

# A Survey of Geometric Algebra and Geometric Calculus

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“The principal argument for the adoption of geometric algebra is that it provides a single, simple mathematical framework which eliminates the plethora of diverse mathematical descriptions and techniques it would otherwise be necessary to learn.”  
[9]

# 1 Foundations

## 1.1 Introduction

Geometric algebra and its extension to geometric calculus unify, simplify, and generalize vast areas of mathematics that involve geometric ideas. They also provide a unified mathematical language for physics, engineering, and the geometrical aspects of computer science (e.g., graphics, robotics, computer vision).

This paper is an introduction to geometric algebra and geometric calculus, presented in the simplest way I could manage, without worrying too much about completeness or rigor. The only prerequisite is an understanding of undergraduate mathematics. In a few inessential sections some knowledge of physics is helpful.

My purpose is to demonstrate some of the scope and power of geometric algebra and geometric calculus. I will illustrate this for linear algebra, multivariable calculus, real analysis, complex analysis, and several geometries: euclidean, noneuclidean, projective, and conformal. I will also outline several applications.

Geometric algebra is nothing less than a new approach to geometry. Geometric objects (points, lines, planes, circles, ... ) are represented by members of an algebra, a *geometric algebra*, rather than by equations relating coordinates. Geometric operations on the objects (rotate, translate, intersect, project, construct the circle through three points, ... ) are then represented by algebraic operations on the objects.

Geometric algebra is *coordinate-free*: coordinates are needed only when specific objects or operations are under consideration.

All this has significant advantages over traditional approaches such as synthetic and analytic geometry and vector, tensor, exterior, and spinor algebra and calculus. The advantages are similar to those of elementary algebra over arithmetic: elementary algebra manipulates symbolic representations of numbers independently of their values, and geometric algebra manipulates symbolic representations of geometric objects independently of their coordinates in some coordinate system.

*At first you will likely find the novelty and scope of the mathematics presented here overwhelming.* This is to be expected: it takes years of serious study to understand the standard approaches to the mathematics discussed here. But after some study I hope that you find, with me, great unity, simplicity, and elegance in geometric algebra.

Readers who want to know more can consult the last section, *Further Study*, of this paper. It includes a listing of many papers on available on the web.

The American physicist and mathematician David Hestenes initiated the modern development of geometric algebra in the 1960's. He built on the work of Hamilton, Grassmann, and Clifford a century or so earlier. After a slow start, geometric algebra has today attracted many workers in many disciplines.

Hestenes was awarded the 2002 Oersted Medal, the American Association of Physics Teachers "most prestigious award". His medal lecture, "Reforming the Mathematical Language of Physics" [5], was published in *The American Journal of Physics*, which has published several papers on geometric algebra. In his lecture, Hestenes claims that "geometric algebra simplifies and clarifies the structure of physics, and ... [thus has] immense implications for physics instruction at all grade levels." I believe that this is equally true of mathematics.

## 1.2 The Geometric Algebra

The most popular algebraic structure today for Euclidean  $n$ -space is the inner product space  $\mathbf{R}^n$ . This section describes a powerful extension of this structure, the geometric algebra  $\mathbf{G}^n$ . In subsequent sections, armed with this algebra, we will unify, simplify, and generalize many areas of mathematics, and give several applications.

**1.2.1. The geometric algebra.**  $\mathbf{G}^n$  is an algebra:

- It has an associative noncommutative product with identity. The product is called the *geometric product*.
- It is a vector space extending the vector space  $\mathbf{R}^n$ . It has dimension  $2^n$ . Its vectors are called *multivectors*.
- The geometric product (left and right) distributes over multivector addition.

We now describe individually these three aspects of  $\mathbf{G}^n$ . This will give us the rules for computing in  $\mathbf{G}^n$ . It will not prove that a mathematical structure satisfying the rules exists. For that, see [8].

**1.2.2. The geometric product.** The geometric product of multivectors can be based on two fundamental rules.

First, for every vector  $\mathbf{u}$  in  $\mathbf{R}^n$ ,

$$\mathbf{u} \mathbf{u} = |\mathbf{u}|^2. \quad (1.1)$$

We shall see shortly that scalars are members of  $\mathbf{G}^n$ . Thus Eq. (1.1) is an equation in  $\mathbf{G}^n$ . It follows that nonzero vectors have an inverse:  $\mathbf{u}^{-1} = \mathbf{u}/|\mathbf{u}|^2$ .

Second, for orthogonal vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $\mathbf{R}^n$ ,

$$\mathbf{v} \mathbf{u} = -\mathbf{u} \mathbf{v}. \quad (1.2)$$

Example:  $(\mathbf{u}\mathbf{v})^2 = \mathbf{u}\mathbf{v}\mathbf{u}\mathbf{v} = -\mathbf{u}\mathbf{u}\mathbf{v}\mathbf{v} = -|\mathbf{u}|^2|\mathbf{v}|^2$ . Thus if  $\mathbf{u}$  and  $\mathbf{v}$  are nonzero, then  $(\mathbf{u}\mathbf{v})^2 < 0$ . Therefore  $\mathbf{u}\mathbf{v}$  is not a scalar or a vector. It is something new, a *bivector*.

**1.2.3. Multivectors.** We illustrate the vector space  $\mathbf{G}^n$  with  $\mathbf{G}^3$ . It is  $2^3 = 8$  dimensional. Let  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  be an orthonormal basis of  $\mathbf{R}^3$ . We have used a specific basis to describe the vector space  $\mathbf{G}^3$ , but all orthonormal bases give the same vector space with the same bivectors and trivectors. Then a basis for  $\mathbf{G}^3$  is

1	spans 0-vectors (scalars)
$\mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{e}_3$	span 1-vectors (vectors)
$\mathbf{e}_1 \mathbf{e}_2 \ \mathbf{e}_2 \mathbf{e}_3 \ \mathbf{e}_1 \mathbf{e}_3$	span 2-vectors (bivectors)
$\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3$	spans 3-vectors (trivectors).

The (one and only) zero element is a  $k$ -vector for all  $k$ .

We might try to form a 4-vector, e.g.,  $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_1$ . But by the product rules, Eqs. (1.1) and (1.2), this is equal to  $\mathbf{e}_2\mathbf{e}_3$ . There are no 4-vectors in  $\mathbf{G}^3$  and, more generally, no  $(n+1)$ -vectors in  $\mathbf{G}^n$ .

We have used a specific basis to describe the vector space  $\mathbf{G}^3$ , but all orthonormal bases give the same vector space with the same bivectors and trivectors.

**1.2.4. Distributivity.** To multiply any two multivectors expanded with respect to the basis of Sec. 1.2.3, use distributivity and the product rules, Eqs. (1.1) and (1.2). For example,  $(1 + \mathbf{e}_1\mathbf{e}_2)(\mathbf{e}_1 - 2\mathbf{e}_2) = -\mathbf{e}_1 - 3\mathbf{e}_2$ .

That's it! That's the geometric algebra. Let's see what we can do with it.

Abbreviate "geometric algebra" to GA and "vector algebra" to VA.

### 1.3 The Inner and Outer Products

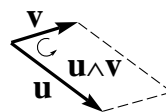
We investigate the geometric product of two given vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $\mathbf{R}^n$ . Let  $\{\mathbf{e}_1, \mathbf{e}_2\}$  be an orthonormal basis for a plane containing the vectors. Let  $\mathbf{u} = a\mathbf{e}_1 + b\mathbf{e}_2$  and  $\mathbf{v} = c\mathbf{e}_1 + d\mathbf{e}_2$ . Then from the product rules, Eqs. (1.1) and (1.2),

$$\mathbf{u}\mathbf{v} = (ac + bd) + (ad - bc)\mathbf{e}_1\mathbf{e}_2. \quad (1.3)$$

**1.3.1. The inner product.** The first term on the right side of Eq. (1.3),  $ac + bd$ , is the usual inner product of  $\mathbf{u}$  and  $\mathbf{v}$ :  $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$ .

**1.3.2. The outer product.** The second term on the right side of Eq. (1.3) is the *outer product* of  $\mathbf{u}$  and  $\mathbf{v}$ . This bivector is denoted  $\mathbf{u} \wedge \mathbf{v}$ . Just as  $\mathbf{u}$  represents an oriented length,  $\mathbf{u} \wedge \mathbf{v}$  represents an oriented area. For the factor  $ad - bc$  is the area of the parallelogram with sides  $\mathbf{u}$  and  $\mathbf{v}$ :  $|\mathbf{u}||\mathbf{v}|\sin\theta$ . And  $\mathbf{i} \equiv \mathbf{e}_1\mathbf{e}_2$  specifies the plane in which the area resides. Thus

$$\mathbf{u} \wedge \mathbf{v} = (ad - bc)\mathbf{e}_1\mathbf{e}_2 = |\mathbf{u}||\mathbf{v}|\sin\theta\mathbf{i}. \quad (1.4)$$



**Fig. 1:** The bivector  $\mathbf{u} \wedge \mathbf{v}$ .

See Fig. 1. (If you want to know why  $\mathbf{e}_1\mathbf{e}_2$  is denoted  $\mathbf{i}$ , square it. We will have much more to say about this.)

The inner product  $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$  is intrinsic to the vectors, not depending on the basis  $\{\mathbf{e}_1, \mathbf{e}_2\}$ . So too with the outer product. For  $|\mathbf{u}||\mathbf{v}|\sin\theta$  is intrinsic. And if we rotate to another basis  $\begin{pmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}$ , then  $\mathbf{i}' = \mathbf{e}'_1\mathbf{e}'_2 = \mathbf{i}$ .

In 3D VA, the oriented area  $\mathbf{u} \wedge \mathbf{v}$  is represented by the cross product  $\mathbf{u} \times \mathbf{v}$ . But VA does not generalize  $\times$  to higher dimensions. Moreover, we shall see that even in 3D,  $\wedge$  is superior to  $\times$  in several respects. Thus the cross product plays only a minor role in GA.

**1.3.3. The fundamental identity.** Rewrite Eq. (1.3) in terms of the inner and outer products to obtain the *fundamental identity*

$$\mathbf{u}\mathbf{v} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v}. \quad (1.5)$$

Forget the coordinates of  $\mathbf{u}$  and  $\mathbf{v}$  which led to this equation. Remember that the geometric product of two vectors is the sum of a scalar and a bivector, both of which have a simple geometric interpretation.

There is no hint in VA that  $\cdot$  and  $\times$  (reformulated to  $\wedge$ ) are parts of a whole: the geometric product, an associative product in which nonzero vectors have an inverse.<sup>1</sup>

**1.3.4. Important miscellaneous facts.** We will use them without comment.

- $\mathbf{u} \wedge \mathbf{v}$ , unlike  $\mathbf{u}\mathbf{v}$ , is always a bivector.
- $\mathbf{u}\mathbf{v}$  is a bivector  $\Leftrightarrow \mathbf{u} \cdot \mathbf{v} = 0 \Leftrightarrow \mathbf{u} \perp \mathbf{v} \Leftrightarrow \mathbf{u}\mathbf{v} = \mathbf{u} \wedge \mathbf{v} \Leftrightarrow \mathbf{u}\mathbf{v} = -\mathbf{v}\mathbf{u}$ .  
In particular, for  $i \neq j$ ,  $\mathbf{e}_i\mathbf{e}_j = \mathbf{e}_i \wedge \mathbf{e}_j$ .
- $\mathbf{u}\mathbf{v}$  is a scalar  $\Leftrightarrow \mathbf{u} \wedge \mathbf{v} = 0 \Leftrightarrow \mathbf{u} \parallel \mathbf{v} \Leftrightarrow \mathbf{u}\mathbf{v} = \mathbf{u} \cdot \mathbf{v} \Leftrightarrow \mathbf{u}\mathbf{v} = \mathbf{v}\mathbf{u}$ .
- $\mathbf{v} \wedge \mathbf{u} = -(\mathbf{u} \wedge \mathbf{v})$ .
- The inner and outer products are the symmetric and antisymmetric parts of the geometric product:  $\mathbf{u} \cdot \mathbf{v} = \frac{1}{2}(\mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u})$  and  $\mathbf{u} \wedge \mathbf{v} = \frac{1}{2}(\mathbf{u}\mathbf{v} - \mathbf{v}\mathbf{u})$ .

<sup>1</sup> Not every nonzero multivector has an inverse. Example: Let  $|\mathbf{u}| = 1$ . If  $1 - \mathbf{u}$  had an inverse, then right multiply  $(1 + \mathbf{u})(1 - \mathbf{u}) = 0$  by the inverse to obtain  $1 + \mathbf{u} = 0$ , a contradiction.

## 1.4 Subspaces

**1.4.1. The pseudoscalar.** The (unit) *pseudoscalar*  $\mathbf{I} = \mathbf{e}_1 \mathbf{e}_2 \cdots \mathbf{e}_n$  is an important member of  $\mathbf{G}^n$ . (We use lower case in 2D:  $\mathbf{i} = \mathbf{e}_1 \mathbf{e}_2$ , as in Sec. 1.3.2.) Since  $n$ -vectors form a 1 dimensional subspace of  $\mathbf{G}^n$ , every  $n$ -vector is a scalar multiple of  $\mathbf{I}$ . If  $\mathbf{e}'_1 \mathbf{e}'_2 \cdots \mathbf{e}'_n = \mathbf{I}'$  for another orthonormal basis, then  $\mathbf{I}' = \pm \mathbf{I}$ , the sign chosen according as the bases have the same or opposite orientation. We saw the 2D case in Sec. 1.3.2. Note that  $\mathbf{I}^{-1} = \mathbf{e}_n \cdots \mathbf{e}_2 \mathbf{e}_1$ . Repeated application of Eq. (1.2) shows that  $\mathbf{I}^{-1} = (-1)^{(n-1)n/2} \mathbf{I}$ . In particular,  $\mathbf{I}^{-1} = \pm \mathbf{I}$ .

**1.4.2. Representing subspaces.** Every oriented subspace of  $\mathbf{R}^n$  has its own unique pseudoscalar, the product of members of any ordered orthonormal basis for the subspace. *GA represents an oriented subspace by its pseudoscalar.* For example,  $\mathbf{e}_1$ ,  $\mathbf{e}_1 \mathbf{e}_2$ , and  $\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3$  represent oriented subspaces of  $\mathbf{R}^3$ .

Representing subspaces of  $\mathbf{R}^n$  as members of  $\mathbf{G}^n$  enables us to perform geometric operations on the subspaces algebraically. We will see several examples in Sec. 1.4.5.

**1.4.3. Blades.** Nonzero scalar multiples of pseudoscalars are called *blades*. A positive multiple represents the same oriented subspace as the pseudoscalar. A negative multiple represents the same subspace with the opposite orientation. For example,  $2\mathbf{e}_1 \mathbf{e}_2$  and  $-\mathbf{e}_1 \mathbf{e}_2 = \mathbf{e}_2 \mathbf{e}_1$  represent opposite orientations of a 2D subspace.

A blade can be expressed (in many ways) as a product of nonzero orthogonal vectors:  $\mathbf{B} = \mathbf{b}_1 \mathbf{b}_2 \cdots \mathbf{b}_k$ .  $\mathbf{B}$  is a *k-blade*, or a blade of *grade k*; it represents an oriented  $k$ -dimensional subspace. It has an inverse  $\mathbf{B}^{-1} = \mathbf{b}_k \cdots \mathbf{b}_2 \mathbf{b}_1 / |\mathbf{b}_k|^2 \cdots |\mathbf{b}_2|^2 |\mathbf{b}_1|^2$ . Nonzero scalars are 0-blades. They represent the subspace  $\{0\}$ .

We use **bold** to designate blades, with lower case reserved for vectors. (Exception: i.) We use upper case *italic* ( $A, B, C, \dots$ ) to denote general multivectors.

**1.4.4. Extend inner and outer products.** We extend the definition of the inner and outer products of vectors from Sec. 1.3 to all of  $\mathbf{G}^n$ .

Let  $\langle C \rangle_j$  denote the  $j$ -vector part of the multivector  $C$ . For example, from the fundamental identity Eq. (1.5),  $\langle \mathbf{a} \mathbf{b} \rangle_0 = \mathbf{a} \cdot \mathbf{b}$ ,  $\langle \mathbf{a} \mathbf{b} \rangle_1 = 0$ , and  $\langle \mathbf{a} \mathbf{b} \rangle_2 = \mathbf{a} \wedge \mathbf{b}$ .

