Examples of Geometric Algebra

Alan Bromborsky
Army Research Lab (Retired)
abrombo@verizon.net

January 6, 2009

Newtonian Mechanics

The space of newtonian mechanics is $\mathcal{G}(3,0)$ the euclidian 3-dimensional space. The basis blades for $\mathcal{G}(3,0)$ are:

Grade			
0	1	2	3
1	e_1	e_1e_2	$e_1e_2e_3$
	e_2	e_1e_3	
	e_3	e_2e_3	

Note that e_1e_2, e_1e_3, and e_2e_3 are proportional to the quaternions $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. Where

$$\mathbf{i} = e_3e_2 \quad \mathbf{j} = e_1e_3 \quad \mathbf{k} = e_2e_1 \tag{1}$$

and

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1 \tag{2}$$

$\mathcal{G}\left(3,0\right)$ **Euclidian 3-Space**

The multiplication table for the $\mathcal{G}\left(3,0\right)$ basis blades is

	1	e_1	e_2	e_3	e_1e_2	e_1e_3	e_2e_3	$e_1e_2e_3$
1	1	e_1	e_2	e_3	e_1e_2	e_1e_3	e_2e_3	$e_1e_2e_3$
e_1	e_1	1	e_1e_2	e_1e_3	e_2	e_3	$e_1e_2e_3$	e_2e_3
e_2	e_2	$-e_1e_2$	1	e_2e_3	$-e_1$	$-e_1e_2e_3$	e_3	$-e_1e_3$
e_3	e_3	$-e_1e_3$	$-e_2e_3$	1	$e_1e_2e_3$	$-e_1$	$-e_2$	e_1e_2
e_1e_2	e_1e_2	$-e_2$	e_1	$e_1e_2e_3$	-1	$-e_2e_3$	e_1e_3	$-e_3$
e_1e_3	e_1e_3	$-e_3$	$-e_1e_2e_3$	e_1	e_2e_3	-1	$-e_1e_2$	e_2
e_2e_3	e_2e_3	$e_1e_2e_3$	$-e_3$	e_2	$-e_1e_3$	e_1e_2	-1	$-e_1$
$e_1e_2e_3$	$e_1e_2e_3$	e_2e_3	$-e_1e_3$	e_1e_2	$-e_3$	e_2	$-e_1$	-1

Note that the squares of all the grade 2 and 3 basis blades are -1. The highest rank basis blade (in this case $e_1e_2e_3$) is ususally denoted by I and is called the pseudoscalar.

Rotating Coordinates

In a three dimensional Euclidian space the pseudoscalar $I^2 = -1$ and I commutes (geometric product) with all vectors in the space. If $\hat{\boldsymbol{\omega}}$ is a unit vector in the space then

$$I\hat{\boldsymbol{\omega}} = \hat{\boldsymbol{\omega}} I \tag{3}$$

is a blade that defines a plane of rotation (normal to $\hat{\boldsymbol{\omega}}$) and a general rotation with constant angular velocity ω is given by the rotor

$$R(t) = \cos\frac{\omega t}{2} - \sin\frac{\omega t}{2} I\hat{\boldsymbol{\omega}} \tag{4}$$

and if we wish to rotate a general vector function of time $\boldsymbol{r}(t)$ at angular velocity ω about the $\hat{\boldsymbol{\omega}}$ axis the correct expression is

$$\boldsymbol{r'}(t) = R(t)\,\boldsymbol{r}(t)\,R^{\dagger}(t) \tag{5}$$

Now note (defining $\boldsymbol{\omega} = \omega\hat{\boldsymbol{\omega}}$)

$$\begin{aligned} \frac{dR}{dt} =& \frac{\omega}{2}\left(-\sin\left(\frac{\omega t}{2}\right) - \cos\left(\frac{\omega t}{2}\right) I\hat{\boldsymbol{\omega}}\right) && (6) \\ =& -\frac{\omega}{2} RI\hat{\boldsymbol{\omega}} && (7) \\ =& -RI\frac{\boldsymbol{\omega}}{2} && (8) \end{aligned}$$

