c \times R\,, \tag{1.95}$$

where $\times$ denotes the commutator product.

In $\mathcal{R}^3$ any rigid displacement can be expressed as a *screw displacement*, which consists of a rotation about some point composed with a translation along the rotation axis. This is known as *Chasles Theorem*. It is useful in robotics and other applications of mechanics. The theorem is easily proved with (1.94a), which shows us how to find the screw axis at the same time. Beginning with the displacement (1.94a), we note that the vector direction n for the rotation axis R satisfies

$$RnR^{-1} = n \qquad \text{or} \qquad Rn = nR\,. \tag{1.96}$$

This suggests the decomposition $c = c_\parallel + c_\perp$ where $c_\parallel = (c\cdot n)n$. The theorem will be proved if we can find a vector b so that

$$D_c = T_{c_\parallel} T_{c_\perp} R = T_{c_\parallel} R_b\,, \tag{1.97}$$

where R_b is given by (1.95). From the null component of $T_{c_\perp} R = R_b$ we obtain the condition

$$\tfrac{1}{2} c_\perp R = b \times R = \tfrac{1}{2} b(R - R^\dagger)\,.$$

With R normalized to unity, this has the solution

$$b = c_\perp (1 - R^{-2})^{-1} = \tfrac{1}{2} c_\perp \frac{1 - R^2}{1 - \langle\, R^2 \,\rangle}\,. \tag{1.98}$$

This tells us how to find a point b on the screw axis.

Everything we have done in this section applies without change for reflections as well as rotations. Thus, for any invertible vector n, (1.94a) gives us the versor

$$n_c = n + e\,c \cdot n\,, \tag{1.99}$$

which represents a reflection in the hyperplane through c with normal n. Note that we have a symmetric product $c \cdot n = n \cdot c$ in (1.99) instead of the skew-symmetric product $c \times R = -R \times c$ in (1.95); this is an automatic consequence of the fact that e anticommutes with vectors n and c but commutes with R.

We have shown how the degenerate model for Euclidean space simplifies *affine geometry*. In Chapter 2 we shall see how it can be generalized to a more powerful model for Euclidean geometry.

Dual Quaternions

Clifford originated the idea of extending the real number system to include an element ϵ with the nilpotence property $\epsilon^2 = 0$. Then any number can be written in the form $\alpha + \epsilon\beta$, where α and β are ordinary numbers. He called them *dual numbers*. Clearly, this has no relation to our use of the term "dual" in geometric algebra, so we employ it only in this brief subsection to explain its connection to the present work.

A *dual quaternion* has the form $Q_1 + \epsilon Q_2$, where, Q_1 and Q_2 are ordinary quaternions. There has been renewed interest in dual quaternions recently, especially for applications to rigid motions in robotics. A direct relation to the present approach stems from the fact that the quaternions can be identified with the spinors in $\mathcal{R}_3$. To demonstrate, it will suffice to consider (1.94a). The so-called "vector part" of a quaternion actually corresponds to a bivector in $\mathcal{R}_3$. The dual of every vector c in $\mathcal{R}_3$ is a bivector $C = cI^\dagger = -cI$, where I is the unit pseudoscalar. Therefore, writing $\epsilon = eI = -Ie$, we can put (1.94a) in the equivalent "dual quaternion form"

$$R_c = R + \epsilon C \times R\,, \tag{1.100}$$

where $\epsilon^2 = 0$, and ϵ commutes with both C and R. In precisely the same way, for $\mathcal{E}^3$ the representation of points by (1.92) can be reexpressed as a dual quaternion, so we have a *dual quaternion model* of $\mathcal{E}^3$.

The drawback of quaternions is that they are limited to 3-D applications, and even there they fail to make the important distinction between vectors and bivectors. This can be remedied somewhat by resorting to complex quaternions, because they are isomorphic to the whole algebra $\mathcal{R}_3$. However, the quaternion nomenclature (dual complex or otherwise) is not well-tailored to geometric applications, so we advise against it. It should be clear that geometric algebra retains all of the advantages and none of the drawbacks of quaternions, while extending the range of applications enormously.

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