**Inner product.** Let  $\mathbf{A}$  be a  $j$ -blade and  $\mathbf{B}$  a  $k$ -blade. Define:<sup>2</sup>

$$\mathbf{A} \cdot \mathbf{B} = \langle \mathbf{A} \mathbf{B} \rangle_{k-j}. \quad (1.6)$$

(The notation implies that if  $j > k$ , then  $\mathbf{A} \cdot \mathbf{B} = 0$ .) Examples:

$$\begin{aligned} (\mathbf{e}_1 \mathbf{e}_2) \cdot (\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3) &= \langle \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \rangle_{3-2} = -\mathbf{e}_3, \\ (\mathbf{e}_1) \cdot (\mathbf{e}_2 \mathbf{e}_3) &= \langle \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \rangle_{2-1} = 0. \end{aligned} \quad (1.7)$$

**Outer product.** Define:

$$\mathbf{A} \wedge \mathbf{B} = \langle \mathbf{A} \mathbf{B} \rangle_{j+k}. \quad (1.8)$$

(The notation implies that if  $j + k > n$ , then  $\mathbf{A} \wedge \mathbf{B} = 0$ .) Examples:

$$\begin{aligned} (\mathbf{e}_1 \mathbf{e}_2) \wedge (\mathbf{e}_3) &= \langle \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \rangle_{2+1} = \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3, \\ (\mathbf{e}_1 \mathbf{e}_2) \wedge (\mathbf{e}_2 \mathbf{e}_3) &= \langle \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_2 \mathbf{e}_3 \rangle_{2+2} = 0. \end{aligned} \quad (1.9)$$

<sup>2</sup>  $|k-j|$  is used more often than  $k-j$  in the definition of the inner product, Eq. (1.6). [1] Dorst has argued the advantages of the inner product as defined here. [2]

The outer product is associative. (The cross product is not.) And  $\mathbf{B} \wedge \mathbf{A} = (-1)^{jk}(\mathbf{A} \wedge \mathbf{B})$ .

Extend the inner and outer products from blades to all of  $\mathbf{G}^n$  by distributivity.

**Duality.** Define the *dual* of a multivector  $A$ :

$$A^* = A/\mathbf{I} (\equiv \mathbf{A}\mathbf{I}^{-1}). \quad (1.10)$$

The inner and outer products are dual on blades:

$$(\mathbf{A} \cdot \mathbf{B})^* = \mathbf{A} \wedge \mathbf{B}^*, \quad (\mathbf{A} \wedge \mathbf{B})^* = \mathbf{A} \cdot \mathbf{B}^*. \quad (1.11)$$

To see, e.g., the second equality, let  $\mathbf{A}$  be a  $j$ -blade and  $\mathbf{B}$  be a  $k$ -blade. Then

$$\mathbf{A} \cdot \mathbf{B}^* = \langle \mathbf{A}(\mathbf{B}\mathbf{I}^{-1}) \rangle_{(n-k)-j} = \langle (\mathbf{A}\mathbf{B})\mathbf{I}^{-1} \rangle_{n-(j+k)} = \langle \mathbf{A}\mathbf{B} \rangle_{j+k} \mathbf{I}^{-1} = (\mathbf{A} \wedge \mathbf{B})^*.$$

**1.4.5. Algebra of subspaces.** We describe the algebraic representation of several geometric operations on subspaces.

We shorten “the subspace represented by blade  $\mathbf{B}$ ”, to “the subspace  $\mathbf{B}$ ”, or simply “ $\mathbf{B}$ ”. For example, set theoretic relationships between blades, e.g., “ $\mathbf{A} \subseteq \mathbf{B}$ ”, refer to the subspaces they represent.

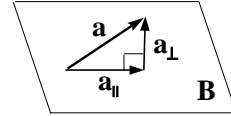
**Complement.** If  $\mathbf{A} \subseteq \mathbf{B}$ , then  $\mathbf{A}\mathbf{B}$  is the orthogonal complement of  $\mathbf{A}$  in  $\mathbf{B}$ . To see this, compute  $\mathbf{A}\mathbf{B}$  using an orthogonal basis for  $\mathbf{A}$  extended to  $\mathbf{B}$ . For example,  $(\mathbf{e}_1)(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3) = \mathbf{e}_2\mathbf{e}_3$ , the orthogonal complement of  $\mathbf{e}_1$  in  $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ . Note that when  $\mathbf{A} \subseteq \mathbf{B}$ ,  $\mathbf{A}\mathbf{B} = \mathbf{A} \cdot \mathbf{B}$ , so that  $\mathbf{A} \cdot \mathbf{B}$  is also the orthogonal complement.

In particular,  $\mathbf{A}\mathbf{I}$  is the orthogonal complement of  $\mathbf{A}$  in  $\mathbf{R}^n$ . Since  $\mathbf{I}^{-1} = \pm\mathbf{I}$ ,  $\mathbf{A}^* = \mathbf{A}/\mathbf{I}$  is also the orthogonal complement of  $\mathbf{A}$  in  $\mathbf{R}^n$ .

The  $*$  operator of  $\mathbf{GA}$  is the  $\perp$  operator of  $\mathbf{VA}$ . Because  $\mathbf{VA}$  does not represent general subspaces, it cannot implement  $\perp$  as an algebraic operation.

In  $\mathbf{R}^3$  the vectors  $(\mathbf{u} \wedge \mathbf{v})^*$  and  $\mathbf{u} \times \mathbf{v}$  are parallel, as both are orthogonal to the oriented plane  $\mathbf{u} \wedge \mathbf{v}$ . From Eq. (1.4),  $(\mathbf{u} \wedge \mathbf{v})^* = \mathbf{u} \times \mathbf{v}$ .

**Projection and rejection.** Let  $\mathbf{B} = \mathbf{b}_1\mathbf{b}_2 \cdots \mathbf{b}_k$ , a  $k$ -blade. Decompose a vector  $\mathbf{a}$  with respect to the subspace  $\mathbf{B}$ :  $\mathbf{a} = \mathbf{a}_\perp + \mathbf{a}_\parallel$ , where  $\mathbf{a}_\perp \perp \mathbf{B}$ ,  $\mathbf{a}_\parallel \in \mathbf{B}$ . See Fig. 2. By the definition of a blade, Sec. 1.4.3,  $\mathbf{a}_\perp\mathbf{B}$  is a  $(k+1)$ -blade (or zero). And since  $\mathbf{a}_\parallel$  is a linear combination of the  $\mathbf{b}_i$ ,  $\mathbf{a}_\parallel\mathbf{B}$  is a  $(k-1)$ -vector. Thus from the definitions of the inner and outer products, Eqs. (1.6) and (1.8),



**Fig. 2:** Projection & rejection.

$$\mathbf{a}_\perp \cdot \mathbf{B} = 0, \quad \mathbf{a}_\parallel \cdot \mathbf{B} = \mathbf{a}_\parallel\mathbf{B}, \quad \mathbf{a}_\perp \wedge \mathbf{B} = \mathbf{a}_\perp\mathbf{B}, \quad \mathbf{a}_\parallel \wedge \mathbf{B} = 0. \quad (1.12)$$

From Eqs. (1.12),  $\mathbf{a}_\parallel\mathbf{B} = \mathbf{a}_\parallel \cdot \mathbf{B} = (\mathbf{a}_\perp + \mathbf{a}_\parallel) \cdot \mathbf{B} = \mathbf{a} \cdot \mathbf{B}$ . Right multiply by  $\mathbf{B}^{-1}$ :

$$\mathbf{a}_\parallel = (\mathbf{a} \cdot \mathbf{B})/\mathbf{B}, \quad (1.13)$$

the *projection* of  $\mathbf{a}$  onto  $\mathbf{B}$ . Similarly, we have the *rejection* of  $\mathbf{a}$  from  $\mathbf{B}$ :

$$\mathbf{a}_\perp = (\mathbf{a} \wedge \mathbf{B})/\mathbf{B}. \quad (1.14)$$

$\mathbf{VA}$  has no analogs of these simple formulas, except when  $\mathbf{B} = \mathbf{b}$ , a vector, in the projection formula Eq. (1.13).

The distance from the endpoint of  $\mathbf{a}$  to  $\mathbf{B}$  is  $|\mathbf{a}_\perp| = |(\mathbf{a} \wedge \mathbf{B})/\mathbf{B}|$ . If  $\mathbf{B}$  is a hyperplane, e.g., a plane in  $\mathbf{R}^3$ , then it divides  $\mathbf{R}^n$  into two sides. Then  $\mathbf{a} \wedge \mathbf{B}$  is an  $n$ -vector. The scalar  $(\mathbf{a} \wedge \mathbf{B})^*/|\mathbf{B}|$  is a *signed* distance from the endpoint of  $\mathbf{a}$  to  $\mathbf{B}$ .

**Span.** The examples of the outer product in Eqs. (1.9) are prototypical:

$$\mathbf{A} \wedge \mathbf{B} = \begin{cases} \text{span}(\mathbf{A}, \mathbf{B}), & \text{if } \mathbf{A} \cap \mathbf{B} = \{0\}; \\ \{0\}, & \text{if } \mathbf{A} \cap \mathbf{B} \neq \{0\}. \end{cases} \quad (1.15)$$

(The *join* operation, discussed in Sec. 4.1.3, always produces the span.)

To prove Eqs. (1.15), let  $\mathbf{C} = \mathbf{A} \cap \mathbf{B}$ . (Recall that  $\mathbf{C} = 1$  represents the subspace  $\{0\}$ .) Let  $\mathbf{A}' = \mathbf{A}/\mathbf{C}$ . Since  $\mathbf{A}/\mathbf{C} = k\mathbf{C}/\mathbf{A}$ , where  $k$  is a nonzero constant,  $\mathbf{A}'$  is the orthogonal complement of  $\mathbf{C}$  in  $\mathbf{A}$ . Similarly, let  $\mathbf{B}' = \mathbf{C}\mathbf{B}$  be the orthogonal complement of  $\mathbf{C}$  in  $\mathbf{B}$ . Then  $\mathbf{A}\mathbf{B} = \mathbf{A}'\mathbf{B}'$ .

Let  $\text{grade}(\mathbf{A}') = j'$  and  $\text{grade}(\mathbf{B}') = k'$ . Set  $\mathbf{B}' = \mathbf{b}'_1\mathbf{b}'_2 \cdots \mathbf{b}'_{k'}$ , where the  $\mathbf{b}'_i$  are nonzero orthogonal vectors. Decompose  $\mathbf{b}'_1 = \mathbf{b}'_{1\perp} + \mathbf{b}'_{1\parallel}$  into its rejection and projection with respect to  $\mathbf{A}'$ . Next decompose  $\mathbf{b}'_2 = \mathbf{b}'_{2\perp} + \mathbf{b}'_{2\parallel}$  with respect to  $\mathbf{A}'\mathbf{b}'_1$ . Continuing,  $\mathbf{A}\mathbf{B} = \mathbf{A}'(\mathbf{b}'_{1\perp} + \mathbf{b}'_{1\parallel})(\mathbf{b}'_{2\perp} + \mathbf{b}'_{2\parallel}) \cdots (\mathbf{b}'_{k'\perp} + \mathbf{b}'_{k'\parallel})$ . Multiply:  $\mathbf{A}\mathbf{B} = \mathbf{A}'\mathbf{b}'_{1\perp}\mathbf{b}'_{2\perp} \cdots \mathbf{b}'_{k'\perp} + \text{other terms}$ . The first term is a blade of grade  $j' + k'$ . The other terms are of lesser grade. And  $j' + k' = j + k$  if and only if  $\mathbf{A} \cap \mathbf{B} = \{0\}$ . ■

Eqs. (1.11) now show that a nonzero  $\mathbf{A} \cdot \mathbf{B}$  is also a blade.

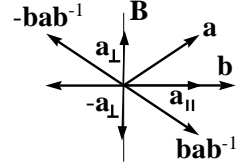
**Reflection.** The reflection of  $\mathbf{a}$  in a hyperplane  $\mathbf{B}$  is

$$-\mathbf{b}\mathbf{a}\mathbf{b}^{-1}, \quad (1.16)$$

where  $\mathbf{b}$  is a vector orthogonal to  $\mathbf{B}$ . See Fig. 3.

To see this, decompose  $\mathbf{a}$  with respect to  $\mathbf{b}$ :  $\mathbf{a} = \mathbf{a}_\parallel + \mathbf{a}_\perp$ . The reflection of  $\mathbf{a}$  in  $\mathbf{b}$  is  $\mathbf{a}_\parallel - \mathbf{a}_\perp$ . To compute it, apply Eqs. (1.13) and (1.14) to  $\mathbf{a}$  and  $\mathbf{b}$ :

$$\begin{aligned} \mathbf{a}_\parallel - \mathbf{a}_\perp &= (\mathbf{a} \cdot \mathbf{b})/\mathbf{b} - (\mathbf{a} \wedge \mathbf{b})/\mathbf{b} \\ &= \{(\mathbf{b} \cdot \mathbf{a}) + (\mathbf{b} \wedge \mathbf{a})\} \mathbf{b}^{-1} \\ &= \mathbf{b}\mathbf{a}\mathbf{b}^{-1}. \end{aligned}$$



**Fig. 3:** The reflection of  $\mathbf{a}$  in hyperplane  $\mathbf{B}$ .

From Fig. 3, the reflection of  $\mathbf{a}$  in  $\mathbf{B}$  is the negative of this, as given by Eq. (1.16).

**1.4.6. Subspace membership.** Since  $\mathbf{a} \in \mathbf{B} \Leftrightarrow \mathbf{a}_\perp = 0$ , Eq. (1.14) provides an algebraic test for subspace membership:

$$\mathbf{a} \in \mathbf{B} \Leftrightarrow \mathbf{a} \wedge \mathbf{B} = 0.$$

Similarly, from Eq. (1.13),  $\mathbf{a} \perp \mathbf{B} \Leftrightarrow \mathbf{a} \cdot \mathbf{B} = 0$ .

Since  $\mathbf{a} \in \mathbf{B} \Leftrightarrow \mathbf{a} \perp \mathbf{B}^*$ , we also have  $\mathbf{a} \in \mathbf{B} \Leftrightarrow \mathbf{a} \cdot \mathbf{B}^* = 0$ . Call  $\mathbf{B}$  a *direct* representation of the subspace, and  $\mathbf{B}^*$  a *dual* representation. We will use both.

**1.4.7. Extend fundamental identity.** Add the Eqs. (1.12) to extend Eq. (1.5):

$$\mathbf{a}\mathbf{B} = \mathbf{a} \cdot \mathbf{B} + \mathbf{a} \wedge \mathbf{B}, \quad (1.17)$$

where  $\mathbf{a} \cdot \mathbf{B}$  is a  $(k-1)$ -vector and  $\mathbf{a} \wedge \mathbf{B}$  is a  $(k+1)$ -vector. In general,  $\mathbf{A}\mathbf{B} \neq \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \wedge \mathbf{B}$ . Example:  $\mathbf{A} = \mathbf{e}_1\mathbf{e}_2$  and  $\mathbf{B} = \mathbf{e}_2\mathbf{e}_3$ . The general product  $\mathbf{A}\mathbf{B}$  contains multivectors of grade  $|k-j|, |k-j+2|, \dots, |k+j-2|, |k+j|$ .

## 1.5 $k$ -Volumes

**1.5.1.  $k$ -Volumes.** We noted in Sec. 1.4.3 that a blade can be expressed as a product of nonzero orthogonal vectors:  $\mathbf{B} = \mathbf{b}_1 \mathbf{b}_2 \cdots \mathbf{b}_k$ . The *norm* of  $\mathbf{B}$  is  $|\mathbf{B}| = |\mathbf{b}_1| |\mathbf{b}_2| \cdots |\mathbf{b}_k|$ . This is the  $k$ -volume of the rectangular parallelepiped spanned by  $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_k$ . In this way  $|\mathbf{B}|$  represents a  $k$ -volume in the subspace  $\mathbf{B}$ . The  $k$ -volume has neither a shape nor a position in  $\mathbf{B}$ . For example,  $|\mathbf{b}_1 \mathbf{b}_2| = |(2\mathbf{b}_1)(\frac{1}{2}\mathbf{b}_2)|$ .

The projection formula Eq. (1.13) generalizes: If  $\text{grade}(\mathbf{A}) \leq \text{grade}(\mathbf{B})$ , then the projection of  $\mathbf{A}$  onto  $\mathbf{B}$  is  $P_{\mathbf{B}}(\mathbf{A}) = (\mathbf{A} \cdot \mathbf{B})/\mathbf{B}$ .<sup>3</sup> Thus the angle between  $\mathbf{A}$  and  $\mathbf{B}$  is given by  $\cos \theta = |P_{\mathbf{B}}(\mathbf{A})|/|\mathbf{A}| = |\mathbf{A} \cdot \mathbf{B}|/|\mathbf{A}| |\mathbf{B}|$ .<sup>4</sup> For example, the angle between planes  $\mathbf{A} = \mathbf{e}_1(\mathbf{e}_2 + \mathbf{e}_3)$  and  $\mathbf{B} = \mathbf{e}_1 \mathbf{e}_2$  is  $\pi/4$ .