Likewise

$$\frac{dR^\dagger}{dt} = I\frac{\boldsymbol{\omega}}{2}R^\dagger \qquad (9)$$

Thus (Note that $\boldsymbol{\omega} \times \boldsymbol{r} = -I\,(\boldsymbol{\omega} \wedge \boldsymbol{r})$)

$$\begin{aligned}
\frac{d\boldsymbol{r'}(t)}{dt} =& \frac{dR}{dt}\boldsymbol{r}R^{\dagger} + R\frac{d\boldsymbol{r}}{dt}R^{\dagger} + R\boldsymbol{r}\frac{dR^{\dagger}}{dt} \\
=& R\left(-\frac{I\boldsymbol{\omega}\boldsymbol{r}}{2} + \frac{\boldsymbol{r}I\boldsymbol{\omega}}{2} + \frac{d\boldsymbol{r}}{dt}\right)R^{\dagger} \\
=& R\left(-\frac{I}{2}\left(\boldsymbol{\omega}\boldsymbol{r} - \boldsymbol{r}\boldsymbol{\omega}\right) + \frac{d\boldsymbol{r}}{dt}\right)R^{\dagger} \\
=& R\left(-I\left(\boldsymbol{\omega} \wedge \boldsymbol{r}\right) + \frac{d\boldsymbol{r}}{dt}\right)R^{\dagger} \\
=& R\left(\boldsymbol{\omega} \times \boldsymbol{r} + \frac{d\boldsymbol{r}}{dt}\right)R^{\dagger} \qquad (10)
\end{aligned}$$

The expression in the parenthesis in equation 10 is the derivative in the rotating coordinate system.

Two Body Problem

All of the follow is in $\mathcal{G}\left(3,0\right)$. For two bodies of masses m_1 and m_2 with central forces the equations of motion of the two masses are

$$m_1\ddot{\boldsymbol{x}}_1 = \boldsymbol{f} \tag{11}$$

$$m_2\ddot{\boldsymbol{x}}_2 = -\boldsymbol{f} \tag{12}$$

In the center of mass coordinate system

$$\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2 \tag{13}$$

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \tag{14}$$

and the equation of motion is

$$\mu \ddot{\boldsymbol{x}} = \boldsymbol{f} \tag{15}$$

If $\boldsymbol{f}$ is a square law force the equation is

$$\mu \ddot{\boldsymbol{x}} = -k \frac{\hat{\boldsymbol{x}}}{r^2} = -k \frac{\boldsymbol{x}}{r^3} \tag{16}$$

If k is positive the force is attractive, negative is repulsive. The conserved angular momentum is

$$L = \mu \boldsymbol{x} \wedge \dot{\boldsymbol{x}} \tag{17}$$

and the total energy is

$$E = \frac{\mu}{2} \dot{\boldsymbol{x}}^2 - \frac{k}{r} \tag{18}$$

Spinor Equations of Motion: Kustaanheimo-Stiefel Transformation

Let $\boldsymbol{x}(t)$ be given by the spinor equation

$$\boldsymbol{x}(t) = U(t)\,\mathbf{e}_1 U^{\dagger}(t) \tag{19}$$

so that

$$\dot{\boldsymbol{x}} = \dot{U}\mathbf{e}_1 U^{\dagger} + U\mathbf{e}_1 \dot{U}^{\dagger} \tag{20}$$

and

$$r = |\boldsymbol{x}| = UU^{\dagger} \tag{21}$$

Since $U = \langle U \rangle_0 + \langle U \rangle_2$, U has four degrees of freedom while $\boldsymbol{x}$ only has three which means we can constrain U with a gauge condition. Note that:

$$\dot{U} = \left\langle \dot{U} \right\rangle_0 + \left\langle \dot{U} \right\rangle_2 \tag{22}$$

$$\mathbf{e}_1 = \langle \mathbf{e}_1 \rangle_1 \tag{23}$$

$$\dot{U}\mathbf{e}_1 = \left\langle \dot{U}\mathbf{e}_1 \right\rangle_1 + \left\langle \dot{U}\mathbf{e}_1 \right\rangle_3 \tag{24}$$

$$\dot{U}\mathbf{e}_1 U^\dagger = \left\langle \dot{U}\mathbf{e}_1 U^\dagger \right\rangle_1 + \left\langle \dot{U}\mathbf{e}_1 U^\dagger \right\rangle_3 \tag{25}$$

but $\left\langle \dot{U}\mathbf{e}_1 U^\dagger \right\rangle_1$ has three coefficients while $\left\langle \dot{U}\mathbf{e}_1 U^\dagger \right\rangle_3$ has one coefficient so that our gauge condition is

$$\left\langle \dot{U}\mathbf{e}_1 U^\dagger \right\rangle_3 = 0. \tag{26}$$

This means that $\dot{U}\mathbf{e}_1 U^\dagger$ is a vector which is equal to its own reverse and

$$\dot{\boldsymbol{x}} = 2\dot{U}\mathbf{e}_1 U^\dagger = 2U\mathbf{e}_1 \dot{U}^\dagger \tag{27}$$