**1.5.2. Theorem.** Let  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$  be linearly independent vectors. Then  $\mathbf{B} = \mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \cdots \wedge \mathbf{u}_k$  is a  $k$ -blade.

**Proof.** By induction. We perform a GA version of Gram-Schmidt orthogonalization on the  $\mathbf{u}_i$ . Note how the rejection operator of GA simplifies the VA version.

The  $k = 1$  case is clear. For ease of notation, we induct from  $k = 2$  to  $k = 3$ . Thus we assume that  $\mathbf{u}_1 \wedge \mathbf{u}_2$  is a 2-blade, i.e.,  $\mathbf{u}_1 \wedge \mathbf{u}_2 = \mathbf{b}_1 \mathbf{b}_2$  with  $\mathbf{b}_1 \perp \mathbf{b}_2$ . See Fig. 4.

Let  $\mathbf{b}_3 = \{\mathbf{u}_3 \wedge (\mathbf{u}_1 \wedge \mathbf{u}_2)\}/(\mathbf{u}_1 \wedge \mathbf{u}_2)$  be the rejection (Eq. (1.14)) of  $\mathbf{u}_3$  from the subspace  $\mathbf{u}_1 \wedge \mathbf{u}_2$ .<sup>5</sup> Since  $\mathbf{u}_1, \mathbf{u}_2$ , and  $\mathbf{u}_3$  are linearly independent,  $\mathbf{b}_3 \neq 0$ . We have

$$\mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \mathbf{u}_3 = (\mathbf{u}_1 \wedge \mathbf{u}_2) \wedge \mathbf{b}_3 = (\mathbf{b}_1 \mathbf{b}_2) \wedge \mathbf{b}_3 = \mathbf{b}_1 \mathbf{b}_2 \mathbf{b}_3, \quad (1.18)$$

a 3-blade. The last equality follows from the definition of  $\wedge$ , Eq. (1.8).

Expressing a blade as an outer product of linearly independent vectors is often more convenient than expressing it as a geometric product of nonzero orthogonal vectors.

**Corollary.** The  $k$ -volume spanned by the  $\mathbf{u}_i$  is  $|\mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \cdots \wedge \mathbf{u}_k|$ . (We first saw this in Fig. 1, where an outer product of two vectors represents an area in a plane.)

**Proof.** By construction the  $\mathbf{b}_i$  span the same  $k$ -volume as the  $\mathbf{u}_i$ . (See Fig. 4.) The result now follows from Sec. 1.5.1 and Eq. (1.18).

We are beginning to see some of the power of GA promised in the introduction. Geometric objects are represented directly as members of the algebra. Examples: Oriented subspaces and  $k$ -volumes are represented by blades. Geometric relationships are expressed algebraically. Examples:  $\mathbf{x} \in \mathbf{B} \Leftrightarrow \mathbf{x} \wedge \mathbf{B} = 0$ ,  $\mathbf{x} \perp \mathbf{B} \Leftrightarrow \mathbf{x} \cdot \mathbf{B} = 0$ . Geometric operations on objects are represented by algebraic operations. Examples:  $\mathbf{A}^* = \mathbf{A}/\mathbf{I}$ ,  $\mathbf{a}_{\perp} = (\mathbf{a} \wedge \mathbf{B})/\mathbf{B}$ ,  $\mathbf{a}_{\parallel} = (\mathbf{a} \cdot \mathbf{B})/\mathbf{B}$ . The result of an operation can be substituted in other expressions, which can then be manipulated algebraically. Example: Our use of Eqs. (1.13) and (1.14) in the derivation of Eq. (1.16).

All this without using coordinates. Geometric algebra is *coordinate-free*: coordinates are needed only when specific objects or operations are under consideration.

These features of GA fit naturally into the modern techniques of object oriented computer programming.

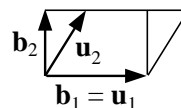


Fig. 4:  $\mathbf{u}_1 \wedge \mathbf{u}_2 = \mathbf{b}_1 \mathbf{b}_2$ .

<sup>3</sup> If  $\mathbf{A} \cdot \mathbf{B} \neq 0$ , then  $P_{\mathbf{B}}(\mathbf{A})$  is a  $j$ -blade in  $\mathbf{B}$  ( $j = \text{grade}(\mathbf{A})$ ). If  $\mathbf{A} \cdot \mathbf{B} = 0$ , then  $P_{\mathbf{B}}(\mathbf{A}) = 0$ . This says that the  $j$ -volume of the projection is 0. For example,  $P_{\mathbf{e}_1 \mathbf{e}_2}(\mathbf{e}_2 \mathbf{e}_3) = 0$ .

<sup>4</sup> Always  $\mathbf{A} \cdot \mathbf{B} \subseteq \mathbf{B}$ . Thus  $|(\mathbf{A} \cdot \mathbf{B})/\mathbf{B}| = |\mathbf{A} \cdot \mathbf{B}|/|\mathbf{B}|$ .

<sup>5</sup> To compute  $\mathbf{b}_3$ , use  $(\mathbf{u}_1 \wedge \mathbf{u}_2)^{-1} = (\mathbf{b}_1 \mathbf{b}_2)^{-1} = \mathbf{b}_2^{-1} \mathbf{b}_1^{-1} = \mathbf{b}_2 \mathbf{b}_1 / |\mathbf{b}_1|^2 |\mathbf{b}_2|^2$ .

## 2 Algebra

### 2.1 Complex Numbers

**2.1.1. Complex numbers.** Let  $\mathbf{i}$  be the pseudoscalar of a plane in  $\mathbf{R}^n$ . A *complex number* in the plane is a scalar + bivector:  $a + \mathbf{i}b$ . Since  $\mathbf{i}^2 = -1$ , these GA complex numbers are isomorphic to the usual complex numbers. But be careful: GA complex numbers are not points in a plane or 2D vectors.

Let  $\theta$  be the angle between vectors  $\mathbf{u}$  and  $\mathbf{v}$ . The fundamental identity, Eq. (1.5), shows that the product of two vectors is a complex number:

$$\mathbf{u}\mathbf{v} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta + \mathbf{i} |\mathbf{u}| |\mathbf{v}| \sin \theta.$$

Define  $e^{i\theta} = \cos \theta + \mathbf{i} \sin \theta$ . Write the complex number  $\mathbf{u}\mathbf{v}$  in *polar form*:

$$\mathbf{u}\mathbf{v} = |\mathbf{u}| |\mathbf{v}| e^{i\theta}. \quad (2.1)$$

Every complex number  $a + \mathbf{i}b$  can be put in the polar form  $re^{i\theta}$  by setting  $r = \sqrt{a^2 + b^2}$ ,  $\cos \theta = a/r$ , and  $\sin \theta = b/r$ .

The pseudoscalar  $\mathbf{i}$  is independent of any orthonormal basis for its plane, so GA complex numbers have no geometrically irrelevant real and imaginary axes. Thus  $\theta$  in  $re^{i\theta}$  is not an angle with respect to a real axis; it is intrinsic to the number.

The usual complex number  $i$  is not needed. It is not part of geometric algebra.

**2.1.2. 3D complex numbers.** Let  $\mathbf{H} = \{h_0 + \mathbf{H} \in \mathbf{G}^3 \mid h_0 \text{ a scalar, } \mathbf{H} \text{ a bivector}\}$ .

**Theorem.**  $\mathbf{H}$  is (i) the set of complex numbers in  $\mathbf{G}^3$ , and (ii) a subalgebra of  $\mathbf{G}^3$ .

**Proof.** Consider the multivector basis  $\{1, \mathbf{i}_1 = \mathbf{e}_3\mathbf{e}_2, \mathbf{i}_2 = \mathbf{e}_1\mathbf{e}_3, \mathbf{i}_3 = \mathbf{e}_2\mathbf{e}_1\}$  for  $\mathbf{H}$ .

(i) Since  $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$  is a basis for bivectors in  $\mathbf{G}^3$ , every complex number in  $\mathbf{G}^3$  is in  $\mathbf{H}$ . For the converse, let  $\mathbf{H} = a_1\mathbf{i}_1 + a_2\mathbf{i}_2 + a_3\mathbf{i}_3$  be a bivector in  $\mathbf{G}^3$ . Then  $\mathbf{H}^* = -(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3)$ , a vector. Thus  $\mathbf{H} = -\mathbf{H}^{**} = \mathbf{i}h$ , where  $h$  is a scalar and  $\mathbf{i}$  is the plane orthogonal to  $\mathbf{H}^*$ . Therefore  $h_0 + \mathbf{H} = h_0 + \mathbf{i}h$  is a complex number.

(ii) To see that  $\mathbf{H}$  is a subalgebra of  $\mathbf{G}^3$ , observe the following products of members of the basis for  $\mathbf{H}$  above:  $\mathbf{i}_1^2 = \mathbf{i}_2^2 = \mathbf{i}_3^2 = -1$ ,  $\mathbf{i}_1\mathbf{i}_2 = \mathbf{i}_3$ ,  $\mathbf{i}_2\mathbf{i}_3 = \mathbf{i}_1$ ,  $\mathbf{i}_3\mathbf{i}_1 = \mathbf{i}_2$ . ■

The identities in (ii) characterize the quaternions. Thus the subalgebra  $\mathbf{H}$  of  $\mathbf{G}^3$  is the quaternion algebra.

Traditionally, quaternions have been considered as scalar + vector. So considered, they have not been satisfactorily united with vectors in a common mathematical system [3, 11]. Considered here as scalar + bivector, they are united with vectors in GA.

For many, quaternions are a 19<sup>th</sup> century mathematical curiosity. But many roboticists, aerospace engineers, and gamers know better: quaternions are the best way to represent rotations in 3D, as we will see in Sec. 2.2.1.

Complex numbers and quaternions are only two of the many algebraic systems embedded in GA. We shall see several more examples. Having these systems embedded in a common structure reveals and clarifies relationships between them. For example, we have seen that the product of two vectors is a complex number.

The exterior (Grassmann) algebra is another system embedded in GA: it consists of multivectors with the operations scalar multiplication, +, and  $\wedge$ . It is thus only a part of GA, not using the inner or geometric products. Taking advantage of these products simplifies the exterior algebra. For example, the Hodge star dual is the GA dual, Eq. (1.10), up to a sign. This algebraic definition allows easy manipulation.

## 2.2 Rotations

**2.2.1. Rotations in  $\mathbf{R}^3$ .** Orthogonal matrices and Euler angles are among the many representations of rotations in  $\mathbf{R}^3$  in use today. GA provides a better representation.

In GA the rotation of a vector  $\mathbf{u}$  by an angle  $\theta$  around an axis  $\mathbf{n}$  is given by

$$\underline{R}(\mathbf{u}) = e^{-\mathbf{n}\mathbf{I}\theta/2} \mathbf{u} e^{\mathbf{n}\mathbf{I}\theta/2}. \quad (2.2)$$

(An underbar denotes a linear transformation.) The formula expresses the rotation simply and directly in terms of  $\theta$  and  $\mathbf{n}$ . We prove this in Sec. 2.2.3. We write  $\underline{R}(\mathbf{u}) = R\mathbf{u}R^{-1}$ , where the unit complex number  $R = e^{-\mathbf{n}\mathbf{I}\theta/2}$  represents the rotation.

In  $\mathbf{R}^3$ , the unit complex numbers are precisely the unit quaternions (Sec. 2.1.2). Thus in  $\mathbf{R}^3$ , unit quaternions represent rotations.

**2.2.2. Rotations compose simply.** Follow the rotation  $R_1$  with the rotation  $R_2$ . Then their composition is represented by the product  $R = R_2R_1$ :

$$M \rightarrow R_2(R_1MR_1^{-1})R_2^{-1} = RMR^{-1}.$$

Since the product of unit quaternions is another, we have a simple proof of the important fact that the composition of 3D rotations is a rotation.

As an example, a rotation by  $90^\circ$  around the  $\mathbf{e}_1$  axis followed by a rotation of  $90^\circ$  around the  $\mathbf{e}_2$  axis is a rotation of  $120^\circ$  around  $\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$ :

$$\begin{aligned} & (\cos 45^\circ - \mathbf{e}_2\mathbf{I} \sin 45^\circ) (\cos 45^\circ - \mathbf{e}_1\mathbf{I} \sin 45^\circ) \\ &= \cos 60^\circ - \frac{\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3}{\sqrt{3}} \mathbf{I} \sin 60^\circ. \end{aligned}$$

In the general case, the composite quaternion is  $\cos(\theta/2) + \mathbf{n}\mathbf{I} \sin(\theta/2)$ , where  $\mathbf{n}$  is a unit vector, the axis of the composite rotation.

Quaternions are superior to orthogonal matrices in representing 3D rotations: (i) It is easier to determine the quaternion representation of a rotation than the matrix representation,<sup>6</sup> (ii) It is more efficient to multiply quaternions than matrices, (iii) It is more efficient to determine  $\mathbf{n}$  directly from  $\mathbf{Q}$  than to compute it as an eigenvector of the product matrix, and (iv) If a quaternion product is not quite normalized due to rounding errors, then divide by its norm to make it so; if a product of orthogonal matrices is not orthogonal, then use Gram-Schmidt orthonormalization, which is expensive and not canonical.

**2.2.3. Rotations in  $\mathbf{R}^n$ .** In GA, an *angle* is a bivector  $\mathbf{i}\theta$ :  $\mathbf{i}$  specifies the plane of the angle and  $\theta$  specifies its size.<sup>7</sup>

In  $\mathbf{R}^n$ , a rotation can be specified by an angle  $\mathbf{i}\theta$ :  $\mathbf{i}$  specifies the plane of the rotation and  $\theta$  specifies the amount of rotation. (Only in 3D does a rotation have an axis, the unique direction normal to the plane of the rotation.)

Let the rotation carry the vector  $\mathbf{u}$  to the vector  $\mathbf{v}$ . First suppose that  $\mathbf{u} \in \mathbf{i}$ . Left multiply  $\mathbf{u}\mathbf{v} = |\mathbf{u}||\mathbf{v}|e^{i\theta}$ , Eq. (2.1), by  $\mathbf{u}$  and use  $\mathbf{u}^2 = |\mathbf{u}|^2 = |\mathbf{u}||\mathbf{v}|$  to obtain  $\mathbf{v} = \mathbf{u}e^{i\theta}$ ;  $e^{i\theta}$  rotates  $\mathbf{u}$  to  $\mathbf{v}$ . The analog in standard complex algebra is familiar.

<sup>6</sup> *Rodrigues' rotation formula* gives the matrix representation of a 3D rotation [10].

<sup>7</sup> As an aside, let  $\mathbf{i}\theta$  be the angle between a vector  $\mathbf{a}$  and a blade  $\mathbf{B}$ . (See Fig. 2.) Then  $\mathbf{i}\tan(\theta) = \mathbf{a}_\perp \mathbf{a}_\parallel^{-1}$  (opposite over adjacent). To see this, note that  $\mathbf{a}_\perp \mathbf{a}_\parallel^{-1}$ , the product of orthogonal vectors, is a bivector in the plane  $\mathbf{i}$  they span. And  $|\mathbf{a}_\perp \mathbf{a}_\parallel^{-1}| = |\mathbf{a}_\perp| |\mathbf{a}_\parallel|^{-1} = \tan \theta$ , where  $\theta$  is the scalar angle between  $\mathbf{a}$  and  $\mathbf{B}$ .

Now consider a general  $\mathbf{u}$ . Decompose  $\mathbf{u}$  with respect to  $\mathbf{i}$ :  $\mathbf{u} = \mathbf{u}_\perp + \mathbf{u}_\parallel$  ( $\mathbf{u}_\perp \perp \mathbf{i}$ ,  $\mathbf{u}_\parallel \in \mathbf{i}$ ). The rotation rotates  $\mathbf{u}_\parallel$  as above but does not affect  $\mathbf{u}_\perp$ . Thus

$$\begin{aligned} \mathbf{v} &= \mathbf{u}_\perp + \mathbf{u}_\parallel e^{i\theta} \\ &= \mathbf{u}_\perp e^{-i\theta/2} e^{i\theta/2} + \mathbf{u}_\parallel e^{i\theta/2} e^{i\theta/2} \\ &= e^{-i\theta/2} \mathbf{u}_\perp e^{i\theta/2} + e^{-i\theta/2} \mathbf{u}_\parallel e^{i\theta/2} \quad (\text{since } \mathbf{u}_\perp \mathbf{i} = \mathbf{i} \mathbf{u}_\perp \text{ and } \mathbf{u}_\parallel \mathbf{i} = -\mathbf{i} \mathbf{u}_\parallel) \\ &= e^{-i\theta/2} \mathbf{u} e^{i\theta/2}. \end{aligned} \tag{2.3}$$

This reduces to the formula for rotations in 3D, Eq. (2.2). For in  $\mathbf{G}^3$ ,  $\mathbf{n} = \mathbf{i}^* = \mathbf{i}/\mathbf{I}$ .

**2.2.4. Rotations extend to automorphisms.** Write Eq. (2.3) as  $\underline{R}(\mathbf{u}) = R\mathbf{u}R^{-1}$ , where  $R = e^{-i\theta/2}$ . Extend to  $\underline{R}(M) = RMR^{-1}$  for all  $M \in \mathbf{G}^n$ . We show that  $\underline{R}$  is an automorphism of  $\mathbf{G}^n$ . It preserves the geometric product:

$$\underline{R}(M_1 M_2) = R(M_1 M_2)R^{-1} = (R M_1 R^{-1}) (R M_2 R^{-1}) = \underline{R}(M_1) \underline{R}(M_2).$$

Next let  $\mathbf{u}_1 \mathbf{u}_2 \cdots \mathbf{u}_k$  be a product of orthogonal nonzero vectors, a  $k$ -blade. Then

$$\underline{R}(\mathbf{u}_1 \mathbf{u}_2 \cdots \mathbf{u}_k) = \underline{R}(\mathbf{u}_1) \underline{R}(\mathbf{u}_2) \cdots \underline{R}(\mathbf{u}_k),$$

a product of  $k$  orthogonal nonzero vectors, is also a  $k$ -blade. Thus  $\underline{R}$  preserves grades.

The definitions of  $\cdot$  and  $\wedge$ , Eqs. (1.6) and (1.8), involve only the geometric product and taking grades. Thus  $\underline{R}$  also preserves  $\cdot$  and  $\wedge$ .

Subspaces are now as easy to rotate as vectors. For example, since  $\mathbf{u}_1 \wedge \mathbf{u}_2$  rotated is  $\underline{R}(\mathbf{u}_1) \wedge \underline{R}(\mathbf{u}_2)$  and  $\underline{R}(\mathbf{u}_1 \wedge \mathbf{u}_2) = \underline{R}(\mathbf{u}_1) \wedge \underline{R}(\mathbf{u}_2)$ ,  $\underline{R}$  rotates  $\mathbf{u}_1 \wedge \mathbf{u}_2$ .

**2.2.5. The orthogonal group.** In  $n$ D,  $n > 3$ , rotations of the form Eq. (2.3) are not closed under composition. They generate the orientation and inner product preserving special orthogonal group  $\text{SO}(n)$ . Given  $e^{-i\theta/2}$ , choose, using Eq. (2.1), unit vectors  $\mathbf{b}_1$  and  $\mathbf{b}_2$  so that  $e^{-i\theta/2} = \mathbf{b}_1 \mathbf{b}_2$ . Then the rotation Eq. (2.3) can be written  $\mathbf{u} \rightarrow (\mathbf{b}_1 \mathbf{b}_2) \mathbf{u} (\mathbf{b}_1 \mathbf{b}_2)^{-1}$ . We may drop the normalization  $|\mathbf{b}_i| = 1$  in this formula. Thus a member of  $\text{SO}(n)$  can be represented by a product of an even number of nonzero vectors  $B = \mathbf{b}_1 \mathbf{b}_2 \cdots \mathbf{b}_{2k}$ :  $\mathbf{u} \rightarrow B \mathbf{u} B^{-1}$  (Sec. 2.2.2).

Rotations and reflections in hyperplanes generate the inner product preserving orthogonal group  $\text{O}(n)$ . From Eq. 1.16, the reflection of  $\mathbf{u}$  in the hyperplane normal to  $\mathbf{b}$  is  $-\mathbf{b} \mathbf{u} \mathbf{b}^{-1}$ . Thus a member of  $\text{O}(n)$  is represented by a product  $B$  of nonzero vectors:  $\mathbf{u} \rightarrow \pm B \mathbf{u} B^{-1}$ . At most  $n$  vectors are needed (Cartan-Dieudonné theorem).

Eq. (2.3) represents a rotation of  $2\pi$  by  $e^{-i2\pi/2} = -1$ . The products  $B$  above represent the simply connected double covering group of  $\text{O}(n)$ , with  $\pm B$  representing the same element of  $\text{O}(n)$ . This is another algebraic system embedded in GA. Matrices do not represent the double cover.

“A particular area where geometric algebra provides a unifying language is in the description of rotations. The most fundamental modern treatments such as those of Abraham, Marsden, Arnol'd and Simo use differential topology and describe rotations in terms of the Lie group  $\text{SO}(3)$ . A rotation is thus an element of a differentiable manifold, and combinations of rotations are described using the group action. Infinitesimal rotations and rotational velocities live in the tangent bundle  $\text{TSO}(3)$ , a differentiable manifold with distinctly non-trivial topology, from where they can be transported to the tangent space at the identity, identifiable with the Lie algebra  $\mathfrak{so}(3)$ . As throughout all of the differentiable topology formulation of mechanics, a proliferation of manifolds occurs. ... In geometric algebra there is no such proliferation of manifolds: the mathematical arena consists only of elements of the algebra and nothing more.” [9]

**2.2.6. Pauli's electron theory.** In 1927 Wolfgang Pauli published a quantum theory of an electron interacting with an electromagnetic field. Pauli's theory does not take Einstein's special relativity theory into account. A year later Paul Dirac published a relativistic quantum theory of the electron. We compare the VA and GA formulations of Pauli's theory in this section and of Dirac's theory in Sec. 2.4.5.

An electron has a property called *spin*, with a fixed value  $\frac{1}{2}$ . For our purposes think of the electron as spinning about an axis  $\mathbf{s}$ . The Pauli and Dirac theories describe the position and spin axis of spin- $\frac{1}{2}$  particles in an electromagnetic field. The field can change the position and the axis. We consider here only the case of a particle at rest in a uniform but time varying magnetic field. This case is important, for example, in the theory of nuclear magnetic resonance.

In the VA formulation of classical electromagnetism, a magnetic field is represented by a vector  $\mathbf{b} \in \mathbf{R}^3$ . In the  $\mathbf{G}^3$  formulation, the field is represented by the bivector  $\mathbf{B} = -\mathbf{b}^*$ , the plane orthogonal to  $\mathbf{b}$ . We saw in Sec. 2.2.3 that  $\mathbf{B}$  and its scalar multiples specify rotations.

The basic physical fact is that from time  $t$  to  $t + dt$ ,  $\mathbf{s}$  rotates through the angle  $\gamma \mathbf{B}(t) dt$ , where  $\gamma$  is a constant. (Thus if  $\mathbf{B}$  is constant in time, then  $\mathbf{s}$  precesses with respect to the plane  $\mathbf{B}$  with constant angular speed  $\gamma|\mathbf{B}|$ .) From Sec. 2.2.3, the rotation is represented by the unit quaternion  $e^{-\frac{1}{2}\gamma \mathbf{B}(t) dt}$ .

In Pauli's theory the spin is not represented by  $\mathbf{s}(t)$ , but by the unit quaternion  $\psi(t)$  which rotates  $\mathbf{s}(0)$  to  $\mathbf{s}(t)$ . From Sec. 2.2.2 we have the composition

$$\psi(t + dt) = e^{-\frac{1}{2}\gamma \mathbf{B}(t) dt} \psi(t) \approx \left(1 - \frac{\gamma}{2} \mathbf{B}(t) dt\right) \psi(t).$$

A little manipulation gives the GA version of *Pauli's equation* (for the spin):  $\psi' = -\frac{\gamma}{2} \mathbf{B} \psi$ .

The formulation of Pauli's theory just outlined uses GA, a general purpose mathematical language. The VA formulation, the one taught and used almost everywhere, requires one to learn specialized mathematics. It represents a spin as a unit complex vector  $\begin{pmatrix} a+bi \\ c+di \end{pmatrix}$ . These vectors are isomorphic to the unit quaternions, and so represent rotations. But there is no hint of this in elementary treatments of the VA theory.

The VA formulation of Pauli's theory uses the three complex *Pauli matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Real linear combinations of these matrices, together with the identity matrix, generate the *Pauli algebra*. In the theory the  $\sigma_i$  are associated, but not identified, with orthogonal directions in  $\mathbf{R}^3$ . But they should be so identified. For the Pauli algebra is isomorphic to  $\mathbf{G}^3$ , with  $\sigma_j \leftrightarrow \mathbf{e}_j$ !

The obvious-at-a-glance GA identities  $\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 = \mathbf{I}$ ,  $\mathbf{e}_1 \mathbf{e}_2 = \mathbf{I} \mathbf{e}_3$ ,  $\mathbf{e}_1 \mathbf{e}_2 = -\mathbf{e}_2 \mathbf{e}_1$ , and  $\mathbf{e}_1 \mathbf{e}_2 - \mathbf{e}_2 \mathbf{e}_1 = 2\mathbf{I} \mathbf{e}_3$  correspond to important but not-so-obvious matrix identities of the  $\sigma_j$ . (Note that  $\mathbf{I} = \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \leftrightarrow \sigma_1 \sigma_2 \sigma_3 = i\mathbf{I}$ , where  $\mathbf{I}$  is the identity matrix.)

The VA formulation uses the "vector" of matrices  $\boldsymbol{\sigma} \equiv (\sigma_1, \sigma_2, \sigma_3)$  to associate a vector  $\mathbf{u} = u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 + u_3 \mathbf{e}_3$  with the matrix  $\boldsymbol{\sigma} \cdot \mathbf{u} \equiv u_1 \sigma_1 + u_2 \sigma_2 + u_3 \sigma_3$ . The identity  $(\boldsymbol{\sigma} \cdot \mathbf{u})(\boldsymbol{\sigma} \cdot \mathbf{v}) = (\mathbf{u} \cdot \mathbf{v}) \mathbf{I} + i \boldsymbol{\sigma} \cdot (\mathbf{u} \times \mathbf{v})$  plays an important role. It is a clumsy way to express a fundamental geometrical fact. For the identity is a matrix representation of our fundamental GA identity  $\mathbf{u} \mathbf{v} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v}$ , Eq. (1.5).

## 2.3 Linear Algebra

Here is a sampling of GA ideas in linear algebra.

**2.3.1. Linear independence.** An inspection of the proof of Theorem 1.5.2 shows that vectors  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$  are linearly independent if and only if  $\mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \dots \wedge \mathbf{u}_k \neq 0$ . Geometrically, this states that the vectors span a parallelepiped of nonzero  $k$ -volume.

**2.3.2. Outermorphisms.** A linear transformation  $\underline{f}$  on  $\mathbf{R}^n$  extends uniquely to an *outermorphism*: a linear transformation  $\underline{f}$  on all of  $\mathbf{G}^n$  with  $\underline{f}(A \wedge B) = \underline{f}(A) \wedge \underline{f}(B)$  for all multivectors  $A$  and  $B$ . We have  $\underline{f}g = \underline{f}g$  and  $\underline{f}^{-1} = \underline{f}^{-1}$ .

For vectors  $\mathbf{u}$  and  $\mathbf{v}$ ,  $\underline{f}(\mathbf{u} \wedge \mathbf{v}) = \underline{f}(\mathbf{u}) \wedge \underline{f}(\mathbf{v})$  expresses algebraically the geometric fact that a linear transformation maps the parallelogram and plane determined by  $\mathbf{u}$  and  $\mathbf{v}$  to those determined by  $\underline{f}(\mathbf{u})$  and  $\underline{f}(\mathbf{v})$ . There is no analog in VA, even in  $\mathbf{R}^3$  when representing planes by normal vectors. For example, let  $\underline{f}$  transform the plane  $\mathbf{e}_1 \wedge \mathbf{e}_2$  to some other plane and  $\mathbf{e}_3$  to itself. Then  $\underline{f}(\mathbf{e}_1 \times \mathbf{e}_2) \neq \underline{f}(\mathbf{e}_1) \times \underline{f}(\mathbf{e}_2)$ .

Outermorphisms are grade preserving. For we have

$$\underline{f}(\mathbf{e}_1 \wedge \dots \wedge \mathbf{e}_k) = \underline{f}(\mathbf{e}_1) \wedge \dots \wedge \underline{f}(\mathbf{e}_k).$$

If  $\underline{f}(\mathbf{e}_1), \dots, \underline{f}(\mathbf{e}_k)$  are linearly independent, then from Theorem 1.5.2, the right side is a non-zero  $k$ -vector. If they are linearly dependent, then the right side is 0, again a  $k$ -vector.

**2.3.3. Determinants.** Since  $\mathbf{I}$  is an  $n$ -vector and  $\underline{f}$  is grade preserving,  $\underline{f}(\mathbf{I})$  is also an  $n$ -vector. It is thus a multiple of  $\mathbf{I}$ :  $\underline{f}(\mathbf{I}) = \det(\underline{f})\mathbf{I}$ , where we have defined the *determinant*  $\det(\underline{f})$  of  $\underline{f}$ . This simple definition tells us what the determinant of a linear transformation *is*: the factor by which it multiplies  $n$ -volumes. Compare this to the usual algebraic definition of the determinant of the *matrix* of the transformation.

The GA definition makes obvious the product rule:  $\det(\underline{f}g) = \det(\underline{f})\det(g)$ .

**2.3.4. Eigenblades.** The linear transformation  $\underline{f}(\mathbf{e}_1) = 2\mathbf{e}_2$ ,  $\underline{f}(\mathbf{e}_2) = -3\mathbf{e}_1$  has no real eigenvalues. It does have complex eigenvalues, but they provide no geometric insight. In GA,  $\underline{f}$  has *eigenblade*  $\mathbf{e}_1 \wedge \mathbf{e}_2$  with eigenvalue 6:  $\underline{f}(\mathbf{e}_1 \wedge \mathbf{e}_2) = 6(\mathbf{e}_1 \wedge \mathbf{e}_2)$ . The transformation multiplies areas in the plane  $\mathbf{e}_1 \wedge \mathbf{e}_2$  by 6.

**2.3.5. Cramer's rule.** Problem: In  $\mathbf{R}^4$  solve  $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4$  for, say,  $c_2$ . Solution: Outer multiply the equation by  $\mathbf{u}_1$  on the left and  $\mathbf{u}_3 \wedge \mathbf{u}_4$  on the right. Then use  $\mathbf{u}_i \wedge \mathbf{u}_i = 0$ :

$$\mathbf{u}_1 \wedge \mathbf{v} \wedge \mathbf{u}_3 \wedge \mathbf{u}_4 = c_2 (\mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \mathbf{u}_3 \wedge \mathbf{u}_4).$$

If the  $\mathbf{u}_i$  are linearly independent, then the outer product on the right is invertible. Then we have a *symbolic* formula for  $c_2$  which can be further manipulated.

**2.3.6. Fourier expansion.** Let  $\mathbf{e}_J$  denote a product of  $\mathbf{e}_j$ 's with increasing indexes. For example, if  $J = \{2, 4\}$ , then  $\mathbf{e}_J = \mathbf{e}_2 \mathbf{e}_4$ . Since the set of all  $\mathbf{e}_J$  is a basis of  $\mathbf{G}^n$  and  $\mathbf{e}_J^{-1} = \pm \mathbf{e}_J$ , the set of all  $\mathbf{e}_J^{-1}$  is also a basis. Multiply the expansion  $A = \sum_J \mathbf{e}_J^{-1} a_J$  on the left by  $\mathbf{e}_K$ , take the scalar part, and use  $\langle \mathbf{e}_K \mathbf{e}_J^{-1} \rangle_0 = \delta_K^J$  to give  $\langle \mathbf{e}_K A \rangle_0 = a_K$ . This gives the *Fourier expansion*  $A = \sum_J \mathbf{e}_J^{-1} \langle \mathbf{e}_J A \rangle_0$ .

## 2.4 The Spacetime Algebra

**2.4.1. Indefinite metrics.** GA readily extends from  $\mathbf{R}^n$  to vector spaces with an indefinite metric. In  $\mathbf{R}^{p,q}$  every orthonormal basis has  $p$   $\mathbf{e}_i$ 's with  $\mathbf{e}_i \cdot \mathbf{e}_i = 1$  and  $q$   $\mathbf{e}_i$ 's with  $\mathbf{e}_i \cdot \mathbf{e}_i = -1$ . Correspondingly, in  $\mathbf{G}^{p,q}$  there are  $p$   $\mathbf{e}_i$ 's with  $\mathbf{e}_i \cdot \mathbf{e}_i = \mathbf{e}_i^2 = 1$  and  $q$   $\mathbf{e}_i$ 's with  $\mathbf{e}_i \cdot \mathbf{e}_i = \mathbf{e}_i^2 = -1$ . Most of the properties of  $\mathbf{G}^n$  remain valid in  $\mathbf{G}^{p,q}$ . In particular, we still have Eq. (1.2):  $\mathbf{v}\mathbf{u} = -\mathbf{u}\mathbf{v}$  for orthogonal vectors  $\mathbf{u}$  and  $\mathbf{v}$ .

**2.4.2. The spacetime algebra.** Special relativity assigns *spacetime coordinates*  $(t, x, y, z)$  to a definite time and place. In VA, spacetime is represented by  $\mathbf{R}^{1,3}$ . An orthonormal basis has a time direction  $\mathbf{e}_0$  with  $\mathbf{e}_0 \cdot \mathbf{e}_0 = 1$  and three space directions  $\mathbf{e}_i$  with  $\mathbf{e}_i \cdot \mathbf{e}_i = -1$ . In GA, spacetime is represented by the *spacetime algebra*  $\mathbf{G}^{1,3}$ .

To get very far in 4D relativistic physics, methods beyond VA must be employed. Usually this is the tensor algebra or exterior algebra of  $\mathbf{R}^{1,3}$ , which are very different from anything a student has seen before in VA. There is no need for this if  $\mathbf{G}^{1,3}$  is employed.

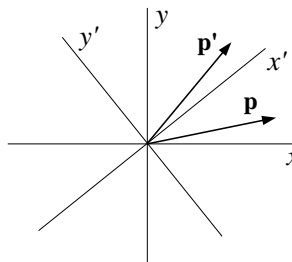
**2.4.3. Boosts.** Consider a second coordinate system  $(t', x', y', z')$  whose spatial points  $(x', y', z')$  move with respect to those of the first with velocity  $\mathbf{v}$ . The origins  $(0, 0, 0, 0)$  of the two systems coincide. And their spatial axes are not rotated with respect to each other.

You can think of the  $\{\mathbf{e}'_i\}$  basis as moving with the new coordinate system. However, translating a vector does not change it, and the  $\mathbf{e}'_i$  are, with the  $\mathbf{e}_i$ , fixed vectors in  $\mathbf{R}^{1,3}$ .

The transformation  $\mathbf{p} = \sum_{i=0}^3 a_i \mathbf{e}_i \rightarrow \mathbf{p}' = \sum_{i=0}^3 a_i \mathbf{e}'_i$  maps  $\mathbf{p}$  to the vector  $\mathbf{p}'$  which “looks the same” in the moving system. This active transformation is called a *boost*, or *Lorentz transformation*. Fig. 5 illustrates an analogy to a rotation in a plane.

The analogy is appropriate. For it can be shown that the boost is a rotation in the  $\mathbf{e}_0\mathbf{v}$  plane:  $\mathbf{p} \rightarrow e^{-\mathbf{e}_0\hat{\mathbf{v}}\alpha/2} \mathbf{p} e^{\mathbf{e}_0\hat{\mathbf{v}}\alpha/2}$ . (C. f., Eq. (2.3).) Here we have set  $\mathbf{v} = \tanh(\alpha)\hat{\mathbf{v}}$ , where  $\hat{\mathbf{v}}^2 = -1$ .<sup>8</sup> And the exponential function is defined for all multivectors  $M$  by  $e^M = \sum_{i=0}^{\infty} M^i/i!$ . You can verify that  $e^{\mathbf{e}_0\hat{\mathbf{v}}\alpha/2} = \cosh(\alpha/2) + \mathbf{e}_0\hat{\mathbf{v}}\sinh(\alpha/2)$ .

VA represents boosts with  $4 \times 4$  orthogonal matrices on  $\mathbf{R}^{1,3}$ . The GA exponential function representation has advantages similar to the GA exponential function representation of rotations in  $\mathbf{R}^3$ . (See Sec. 2.2.2.) We will see an example in the next section, where we compute the composition of two boosts. The results obtained there in a few lines are much more difficult to obtain with VA, so difficult that it is a rare relativity text which derives them.



**Fig. 5:** An active “boost”  $\mathbf{p} \rightarrow \mathbf{p}'$  in a plane.

<sup>8</sup> We use units of space and time in which the speed of light  $c = 1$ . For example, time might be measured in seconds and distance in light-seconds. Since speeds of material objects are less than  $c$ ,  $-1 < \mathbf{v}^2 \leq 0$ . Thus  $\tanh(\alpha) = (-\mathbf{v}^2)^{\frac{1}{2}}$ .

**2.4.4. Composition of boosts.** By definition, a boost is free of any spatial rotation. Perhaps surprisingly, a composition of boosts is not. The composition of boosts  $e^{-\mathbf{e}_0 \hat{\mathbf{v}} \beta / 2}$  and  $e^{-\mathbf{e}_0 \hat{\mathbf{u}} \alpha / 2}$  can be written as a spatial rotation followed by a boost:

$$e^{-\mathbf{e}_0 \hat{\mathbf{v}} \beta / 2} e^{-\mathbf{e}_0 \hat{\mathbf{u}} \alpha / 2} = e^{-\mathbf{e}_0 \hat{\mathbf{w}} \delta / 2} e^{-i\theta / 2}. \quad (2.4)$$

Active transformations leave reference frames unchanged, so we can express the boosts in Eq. (2.4) in terms of the basis vector  $\mathbf{e}_0$  and vectors orthogonal to it ( $\hat{\mathbf{u}}$ ,  $\hat{\mathbf{v}}$ , and  $\hat{\mathbf{w}}$ ).

The rotation  $e^{-i\theta / 2}$  is called a *Thomas rotation*. It is an important physical effect.

We prove Eq. (2.4) by solving uniquely for  $\hat{\mathbf{w}}$ ,  $\delta$ ,  $\mathbf{i}$ ,  $\theta$  on the right side. Expand the exponentials and equate the terms with an  $\mathbf{e}_0$  factor and those without:

$$\begin{aligned} s_\alpha c_\beta \hat{\mathbf{u}} + c_\alpha s_\beta \hat{\mathbf{v}} &= s_\delta \hat{\mathbf{w}} e^{-i\theta / 2}, \\ c_\alpha c_\beta - s_\alpha s_\beta \hat{\mathbf{v}} \cdot \hat{\mathbf{u}} &= c_\delta e^{-i\theta / 2}, \end{aligned} \quad (2.5)$$

where  $s_\alpha = \sinh(\alpha/2)$ ,  $c_\beta = \cosh(\beta/2)$ , etc. Divide to obtain  $\hat{\mathbf{w}}$  and  $\delta$ :<sup>9</sup>

$$\tanh(\delta/2) \hat{\mathbf{w}} = \frac{s_\alpha c_\beta \hat{\mathbf{u}} + c_\alpha s_\beta \hat{\mathbf{v}}}{c_\alpha c_\beta - s_\alpha s_\beta \hat{\mathbf{v}} \cdot \hat{\mathbf{u}}} = \frac{\tanh(\alpha/2) \hat{\mathbf{u}} + \tanh(\beta/2) \hat{\mathbf{v}}}{1 - \tanh(\alpha/2) \tanh(\beta/2) \hat{\mathbf{v}} \cdot \hat{\mathbf{u}}}. \quad (2.6)$$

When the boosts are parallel, this reduces to the familiar ‘‘addition of velocities’’ formula in special relativity.

Equate the bivector parts and the scalar parts of Eq. (2.5):

$$\begin{aligned} s_\alpha s_\beta \hat{\mathbf{v}} \wedge \hat{\mathbf{u}} &= c_\delta \sin(\theta/2) \mathbf{i}, \\ c_\alpha c_\beta - s_\alpha s_\beta \hat{\mathbf{v}} \cdot \hat{\mathbf{u}} &= c_\delta \cos(\theta/2). \end{aligned}$$

Divide to obtain  $\mathbf{i}$  and  $\theta$ :

$$\tan(\theta/2) \mathbf{i} = \frac{s_\alpha s_\beta \hat{\mathbf{v}} \wedge \hat{\mathbf{u}}}{c_\alpha c_\beta - s_\alpha s_\beta \hat{\mathbf{v}} \cdot \hat{\mathbf{u}}} = \frac{\tanh(\alpha/2) \tanh(\beta/2) \hat{\mathbf{v}} \wedge \hat{\mathbf{u}}}{1 - \tanh(\alpha/2) \tanh(\beta/2) \hat{\mathbf{v}} \cdot \hat{\mathbf{u}}}.$$

The rotation plane  $\mathbf{i}$  is  $\hat{\mathbf{v}} \wedge \hat{\mathbf{u}}$ . To obtain a scalar expression for  $\tan(\theta/2)$ , substitute  $\hat{\mathbf{v}} \wedge \hat{\mathbf{u}} = \sin(\phi) \mathbf{i}$  and  $\hat{\mathbf{v}} \cdot \hat{\mathbf{u}} = -\cos(\phi)$ , where  $\phi$  is the scalar angle from  $\hat{\mathbf{v}}$  to  $\hat{\mathbf{u}}$ .

**2.4.5. Dirac’s electron theory.** The most elegant formulation of Dirac’s relativistic quantum theory of the electron is in the spacetime algebra.

Recall from Sec. 2.2.6 that Pauli’s theory represents spins by 3D rotations, i.e., by members of  $\text{SO}(3)$ . Dirac’s theory represents spins by members of  $\text{SO}(1, 3)$ .

The VA version of Dirac’s equation uses four  $4 \times 4$  complex *Dirac matrices*  $\gamma_0, \gamma_1, \gamma_2, \gamma_3$  associated with (but not identified with) orthogonal directions in spacetime. These matrices generate the *Dirac algebra*. The Dirac algebra is isomorphic to  $\mathbf{G}^{1,3}$ , with  $\gamma_j \leftrightarrow \mathbf{e}_j$ .

Both Pauli and Dirac invented the geometric algebra for the space(time) in which they were working out of necessity, without realizing that the algebras are not special to quantum theory, but have deep geometric significance and wide applicability. From the perspective of GA, the Pauli and Dirac algebras are uninteresting matrix representations of a geometric algebra, which obscure the physical content of their theories.

<sup>9</sup> To see that the right side of Eq. (2.6) is in fact a vector, multiply its numerator and denominator by  $1 - \tanh(\alpha/2) \tanh(\beta/2) \hat{\mathbf{u}} \cdot \hat{\mathbf{v}}$ . Then use Eq. (1.16) and  $\hat{\mathbf{u}} \cdot \hat{\mathbf{v}} = \frac{1}{2}(\hat{\mathbf{u}} \hat{\mathbf{v}} + \hat{\mathbf{v}} \hat{\mathbf{u}})$ .

## 3 Geometric Calculus

### 3.1 The Derivative

**3.1.1. The norm.** Limits in geometric calculus are with respect to the standard  $\mathbf{R}^{2^n}$  norm. Thus if  $A = \sum_J \mathbf{e}_J^{-1} a_J$  (Fourier expansion, Sec. 2.3.6), then  $|A| = (\sum a_J^2)^{1/2}$ . Of course  $|A+B| \leq |A|+|B|$ . It is easy to see that  $|\mathbf{A}B| = |\mathbf{A}||B|$  and  $|A\mathbf{B}| = |A||\mathbf{B}|$ . But in general  $|A^2| \neq |A|^2$ , as shown by the example  $A = \mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_4$ .

**3.1.2. The vector derivative.** In vector calculus  $\nabla = \sum_j \mathbf{e}_j \partial_{\mathbf{e}_j}$ , where  $\partial_{\mathbf{e}_j} = \partial/\partial x_j$  is the directional derivative in the direction  $\mathbf{e}_j$ . We call  $\nabla$  the *vector derivative* because it acts algebraically as a vector: we can multiply it by a scalar field  $f$ , giving the vector field  $\nabla f$ ; dot it with a vector field  $\mathbf{f}$ , giving the scalar field  $\nabla \cdot \mathbf{f}$ ; and cross it with  $\mathbf{f}$ , giving the vector field  $\nabla \times \mathbf{f}$ . But  $\nabla \mathbf{f}$ , a product of vectors, cannot be formed in vector calculus.

In geometric calculus  $\nabla \mathbf{f}$  *does* make sense. This product of vectors is scalar + bivector:  $\nabla \mathbf{f} = \nabla \cdot \mathbf{f} + \nabla \wedge \mathbf{f}$ , just as with our fundamental identity, Eq. (1.5). In this way  $\nabla$  unifies the divergence and curl and generalizes the curl to nD.

The geometric calculus identity  $\nabla^2 \mathbf{f} = \nabla(\nabla \mathbf{f})$  cannot be written in vector calculus. Instead, we must resort to  $\nabla^2 \mathbf{f} = \nabla(\nabla \cdot \mathbf{f}) - \nabla \times (\nabla \times \mathbf{f})$  – and this only in  $\mathbf{R}^3$ .

**3.1.3. Analytic functions.** Let  $f(x, y) = u(x, y) + v(x, y)\mathbf{i}$ , with  $u$  and  $v$  real valued. Then

$$\nabla f = \mathbf{e}_1(u_x + v_x \mathbf{i}) + \mathbf{e}_2(u_y + v_y \mathbf{i}) = \mathbf{e}_1(u_x - v_y) + \mathbf{e}_2(v_x + u_y).$$

From the Cauchy-Riemann equations,  $\nabla f = 0 \Leftrightarrow f$  is analytic. Generalizing, we call a multivector field  $F$  on  $\mathbf{R}^n$  *analytic* if  $\nabla F = 0$ .

This definition leads to a generalization of standard complex analysis to  $n$  (real) dimensions. Many standard results generalize. A simple example: since  $\nabla F = 0 \Rightarrow \nabla^2 F = \nabla(\nabla(F)) = 0$ , analytic functions are harmonic functions. Most important, Cauchy’s theorem and Cauchy’s integral formula generalize, as we shall see.

**3.1.4. The derivative.** It violates the spirit of GA to write  $f$  above as a function of the coordinates  $(x, y)$ . Henceforth we shall think of it, equivalently, as a function of the vector  $\mathbf{x} = \mathbf{e}_1 x + \mathbf{e}_2 y$ , and similarly in higher dimensions. As an example, you can verify that  $\nabla(\mathbf{x} \mathbf{a}) = n \mathbf{a}$ .

With this change of viewpoint we can generalize the derivative from functions of a vector to functions of a multivector (and taking multivector values). First, the *directional derivative* of  $F$  in the “direction”  $A$  is

$$\partial_A F(X) = \lim_{\tau \rightarrow 0} \frac{F(X + \tau A) - F(X)}{\tau}.$$

If  $A$  contains grades for which  $F$  is not defined, then define  $\partial_A F(X) = 0$ . For example, if  $F$  is a function of a vector  $\mathbf{x}$ , then  $\partial_{\mathbf{e}_1 \mathbf{e}_2} F(\mathbf{x}) = 0$ .

The geometric calculus (*multivector*) derivative can now be defined as a generalization of the vector derivative  $\nabla = \sum_j \mathbf{e}_j \partial_{\mathbf{e}_j}$  above:

$$\nabla = \sum_J \mathbf{e}_J^{-1} \partial_{\mathbf{e}_J}.$$

This is a Fourier expansion of  $\nabla$ . (See Sec. 2.3.6.) The derivative  $\nabla$  is independent of the basis  $\{\mathbf{e}_i\}$ .

**3.1.5. Minimization example.** As an example of the use of the derivative, consider the problem of rotating 3D vectors  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$  to best approximate the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  [6]. By this we mean choose a unit quaternion  $R$  to minimize  $\phi(R) = \sum_j (\mathbf{v}_j - R\mathbf{u}_j R^{-1})^2$ . We show that the minimum occurs when  $\nabla\phi(R) = 0$ .

First note that we may drop the constraint  $|R| = 1$ , since  $\phi(aR) = \phi(R)$ . Set  $R = r_0 + \mathbf{e}_3\mathbf{e}_2r_1 + \mathbf{e}_1\mathbf{e}_3r_2 + \mathbf{e}_2\mathbf{e}_1r_3$ . Then  $\phi(R) = \phi(r_0 + \mathbf{e}_3\mathbf{e}_2r_1 + \mathbf{e}_1\mathbf{e}_3r_2 + \mathbf{e}_2\mathbf{e}_1r_3)$  is a real valued function of four real variables. Its minimum occurs when all four partial derivatives  $\partial\phi/\partial r_i = 0$ . We have

$$\nabla\phi(R) = 1\partial\phi/\partial r_0 + \mathbf{e}_2\mathbf{e}_3\partial\phi/\partial r_1 + \mathbf{e}_3\mathbf{e}_1\partial\phi/\partial r_2 + \mathbf{e}_1\mathbf{e}_2\partial\phi/\partial r_3.$$

Thus the minimum occurs when  $\nabla\phi(R) = 0$ .

After learning a few rules about differentiating with respect to a multivector we find that  $\nabla\phi(R) = 0 \Rightarrow \sum_j \mathbf{v}_j \wedge (R\mathbf{u}_j R^{-1}) = 0$ . The right side has an agreeable geometric interpretation: the (bivector) sum of the parallelograms spanned by the  $\mathbf{v}_j$  and the rotated  $\mathbf{u}_j$  is zero. The equation can be solved for  $R$ .

Geometric calculus uses only one derivative,  $\nabla\phi$ , to solve this problem. Vector calculus must break  $\nabla\phi$  into its four components  $\partial\phi/\partial r_i$ . Is generally best not to break a multivector into its components, just as it is generally best not to break a complex number into its real and imaginary parts or a vector into its components.

**3.1.6. Electromagnetism.** Elementary electromagnetic theory is usually formulated in 3D vector calculus. Two vector fields, the electric field  $\mathbf{e}$  and the magnetic field  $\mathbf{b}$ , represent the electromagnetic field. The charge density scalar field  $\rho$  and the current density vector field  $\mathbf{j}$  represent the distribution and motion of charges. *Maxwell's equations* are the heart of the theory:

$$\nabla \cdot \mathbf{e} = 4\pi\rho, \quad \nabla \times \mathbf{e} = -\frac{\partial\mathbf{b}}{\partial t}, \quad \nabla \cdot \mathbf{b} = 0, \quad \nabla \times \mathbf{b} = 4\pi\mathbf{j} + \frac{\partial\mathbf{e}}{\partial t}. \quad (3.1)$$

The spacetime algebra  $\mathbf{G}^{1,3}$  of Sec. 2.4.2 provides a more elegant formulation. A spacetime bivector field  $F$  unifying  $\mathbf{e}$  and  $\mathbf{b}$  represents the electromagnetic field. A spacetime vector field  $J$  unifying  $\rho$  and  $\mathbf{j}$  represents the distribution and motion of charges. Maxwell's four equations become a single equation:  $\nabla F = J$ . What a simple equation: the derivative of one single grade field is another.

Multiplying  $\nabla F = J$  by  $\mathbf{e}_0$  and equating the 0-, 1-, 2-, and 3-vector parts yields the standard Maxwell equations, Eqs. (3.1).

Calculations using the  $\mathbf{G}^{1,3}$  formulation of Maxwell's equations are often easier, and sometimes much easier, than using the  $\mathbf{R}^3$  formulation. This is in part due to the fact that the GA derivative  $\nabla$ , unlike the divergence and curl in Eqs. (3.1), is invertible, as we will see in Sec. 3.2.4.

In geometric calculus the same derivative  $\nabla$  is used in the definition of an analytic function, the minimization example, Maxwell's theory, the full Pauli theory, and the full Dirac theory. That's unification.

## 3.2 The Integral

**3.2.1. The integral.** Let  $S$  be a compact oriented  $s$ -dimensional manifold in  $\mathbf{R}^n$ . Let  $F(\mathbf{x})$  be a multivector valued field on  $S$ . Then we can form the *integral*

$$\int_S \mathbf{dS} F. \quad (3.2)$$

Here  $\mathbf{dS} = |\mathbf{dS}(\mathbf{x})| \mathbf{I}(\mathbf{x})$ , where  $|\mathbf{dS}(\mathbf{x})|$  is an element of  $s$ -volume of  $S$  at  $\mathbf{x}$  and  $\mathbf{I}(\mathbf{x})$  is the pseudoscalar of the tangent space to  $S$  at  $\mathbf{x}$ . For example, if  $S$  is a surface in  $\mathbf{R}^3$ , then  $|\mathbf{dS}| = dA$  is an element of area of  $S$  and  $\mathbf{I}(\mathbf{x})$  is the pseudoscalar of the tangent plane to  $S$  at  $\mathbf{x}$  (a bivector). If  $S$  is a volume in  $\mathbf{R}^3$ , then  $|\mathbf{dS}| = dV$  is an element of volume of  $V$  and  $\mathbf{I}(\mathbf{x}) \equiv \mathbf{I}$  is the pseudoscalar of  $\mathbf{R}^3$ . Note that the order of the factors in the integrand is important, as the geometric product is not commutative.

Eq. (3.2) is a *directed integral*, using the *directed measure*  $\mathbf{dS}$ . The integral  $\int_C f(z) dz$  from complex analysis is a directed integral, a special case of Eq. (3.2).

**3.2.2. The fundamental theorem.** Let  $F$  be defined on an open set containing  $S$ . We define  $\nabla_S F$ , the *tangential derivative* of  $F$  on  $S$ . Let  $T(\mathbf{x})$  be the tangent space to  $S$  at  $\mathbf{x}$ . Then  $\nabla_S F(\mathbf{x}) = \nabla(F|_{T(\mathbf{x})})(0)$ , where 0 is the origin of  $T(\mathbf{x})$ , i.e.,  $\mathbf{x}$ .

### The Fundamental Theorem of (Geometric) Calculus

$$\int_S \mathbf{dS} \nabla_S F = \int_{\partial S} \mathbf{dC} F. \quad (3.3)$$

This is a marvelous theorem. Its scalar part is equivalent to Stokes' theorem for differential forms. Thus the divergence and Stokes' theorems of vector calculus are special cases of Eq. (3.3). A generalization of Cauchy's theorem to a manifold is an obvious special case: If  $F$  is analytic on  $S$ , i.e., if  $\nabla_S F = 0$ , then  $\int_{\partial S} \mathbf{dC} F = 0$ .

The fundamental theorem also generalizes the residue theorem. Let  $\Omega_n$  be the  $(n-1)$ -volume of the boundary of the unit ball in  $\mathbf{R}^n$  (e.g.,  $\Omega_2 = 2\pi$ ). Let  $\delta(\mathbf{x} - \mathbf{x}_k)$  be the Dirac delta distribution concentrated at  $\mathbf{x} = \mathbf{x}_k$ . We say that  $F$  has a *pole* at  $\mathbf{x}_k$  with *residue* the multivector  $R_k$  if  $\nabla_S F(\mathbf{x}) = \Omega_n R_k \delta(\mathbf{x} - \mathbf{x}_k)$  near  $\mathbf{x}_k$ . Eq. (3.3) holds if  $F$  is analytic in  $S$  except at a finite number of poles at the points  $\mathbf{x}_k$ . Thus

$$\int_{\partial S} \mathbf{dC} F = \int_S \mathbf{dS} \nabla_S F = \int_S |\mathbf{dS}| \mathbf{I}(\mathbf{x}) \left( \sum_k \Omega_n R_k \delta(\mathbf{x} - \mathbf{x}_k) \right) = \Omega_n \sum_k \mathbf{I}(\mathbf{x}_k) R_k.$$

If  $S$  is a region of a plane and the  $R_k$  are complex numbers, then this reduces to the usual residue theorem.

With directed integrals, complex analysis becomes a subdiscipline of real analysis: it is the study of functions  $F$  with  $\nabla F = 0$ . Traditional real analysis does not use directed integrals. Unification with complex analysis cannot be achieved without them. For example, consider Cauchy's theorem:  $\oint_C f(z) dz = 0$ . Green's theorem gives the real and imaginary parts of the theorem separately. But the theorem cannot be written as a single formula in vector calculus or with differential forms.

The fundamental theorem can be generalized, with several important corollaries. Eqs. (3.4) and (3.5) below are examples.

**3.2.3.  $F$  from  $\nabla F$  and boundary values.** Let  $F$  be a multivector valued function defined in a region  $V$  of  $\mathbf{R}^n$ . Then for  $\mathbf{x}_0 \in V$ ,

$$F(\mathbf{x}_0) = \frac{(-1)^n}{\Omega_n \mathbf{I}} \left\{ \int_V \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|^n} d\mathbf{V} \nabla F(\mathbf{x}) - \int_{\partial V} \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|^n} d\mathbf{S} F(\mathbf{x}) \right\}. \quad (3.4)$$

In particular,  $F|_V$  is determined by  $\nabla F|_V$  and  $F|_{\partial V}$ . This generalizes Helmholtz's theorem from 3D vector calculus.

If  $F$  is analytic, then Eq. (3.4) becomes

$$F(\mathbf{x}_0) = -\frac{(-1)^n}{\Omega_n \mathbf{I}} \int_{\partial V} \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|^n} d\mathbf{S} F(\mathbf{x}).$$

This is a generalization of Cauchy's integral formula. Geometric calculus enables us to see better how Cauchy's formula fits into the scheme of things: it is a very special case of Eq. (3.4), which applies to all functions in all dimensions.

**3.2.4. Antiderivatives.** Eq. (3.4) can be used to prove that  $F$  has an *antiderivative*  $P$ :  $\nabla P = F$ . In other words,  $\nabla$  is invertible. Clearly, two antiderivatives of  $F$  differ by an analytic function. This generalizes the  $n = 1$  case, where the analytic functions are the constant functions.

For  $n > 2$  an antiderivative is given by

$$P(\mathbf{x}_0) = \frac{1}{(n-2)\Omega_n} \left\{ \int_V \frac{|d\mathbf{V}|}{|\mathbf{x} - \mathbf{x}_0|^{n-2}} \nabla F(\mathbf{x}) - \int_{\partial V} \frac{|d\mathbf{S}|}{|\mathbf{x} - \mathbf{x}_0|^{n-2}} \mathbf{n}(\mathbf{x}) F(\mathbf{x}) \right\}, \quad (3.5)$$

where  $\mathbf{n}(\mathbf{x}) = \pm \mathbf{I}(\mathbf{x})^*$  is the unit outward normal to  $\partial V$ . (In fact,  $\mathbf{n} = \mathbf{I}^{-1} \mathbf{I}(\mathbf{x})$ , where  $\mathbf{I}$  is the pseudoscalar of  $\mathbf{R}^n$ .)

If  $F$  is analytic, then the first integral is zero.

If  $|\nabla F(\mathbf{x})| = O(|\mathbf{x}|^{-2})$  and  $\lim_{\mathbf{x} \rightarrow \infty} \mathbf{x} F(\mathbf{x}) = 0$ , then

$$P(\mathbf{x}_0) = \frac{1}{(n-2)\Omega_n} \int_{\mathbf{R}^n} \frac{|d\mathbf{V}|}{|\mathbf{x} - \mathbf{x}_0|^{n-2}} \nabla F(\mathbf{x}).$$

**3.2.5. Potentials, Fields, Sources.** Three fields  $P, F$ , and  $S$  with  $\nabla P = F$  and  $\nabla F = S$  are called a *potential*, a *field*, and a *source*, respectively. Given a source  $S$  with suitable boundedness,  $P$  and  $F$  always exist. A common situation is  $S = s - \mathbf{s}^*$ , where  $s$  is a scalar field and  $\mathbf{s}$  is a vector field with  $\nabla \cdot \mathbf{s} = 0$ . Then we can take  $P = p - \mathbf{p}^*$ , where  $p$  is a scalar field,  $\mathbf{p}$  is a vector field with  $\nabla \cdot \mathbf{p} = 0$ , and  $F = \mathbf{f}$ , a vector field. Vector calculus cannot form  $S$  or  $P$ . In particular, it cannot use the simple formulas  $F = \nabla P$  and  $S = \nabla F$ . Instead,  $\mathbf{f} = \nabla p - \nabla \times \mathbf{p}$ ,  $s = \nabla \cdot \mathbf{f}$ , and  $\mathbf{s} = \nabla \times \mathbf{f}$ .

## 4 Other Models

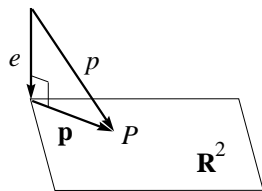
### 4.1 The Homogeneous Model

We have represented subspaces of  $\mathbf{R}^n$  (points, lines, planes, ... through the origin) with blades of  $\mathbf{G}^n$ . This is the *standard model*. But we have not represented *translated* subspaces (points, lines, planes, ... not through the origin). Yet there is nothing special geometrically about the origin.

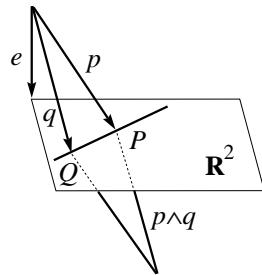
The *homogeneous model* puts subspaces and translated subspaces on an equal footing. The model represents and manipulates all points, lines, planes, ... much as the standard model represents and manipulates those through the origin.

**4.1.1. The homogeneous model.** *The homogeneous model represents a translated  $r$ -dimensional subspace of  $\mathbf{R}^n$  with an  $(r+1)$ -blade of  $\mathbf{G}^{n+1}$ .* It is the coordinate-free geometric algebra version of homogeneous coordinates used in computer graphics and projective geometry.

Extend the orthonormal basis  $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$  of  $\mathbf{R}^n$  with a new vector  $e$  to form  $\mathbf{R}^{n+1}$ , and with it  $\mathbf{G}^{n+1}$ . (Only vectors in  $\mathbf{R}^n$  will be denoted in bold.) The homogeneous model represents the *point*  $P$  at the end of the vector  $\mathbf{p} \in \mathbf{R}^n$  with the *vector*  $p = e + \mathbf{p} \in \mathbf{R}^{n+1}$ . Fig. 6 shows a useful way to visualize this for  $\mathbf{R}^2$ .



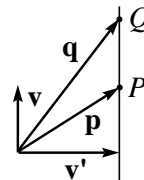
**Fig. 6:** Vector  $p = e + \mathbf{p}$  represents the point  $P$ .



**Fig. 7:** Bivector  $p \wedge q$  represents the oriented length  $PQ$ .

The vector  $p$  is *normalized* in the sense that  $p \cdot e = e \cdot e + \mathbf{p} \cdot e = 1$ . However, the representation is *homogeneous*: for nonzero scalars  $\lambda$ ,  $\lambda(e + \mathbf{p})$  also represents  $P$ .

The bivector  $p \wedge q$  in Fig. 7 represents an oriented length  $PQ$  in the line  $\overline{PQ}$ . Let  $\mathbf{v} = \mathbf{q} - \mathbf{p}$  and  $\mathbf{v}'$  be the vector with endpoint on the line  $\overline{PQ}$  and perpendicular to it. See the figure. Then  $p \wedge q$  determines, and is determined by,  $\mathbf{v}$  and  $\mathbf{v}'$ . This follows from  $\mathbf{v} = e \cdot (p \wedge q)$  and  $p \wedge q = (e + \mathbf{p}) \wedge \mathbf{v} = (e + \mathbf{v}') \wedge \mathbf{v} = (e + \mathbf{v}') \mathbf{v}$ . The equation  $p \wedge q = (e + \mathbf{v}') \wedge \mathbf{v}$  shows that  $p \wedge q$  does not determine  $\mathbf{p}$  or  $\mathbf{q}$ .



The trivector  $p \wedge q \wedge r$  represents an oriented area in the plane determined by points  $P, Q, R$ . Let  $\mathbf{V} = (\mathbf{q} - \mathbf{p}) \wedge (\mathbf{r} - \mathbf{p})$ , which represents twice the area. Define  $\mathbf{v}'$  analogously to above. Then  $p \wedge q \wedge r$  determines, and is determined by,  $\mathbf{V}$  and  $\mathbf{v}'$ . The equation  $p \wedge q \wedge r = (e + \mathbf{v}') \wedge \mathbf{V}$  shows that  $p \wedge q \wedge r$  does not determine  $\mathbf{p}, \mathbf{q}$ , or  $\mathbf{r}$ .

In VA and the standard model of GA, vectors often do double duty, representing oriented line segments and, after designating an origin, the points at their ends. In the homogeneous model oriented line segments and points have different representations.

We now have two geometric algebras for Euclidean  $n$ -space: the standard model  $\mathbf{G}^n$  and the homogeneous model  $\mathbf{G}^{n+1}$ . In both, blades represent geometric objects. In the standard model a vector represents an oriented line segment. In the homogeneous model it represents a point. In the standard model an outer product of vectors represents an oriented area. In the homogeneous model it represents an oriented length. Oriented areas and oriented lengths are different, yet they share a common algebraic structure. We have to learn geometric algebra only once to work with both.

**4.1.2. The Euclidean group.** Rotations and reflections generate the orthogonal group  $\mathbf{O}(n)$ , discussed in Sec. 2.2.5. Include translations to generate the distance preserving Euclidean group. In  $\mathbf{R}^n$  translations are not linear. This is a problem: “The orthogonal group is multiplicative while the translation group is additive, so combining the two destroys the simplicity of both.”[4]

The homogeneous model solves the problem. The subgroup of  $\mathbf{O}(n+1)$  which fixes  $e$  is isomorphic to  $\mathbf{O}(n)$ . For members of this subgroup map  $\mathbf{R}^n$ , the subspace of  $\mathbf{R}^{n+1}$  orthogonal to  $e$ , to  $\mathbf{R}^n$ . And since  $p \cdot q = 1 + \mathbf{p} \cdot \mathbf{q}$ , the map is also orthogonal on  $\mathbf{R}^n$ .

For a fixed  $\mathbf{a} \in \mathbf{R}^n$ , consider the linear transformation  $x \rightarrow x + (x \cdot e)\mathbf{a}$  of  $\mathbf{R}^{n+1}$ . In particular,  $e + \mathbf{p} \rightarrow e + (\mathbf{p} + \mathbf{a})$ . This represents a translation by  $\mathbf{a}$  in  $\mathbf{R}^n$ . In this way translations are *linearized* in the homogeneous model.

**4.1.3. Join and meet.** Geometric objects *join* to form higher dimensional objects. For example, the join of two points is the line between them; the join of intersecting lines is the plane containing them; and the join of a line and a point not on it is the plane containing them.

Geometric objects *meet* in lower dimensional objects. For example, the meet of an intersecting line and plane is their point or line of intersection; and the meet of two intersecting planes is their line of intersection.

GA defines the join and meet of two blades (only) to represent the join and meet of the geometric objects that they represent.

The *join* of blades  $\mathbf{A}$  and  $\mathbf{B}$  is the span of their subspaces. From Eq. (1.15), if  $\mathbf{A} \cap \mathbf{B} = \{0\}$ , then their join  $\mathbf{J} = \mathbf{A} \wedge \mathbf{B}$ . However, there is no general formula for the join in terms of the geometric product, as there is for the inner and outer products (Eqs. (1.6) and (1.8)).

The *meet* of  $\mathbf{A}$  and  $\mathbf{B}$  is the intersection of their subspaces. In this section, let  $\mathbf{X}^* = \mathbf{X}/\mathbf{J}$ , the dual of  $\mathbf{X}$  in join  $\mathbf{J}$  of  $\mathbf{A}$  and  $\mathbf{B}$ . The meet of  $\mathbf{A}$  and  $\mathbf{B}$  is given by

$$\mathbf{A} \vee \mathbf{B} = \mathbf{A}^* \cdot \mathbf{B}. \quad (4.1)$$

Before proving this we give examples of the join and meet.

In the standard model, two lines through the origin join to form a plane. Two examples are  $\mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_1 \mathbf{e}_2$  and  $\mathbf{e}_2 \wedge \mathbf{e}_3 = \mathbf{e}_2 \mathbf{e}_3$ . The two planes meet in a line:

$$(\mathbf{e}_1 \mathbf{e}_2) \vee (\mathbf{e}_2 \mathbf{e}_3) = (\mathbf{e}_1 \mathbf{e}_2)^* \cdot (\mathbf{e}_2 \mathbf{e}_3) = -\mathbf{e}_2.$$

In the homogeneous model, two points join to form a line. Two examples are  $\ell_1 = (e + \mathbf{e}_1) \wedge (e + \mathbf{e}_2)$  and  $\ell_2 = e \wedge (e + \mathbf{e}_1 + \mathbf{e}_2)$ . The two lines meet in a point:

$$\ell_1 \vee \ell_2 = (e \mathbf{e}_2 - e \mathbf{e}_1 + \mathbf{e}_1 \mathbf{e}_2)^* \cdot (e \mathbf{e}_1 + e \mathbf{e}_2) = 2 \left( e + \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2) \right).$$

We now prove Eq. (4.1),  $\mathbf{A} \vee \mathbf{B} = \mathbf{A}^* \cdot \mathbf{B}$ . We work with the two sides of this equation separately, using duality, Eq. (1.11), on the right.

$$\begin{aligned} \mathbf{x} \in \mathbf{A}^* \cdot \mathbf{B} &\Leftrightarrow \mathbf{x} \in (\mathbf{A}^* \wedge \mathbf{B}^*)^* \Leftrightarrow \mathbf{x} \perp \mathbf{A}^* \wedge \mathbf{B}^*, \\ \mathbf{x} \in \mathbf{A} \vee \mathbf{B} &\Leftrightarrow (\mathbf{x} \in \mathbf{A} \text{ and } \mathbf{x} \in \mathbf{B}) \Leftrightarrow (\mathbf{x} \perp \mathbf{A}^* \text{ and } \mathbf{x} \perp \mathbf{B}^*). \end{aligned} \quad (4.2)$$

To complete the proof we show that the right sides above are equivalent. For  $\mathbf{x} \in \mathbf{J}$ ,

$$\mathbf{x} \in \mathbf{A}^* \cap \mathbf{B}^* \Leftrightarrow (\mathbf{x} \in \mathbf{A}^* \text{ and } \mathbf{x} \in \mathbf{B}^*) \Leftrightarrow (\mathbf{x} \perp \mathbf{A} \text{ and } \mathbf{x} \perp \mathbf{B}) \Leftrightarrow \mathbf{x} = 0.$$

Thus from Eq. (1.15),  $\mathbf{A}^* \wedge \mathbf{B}^* = \text{span}(\mathbf{A}^*, \mathbf{B}^*)$ . Therefore the right sides are equivalent.

Eq. (4.2) shows that the dual representation (Sec. 1.4.6) of the meet is  $\mathbf{A}^* \wedge \mathbf{B}^*$ .

**4.1.4. Projective geometry.** The homogeneous model enables us to study projective geometry algebraically in a coordinate-free manner. The blades of  $\mathbf{G}^{n+1}$  represent the points, lines, ... of the projective space  $\mathbf{P}^n$ . Formulating projective geometry within GA should help to integrate it with the rest of mathematics.

In our visualization of the points of the homogeneous model of a plane, Fig. 6, a vector of  $\mathbf{G}^3$  orthogonal to  $e$  does not intersect the embedded  $\mathbf{R}^2$ , and thus does not represent a point in the plane. For  $\mathbf{P}^2$  it represents a point on the line at infinity.

GA provides simple algebraic tests for collinearity and concurrency in the projective plane  $\mathbf{P}^2$ :

- $p, q, r$  collinear  $\Leftrightarrow p \wedge q \wedge r = 0$ . This is easy to see:  $p \wedge q$  represents the line through  $p$  and  $q$ . From Eq. (1.15),  $r$  is on this line if and only if  $p \wedge q \wedge r = 0$ .
- $P, Q, R$  concurrent  $\Leftrightarrow \langle PQR \rangle_0 = 0$ . This is a bit harder. For the bivectors  $P, Q$ , and  $R$ , we have  $(P^* \cdot Q) \cdot R^* = \langle P^*Q \rangle_{2-1} \cdot R^* = -\langle PQR \rangle_0$ . Thus from Eqs. (4.1) and (1.11),

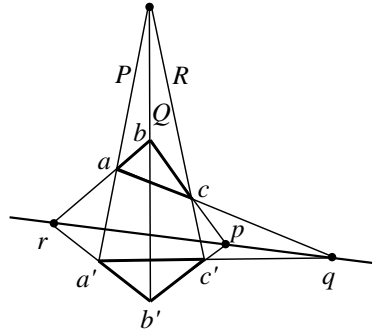
$$P, Q, R \text{ concurrent} \Leftrightarrow (P \vee Q) \wedge R = 0 \Leftrightarrow (P^* \cdot Q) \cdot R^* = 0 \Leftrightarrow \langle PQR \rangle_0 = 0.$$

Fig. 8 illustrates *Desargues' Theorem*, perhaps the most celebrated theorem of projective geometry: Given coplanar triangles  $abc$  and  $a'b'c'$ , construct  $P, Q, R$  and  $p, q, r$ . Then  $P, Q, R$  concurrent  $\Leftrightarrow p, q, r$  collinear.

We indicate an algebraic proof of the theorem. In Fig. 8 we have  $P = a \wedge a'$ , with similar expressions for  $Q$  and  $R$ . Also,  $p = (b \wedge c) \vee (b' \wedge c')$ , with similar expressions for  $q$  and  $r$ .

Let  $J = a \wedge b \wedge c$  and  $J' = a' \wedge b' \wedge c'$ . Then  $J$  and  $J'$  are nonzero multiples of the pseudoscalar. Thus  $JJ'$  is a nonzero scalar.

Desargues' theorem is, from the tests for collinearity and concurrency above, an immediate corollary of the identity  $(p \wedge q \wedge r)^* = JJ' \langle PQR \rangle_0$ , which applies to *all* pairs of triangles. The identity is thus a generalization of the theorem. Unfortunately its proof is too involved to give here.



**Fig. 8:** Desargues theorem.

## 4.2 The Conformal Model

The conformal model is the most powerful geometric algebra known for Euclidean geometry. It represents points, lines, planes, circles, and spheres as blades in the algebra. It represents the Euclidean group in a simple way as a group of automorphisms of the algebra. As a bonus, the model unifies Euclidean, hyperbolic, and elliptic geometries.

**4.2.1. The conformal model.** Extend the orthonormal basis  $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$  of  $\mathbf{R}^n$  with  $e_+$  and  $e_-$ , where  $e_{\pm}^2 = \pm 1$ . Then we have  $\mathbf{R}^{n+1,1}$ , and with it  $\mathbf{G}^{n+1,1}$ . (See Sec. 2.4.1 for indefinite metrics.) More useful than  $e_{\pm}$  are  $e_o = \frac{1}{2}(e_- - e_+)$  and  $e_{\infty} = e_- + e_+$ . Then  $e_o^2 = e_{\infty}^2 = 0$  and  $e_o \cdot e_{\infty} = -1$ .

The conformal model represents the point at the end of the vector  $\mathbf{p} \in \mathbf{R}^n$  with

$$p = e_o + \mathbf{p} + \frac{1}{2}\mathbf{p}^2 e_{\infty} \in \mathbf{R}^{n+1,1} \quad (4.3)$$

and its nonzero scalar multiples. The vector  $p$  is *normalized*: the coefficient of  $e_o$  is 1.

You can check that  $p$  is *null*,  $p^2 = p \cdot p = 0$ , and more generally that

$$p \cdot q = -\frac{1}{2}(\mathbf{p} - \mathbf{q})^2. \quad (4.4)$$

Append  $\infty$ , the *point at infinity*, to  $\mathbf{R}^n$ , and represent it with  $e_{\infty}$ . We now have a representation of  $\bar{\mathbf{R}}^n \equiv \mathbf{R}^n \cup \infty$ .

**4.2.2. Representing geometric objects.** For simplicity we work in 3D.

**Spheres.** Equation (4.4) immediately gives an equation of the sphere with center  $\mathbf{c}$  and radius  $\rho$ :  $x \cdot \mathbf{c} = -\frac{1}{2}\rho^2$ . This is equivalent to  $x \cdot (c - \frac{1}{2}\rho^2 e_{\infty}) = 0$ . Thus the vector  $\sigma = c - \frac{1}{2}\rho^2 e_{\infty}$  is a dual representation of the *sphere*! (Sec. 1.4.6 discusses dual representations.)

We rewrite  $\sigma$  in terms of  $\mathbf{c}$  and a point  $\mathbf{p}$  on the sphere:

$$\sigma = c - \frac{1}{2}\rho^2 e_{\infty} = -(p \cdot e_{\infty})c + (p \cdot c)e_{\infty} = p \cdot (c \wedge e_{\infty}), \quad (4.5)$$

where the last equality is an important general identity for vectors in GA.

The 4-blade  $S = \sigma^*$  is a direct representation of the sphere. A direct representation can also be constructed from four points on the sphere:  $S = p \wedge q \wedge r \wedge s$ . Then the center and radius can be read off from  $\alpha S^* = c - \frac{1}{2}\rho^2 e_{\infty}$ , where  $\alpha$  normalizes  $S^*$ .

**Lines.** A direct representation of the line through points  $\mathbf{p}$  and  $\mathbf{q}$  is  $p \wedge q \wedge e_{\infty}$ . To prove this, let  $x$  represent the point  $\mathbf{x}$ . Then:

$$\begin{aligned} x \wedge (p \wedge q \wedge e_{\infty}) = 0 &\Leftrightarrow x = ap + bq + ce_{\infty} \\ &\Leftrightarrow x = (a+b)e_o + (a\mathbf{p} + b\mathbf{q}) + c'e_{\infty} \\ &\Leftrightarrow \mathbf{x} = \frac{a}{a+b}\mathbf{p} + \frac{b}{a+b}\mathbf{q}, \quad \frac{a}{a+b} + \frac{b}{a+b} = 1. \end{aligned}$$

The representation does not determine  $\mathbf{p}$  or  $\mathbf{q}$ . But it does determine the distance between them:  $(p \wedge q \wedge e_{\infty})^2 = |\mathbf{p} - \mathbf{q}|^2$ .

**Planes.** The representation  $p \wedge q \wedge e_\infty$  of lines gives a test:  $\mathbf{m}, \mathbf{p}, \mathbf{q}$  collinear  $\Leftrightarrow m \wedge p \wedge q \wedge e_\infty = 0$ . If they are not collinear, then they determine a plane. The plane is represented by  $m \wedge p \wedge q \wedge e_\infty$ , a “sphere through infinity”. The area  $A$  of the triangle with vertices  $\mathbf{m}, \mathbf{p}$ , and  $\mathbf{q}$  is given by  $(m \wedge p \wedge q \wedge e_\infty)^2 = 4A^2$ .

A dual representation of the plane through the point  $\mathbf{p}$  and orthogonal to  $\mathbf{n}$  is  $\mathbf{n} + (\mathbf{p} \cdot \mathbf{n})e_\infty$ : its VA equation is  $(\mathbf{x} - \mathbf{p}) \cdot \mathbf{n} = 0$  and  $x \cdot [\mathbf{n} + (\mathbf{p} \cdot \mathbf{n})e_\infty] = (\mathbf{x} - \mathbf{p}) \cdot \mathbf{n}$ . A different form of the dual representation follows from the identity used in Eq. (4.5):

$$\mathbf{n} + (\mathbf{p} \cdot \mathbf{n})e_\infty = -(p \cdot e_\infty) \mathbf{n} + (p \cdot \mathbf{n}) e_\infty = p \cdot (\mathbf{n} \wedge e_\infty). \quad (4.6)$$

If instead we are given  $\mathbf{n}$  and the distance  $d$  of the plane to the origin, then the equation  $\mathbf{x} \cdot \mathbf{n} = d$  gives the dual representation  $\mathbf{n} + de_\infty$ .

**Circles.** A direct representation of the circle through  $\mathbf{p}, \mathbf{q}$ , and  $\mathbf{r}$  is  $p \wedge q \wedge r$ . We can think of the lines  $p \wedge q \wedge e_\infty$  above as circles through infinity.

We determine the dual representation of the circle with center  $\mathbf{c}$ , radius  $\rho$ , and normal  $\mathbf{n}$ . The circle is the intersection of the dual sphere  $c - \frac{1}{2}\rho^2 e_\infty$  (Eq. (4.5)) and the dual plane  $c \cdot (\mathbf{n} \wedge e_\infty)$  (Eq. (4.6)). From the dual representation of the meet, Eq. (4.2), the dual representation of the circle is  $(c - \frac{1}{2}\rho^2 e_\infty) \wedge (c \cdot (\mathbf{n} \wedge e_\infty))$ .

**4.2.3. Representing geometric operations.** Now that we know how to represent Euclidean objects in the conformal model, we turn to representing geometric operations on the objects.

Let  $\underline{Q}$  be an orthogonal transformation of  $\mathbf{R}^{n+1,1}$ . Then  $p \cdot p = 0 \Rightarrow \underline{Q}(p) \cdot \underline{Q}(p) = 0$ . It is not difficult to show that every null vector of  $\mathbf{G}^{n+1,1}$  is of the form  $\lambda e_\infty$  or  $\lambda p$  for some  $\mathbf{p} \in \mathbf{R}^n$ . Thus  $\underline{Q}$  induces a map of  $\bar{\mathbf{R}}^n$  to itself. Such maps form the conformal group of  $\bar{\mathbf{R}}^n$ . (They preserve angles, but not necessarily orientations.)

The conformal group is generated (redundantly) by the following transformations:

- **Rotations.** A rotation around the origin by angle  $i\theta$  is represented in the conformal model just as in the standard model:  $p \rightarrow e^{-i\theta/2} p e^{i\theta/2}$ . (Eq. (2.3))
- **Translations.** The translation  $\mathbf{p} \rightarrow \mathbf{p} + \mathbf{a}$  is represented by  $p \rightarrow e^{-\mathbf{a}e_\infty/2} p e^{\mathbf{a}e_\infty/2}$ . Note that since  $e_\infty \mathbf{a} = -\mathbf{a} e_\infty$  and  $e_\infty^2 = 0$ ,  $e^{\pm \mathbf{a}e_\infty/2} = 1 \pm \mathbf{a} e_\infty/2$  (exactly).
- **Reflections.** A reflection in the hyperplane orthogonal to  $\mathbf{b}$  is represented in the conformal model just as in the standard model:  $p \rightarrow -\mathbf{b} p \mathbf{b}^{-1}$ . (Eq. (1.16))
- **Inversions.** The inversion  $\mathbf{p} \rightarrow \mathbf{p}^{-1}$  is represented in the conformal model by a reflection in the hyperplane normal to  $e_+$ :  $p \rightarrow -e_+ p e_+$ . To normalize the result, divide by the coefficient of  $e_0$  (which is  $\mathbf{p}^2$ ).
- **Dilations.** The dilation  $\mathbf{p} \rightarrow \alpha \mathbf{p}$  is represented in the conformal model by  $p \rightarrow e^{-\ln(\alpha)E/2} p e^{\ln(\alpha)E/2}$ , where  $E = e_\infty \wedge e_0$ . Note that  $e^{\beta E} = \cosh(\beta) + E \sinh(\beta)$ . To normalize the result, divide by the coefficient of  $e_0$  (which is  $1/\alpha$ ).

The subgroup of the conformal group fixing  $e_\infty$  is the Euclidean group of  $\mathbf{R}^n$ . For rotations, translations, and reflections fix  $e_\infty$ , while inversions and dilations do not.

**4.2.4. Conformal transformations as automorphisms.** As we have seen, the representation of a conformal transformation of  $\mathbf{R}^n$  is constructed simply from its geometric description. The representations are of the form  $p \rightarrow \pm VpV^{-1}$  (“ $\pm$ ” for reflections and inversions). They extend to automorphisms of the entire geometric algebra, as with rotations in Sec. 2.2.4. The automorphisms are of the form  $M \rightarrow \pm VMV^{-1}$  (“ $\pm$ ” for reflections and inversions of odd grade  $M$ ). The automorphisms preserve the geometric, inner, and outer products and grades. They compose simply, as in Sec. 2.2.2. The following examples show how all this can be used to advantage.

The rotations in Sec. 4.2.3 are around the origin. To rotate an object around  $\mathbf{a}$ , translate it by  $-\mathbf{a}$ , rotate it around the origin, and it translate back by  $\mathbf{a}$ . Thus a rotation around  $\mathbf{a}$  is represented by  $T_{\mathbf{a}}e^{-i\theta/2}T_{\mathbf{a}}^{-1}$ , where  $T_{\mathbf{a}} = e^{-\mathbf{a}e_{\infty}/2}$ . This says that rotations translate in the same way as geometric objects. There is more. You can quickly check that  $T_{\mathbf{a}}e^{-i\theta/2}T_{\mathbf{a}}^{-1} = e^{-T_{\mathbf{a}}(i\theta)T_{\mathbf{a}}^{-1}/2}$  using the power series expansion of the exponential. This says that the angle  $T_{\mathbf{a}}(i\theta)T_{\mathbf{a}}^{-1}$  specifies the rotation around  $\mathbf{a}$ . It is the translation by  $\mathbf{a}$  of the angle  $i\theta$  specifying the rotation around 0.

We compute the angle  $\theta$  between intersecting lines  $\ell_1$  and  $\ell_2$ . Translate the lines so that they intersect at the origin. They now have representations  $\ell'_1 = e_{\circ} \wedge p_1 \wedge e_{\infty}$  and  $\ell'_2 = e_{\circ} \wedge p_2 \wedge e_{\infty}$ . Because of the simple form of  $\ell'_1$  and  $\ell'_2$ , it is easy to show that

$$(\ell'_1 \cdot \ell'_2)/|\ell'_1||\ell'_2| = (\mathbf{p}_1 \cdot \mathbf{p}_2)/|\mathbf{p}_1||\mathbf{p}_2|,$$

which is  $\cos(\theta)$ . Since the translation preserves inner products and norms, we also have  $(\ell_1 \cdot \ell_2)/|\ell_1||\ell_2| = \cos(\theta)$ .

Now let  $\sigma_1$  and  $\sigma_2$  be coplanar circles intersecting at two points. (Circles intersecting tangentially require a separate analysis.) Translate one intersection point to the origin. Now perform an inversion. This maps the circles to straight lines:  $p \wedge q \wedge e_{\circ} \rightarrow p' \wedge q' \wedge e_{\infty}$ . Thus the angle between  $\sigma_1$  and  $\sigma_2$  is given by the same formula as for the lines:  $\cos(\theta) = (\sigma_1 \cdot \sigma_2)/|\sigma_1||\sigma_2|$ .

**4.2.5. NonEuclidean geometry.** The conformal model can be used for hyperbolic and elliptic geometry [1]. Let  $\mathcal{B}$  be the unit ball centered at the origin of  $\mathbf{R}^n$ .

$\mathcal{B}$  provides a model for hyperbolic geometry:

- The hyperbolic line through  $\mathbf{p}, \mathbf{q} \in \mathcal{B}$  is  $p \wedge q \wedge e_+$ .  
(This is the Euclidean circle through  $\mathbf{p}$  and  $\mathbf{q}$  intersecting  $\partial\mathcal{B}$  orthogonally.)
- The hyperbolic distance  $d$  between  $\mathbf{p}$  and  $\mathbf{q}$  satisfies  $p \cdot q = -2 \sinh^2(d/2)$ .  
(The coefficient of  $e_+$  in  $p$  and  $q$  must be normalized to 1.)
- The hyperbolic group is the subgroup of  $O(n+1, 1)$  fixing  $e_+$ .

$\mathcal{B}$  provides a model for elliptic geometry:

- The elliptic line through  $\mathbf{p}, \mathbf{q} \in \mathcal{B}$  is  $p \wedge q \wedge e_-$ .  
(This is the Euclidean circle through  $\mathbf{p}$  and  $\mathbf{q}$  intersecting  $\partial\mathcal{B}$  at antipodes.)
- The elliptic distance  $d$  between  $\mathbf{p}$  and  $\mathbf{q}$  satisfies  $p \cdot q = -2 \sin^2(d/2)$ .  
(The coefficient of  $e_-$  in  $p$  and  $q$  must be normalized to -1.)
- The elliptic group is the subgroup of  $O(n+1, 1)$  fixing  $e_-$ .

## 5 Useful Identities

$$A \cdot (B \cdot C) = (A \wedge B) \cdot C.$$

$$(\mathbf{A} \cdot B) \cdot \mathbf{C} = \mathbf{A} \wedge (B \cdot \mathbf{C}) \text{ if } \mathbf{A} \subseteq \mathbf{C}$$

$$\mathbf{B} \wedge \mathbf{A} = (-1)^{jk} (\mathbf{A} \wedge \mathbf{B}). \text{ (}\mathbf{A} \text{ is a } j\text{-blade and } \mathbf{B} \text{ is a } k\text{-blade.)}$$

$$\mathbf{a} \wedge A \wedge \mathbf{b} = -\mathbf{b} \wedge A \wedge \mathbf{a}.$$

$$\mathbf{a} \wedge A \wedge \mathbf{a} = 0.$$

$$(\mathbf{a} \wedge \mathbf{b}) \cdot (\mathbf{c} \wedge \mathbf{d}) = (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}).$$

$$\mathbf{a} \cdot (BC) = (\mathbf{a} \cdot B)C + (-1)^k B(\mathbf{a} \cdot C). \text{ (}B \text{ is a } k\text{-vector.)}$$

$$\mathbf{a} \cdot (\mathbf{bc}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{c} - (\mathbf{a} \cdot \mathbf{c})\mathbf{b}.$$

$$\mathbf{a} \cdot (\mathbf{b}_1 \mathbf{b}_2 \cdots \mathbf{b}_k) = \sum_r (-1)^{r+1} (\mathbf{a} \cdot \mathbf{b}_r) \mathbf{b}_1 \cdots \check{\mathbf{b}}_r \cdots \mathbf{b}_k. \text{ (}\check{\mathbf{b}}_r \text{ means omit } \mathbf{b}_r\text{)}$$

$$\text{Thus for a blade } \mathbf{B} = \mathbf{b}_1 \mathbf{b}_2 \cdots \mathbf{b}_k, \mathbf{a} \cdot \mathbf{B} = 0 \Leftrightarrow \mathbf{a} \cdot \mathbf{b}_r = 0 \text{ for all } r.$$

$$\mathbf{a} \cdot (B \wedge C) = (\mathbf{a} \cdot B) \wedge C + (-1)^k B \wedge (\mathbf{a} \cdot C). \text{ (}B \text{ is a } k\text{-vector.)}$$

$$\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{c} - (\mathbf{a} \cdot \mathbf{c})\mathbf{b}.$$

$$\mathbf{a} \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \cdots \wedge \mathbf{b}_k) = \sum_r (-1)^{r+1} (\mathbf{a} \cdot \mathbf{b}_r) \mathbf{b}_1 \wedge \cdots \wedge \check{\mathbf{b}}_r \wedge \cdots \wedge \mathbf{b}_k.$$

$$\text{Thus for a blade } \mathbf{B} = \mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \cdots \wedge \mathbf{b}_k, \mathbf{a} \cdot \mathbf{B} = 0 \Leftrightarrow \mathbf{a} \cdot \mathbf{b}_r = 0 \text{ for all } r.$$

$$\mathbf{a} \wedge (B \cdot C) = (\mathbf{a} \cdot B) \cdot C + (-1)^k B \cdot (\mathbf{a} \wedge C). \text{ (}B \text{ is a } k\text{-vector.)}$$

$$(\mathbf{A} \cdot \mathbf{B})^* = \mathbf{A} \wedge \mathbf{B}^*.$$

$$(\mathbf{A} \wedge \mathbf{B})^* = \mathbf{A} \cdot \mathbf{B}^*.$$

$$\mathbf{A} \vee \mathbf{B} = \mathbf{A}^* \cdot \mathbf{B}, \text{ where } * \text{ is with respect to the join of } \mathbf{A} \text{ and } \mathbf{B}.$$

$$(\mathbf{A} \vee \mathbf{B})^* = \mathbf{A}^* \wedge \mathbf{B}^*.$$

## 6 Further Study

### Web Sites

David Hestenes

[http://modelingnts.la.asu.edu/GC\\_R&D.html](http://modelingnts.la.asu.edu/GC_R&D.html)

Cambridge University Geometric Algebra Research Group

<http://www.mrao.cam.ac.uk/~clifford/>

University of Amsterdam Geometric Algebra Website

<http://www.science.uva.nl/ga/>

### Books

C. Doran and A. Lasenby, *Geometric Algebra for Physicists*

(Cambridge University Press, 2003).

Web page: <http://www.mrao.cam.ac.uk/~cjld1/pages/book.htm>

Development of geometric algebra and calculus for applications to physics.  
Noneuclidean geometry.

D. Hestenes, *New Foundations for Classical Mechanics*

(Kluwer Academic Publishers, 1999).

Web page: <http://modelingnts.la.asu.edu/html/NFCM.html>

Textbook suitable for courses at an intermediate level.

L. Dorst, D. Fontijne, S. Mann, *Geometric Algebra for Computer Science:*

*An Object-Oriented Approach to Geometry* (Morgan Kaufmann, 2007)

Web page: <http://www.geometricalgebra.net/>

Development of geometric algebra for applications to computer science.

D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus*

(Kluwer Academic Publishers, 1984).

Web page: [http://modelingnts.la.asu.edu/html/CA\\_to\\_GC.html](http://modelingnts.la.asu.edu/html/CA_to_GC.html)

The bible. Not for the faint of heart.

### Papers available on the Web

#### Elementary.

*A unified mathematical language for physics and engineering in the 21st century*

[http://www.mrao.cam.ac.uk/~clifford/publications/abstracts/dll\\_millen.html](http://www.mrao.cam.ac.uk/~clifford/publications/abstracts/dll_millen.html)

*Synopsis of Geometric Algebra*

<http://modelingnts.la.asu.edu/html/NFMP.html>

*Geometric Algebra: a computational framework for geometrical applications, I & II*

*Performance and elegance of five models of 3D Euclidean geometry*

<http://www.science.uva.nl/ga/publications/CGnA.html>

*Physical Applications of Geometric Algebra*

This is a preliminary version of Doran and Lasenby's book.

<http://www.mrao.cam.ac.uk/~clifford/ptIIIcourse/>

*An Elementary Construction of the Geometric Algebra*

<http://faculty.luther.edu/~macdonal>

### **History.**

*On the Evolution of Geometric Algebra and Geometric Calculus*

<http://modelingnts.la.asu.edu/html/evolution.html>

*A unified mathematical language for physics and engineering in the 21st century*

[http://www.mrao.cam.ac.uk/~clifford/publications/abstracts/dll\\_millen.html](http://www.mrao.cam.ac.uk/~clifford/publications/abstracts/dll_millen.html)

*Grassmann's Vision*

<http://modelingnts.la.asu.edu/html/GeoAlg.html>

### **Polemics.**

*Oersted Medal Lecture 2002: Reforming the mathematical language of physics*

<http://modelingnts.la.asu.edu/html/overview.html>

*Mathematical Viruses*

<http://modelingnts.la.asu.edu/html/GeoAlg.html>

*Unified Language for Mathematics and Physics*

<http://modelingnts.la.asu.edu/html/GeoCalc.html>

*The inner products of geometric algebra*

<http://www.science.uva.nl/ga/publications/index.html>

### **Linear Algebra and Geometry.**

*The Design of Linear Algebra and Geometry*

*Universal Geometric Algebra*

*Projective Geometry with Clifford Algebra*

<http://modelingnts.la.asu.edu/html/GeoAlg.html>

*(Interactively) Exploring the conformal model of 3D Euclidean geometry*

<http://www.science.uva.nl/ga/tutorials/CGA/index.html>

*Conformal Geometry, Euclidean Space and Geometric Algebra*

Paper dll.sheffield.ps.gz at

<http://www.mrao.cam.ac.uk/~clifford/publications/ps/>

*Old Wine in New Bottles: A new algebraic framework for computational geometry*

<http://modelingnts.la.asu.edu/html/ComputationalGeometry.html>

*A Unified Algebraic Framework for Classical Geometry*

<http://modelingnts.la.asu.edu/html/UAFCG.html>

*Lie Groups as Spin Groups*

<http://modelingnts.la.asu.edu/html/GeoAlg.html>

*A Covariant Approach to Geometry using Geometric Algebra*

Includes noneuclidean geometry.

<http://www-sigproc.eng.cam.ac.uk/ga/Category:Reports>

### **Calculus.**

*Multivector Calculus*

*Multivector Functions*

<http://modelingnts.la.asu.edu/html/GeoCalc.html>

*Geometric Calculus*

<http://modelingnts.la.asu.edu/html/NFMP.html>

### **Spacetime and Electromagnetism.**

*Spacetime Physics with Geometric Algebra*

<http://modelingnts.la.asu.edu/html/overview.html>

*SpaceTime Calculus*

<http://modelingnts.la.asu.edu/html/STC.html>

## Quantum Mechanics.

*Geometric Algebra in Physics Education*

<http://modelingnts.la.asu.edu/html/overview.html>

Various papers

<http://modelingnts.la.asu.edu/html/GAinQM.html>

[http://modelingnts.la.asu.edu/html/Impl\\_QM.html](http://modelingnts.la.asu.edu/html/Impl_QM.html)

## Software

GAViewer, software and tutorials.

[http://www.science.uva.nl/ga/viewer/gaviewer\\_download.html](http://www.science.uva.nl/ga/viewer/gaviewer_download.html)

<http://www.science.uva.nl/ga/tutorials/index.html>

Maple software.

<http://www.mrao.cam.ac.uk/~clifford/pages/software.htm>

I have fixed a few bugs and added several procedures, including one for the inner product used here. It works on Maple V and, I think, higher versions.

<http://faculty.luther.edu/~macdonal>

CluCalc.

<http://www.perwass.de/cbup/clu.html>

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Another version:  
[http://www.science.uva.nl/ga/publications/content\\_talks.html#inner\\_talk](http://www.science.uva.nl/ga/publications/content_talks.html#inner_talk)
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