If no gauge condition is imposed we have

$$\left\langle \dot{U}\mathbf{e}_1 U^\dagger \right\rangle_3 + \left\langle U\mathbf{e}_1 \dot{U}^\dagger \right\rangle_3 = 0 \tag{28}$$

since $\dot{\boldsymbol{x}}$ must be a vector. Now change the variables from t to s defined by

$$\frac{dt}{ds} = r \tag{29}$$

then

$$\dot{U} = \frac{dU}{ds}\frac{ds}{dt} = \frac{1}{r}\frac{dU}{ds} \tag{30}$$

$$\dot{\boldsymbol{x}}r = 2\frac{dU}{ds}\mathbf{e}_1 U^\dagger \tag{31}$$

$$\dot{\boldsymbol{x}} r U \mathbf{e}_1 = 2\frac{dU}{ds}\mathbf{e}_1 U^\dagger U \mathbf{e}_1 = 2r\frac{dU}{ds} \tag{32}$$

$$\dot{\boldsymbol{x}} U \mathbf{e}_1 = 2\frac{dU}{ds} \tag{33}$$

$$\begin{aligned} 2\frac{d^2U}{ds^2} &= \ddot{\boldsymbol{x}} r U \mathbf{e}_1 + \dot{\boldsymbol{x}}\frac{dU}{ds}\mathbf{e}_1 && (34) \\ &= \ddot{\boldsymbol{x}} r U \mathbf{e}_1 + \frac{1}{2}\dot{\boldsymbol{x}}\dot{\boldsymbol{x}} U \mathbf{e}_1 \mathbf{e}_1 && (35) \\ &= \ddot{\boldsymbol{x}} r U \mathbf{e}_1 + \frac{1}{2}\dot{\boldsymbol{x}}^2 U && (36) \\ &= \ddot{\boldsymbol{x}} r U \mathbf{e}_1 \frac{UU^\dagger}{r} + \frac{1}{2}\dot{\boldsymbol{x}}^2 U && (37) \\ &= \left(\ddot{\boldsymbol{x}}\boldsymbol{x} + \frac{1}{2}\dot{\boldsymbol{x}}^2\right) U && (38) \end{aligned}$$

For the inverse sqare law

$$\ddot{\boldsymbol{x}} = -\frac{k}{\mu}\frac{\boldsymbol{x}}{r^3} \tag{39}$$

then

$$\ddot{\boldsymbol{x}}\boldsymbol{x} + \frac{1}{2}\dot{\boldsymbol{x}}^2 = -\frac{k}{\mu}\frac{\boldsymbol{x}^2}{r^3} + \frac{1}{2}\dot{\boldsymbol{x}}^2 = \frac{1}{2}\dot{\boldsymbol{x}}^2 - \frac{k}{\mu}\frac{1}{r} = \frac{E}{\mu} \tag{40}$$

where E is the total energy of the system and the equation of motion is now

$$\frac{d^2U}{ds^2} = \frac{E}{2\mu}U \tag{41}$$

Letting $\Omega = \sqrt{\left|\frac{E}{2\mu}\right|}$ then

$$U(s) = \left\{ \begin{array}{cc} U_0\left(\cos\left(\Omega s\right) + R\sin\left(\Omega s\right)\right) & E < 0 \\ U_0\left(\cosh\left(\Omega s\right) + R\sinh\left(\Omega s\right)\right) & E > 0 \end{array} \right\} \tag{42}$$

Where both U_0 and R are constant even multivectors (constant spinors).

Now require that

$$\boldsymbol{x}(0) = r_0\mathbf{e}_1 = U(0)\,\mathbf{e}_1 U^\dagger(0) = U_0\mathbf{e}_1 U_0^\dagger \tag{43}$$

for this to be true in three dimensions we must have $U_0 = \sqrt{r_0}$ a simple scalar so that (we will treat the case of a bound orbit, $E < 0$)

$$U(s) = \sqrt{r_0}\left(\cos\left(\Omega s\right) + \left(\left\langle R\right\rangle_0 + \left\langle R\right\rangle_2\right)\sin\left(\Omega s\right)\right) \tag{44}$$

since $r(s) = U(s)\,U^\dagger(s)$ if we want $r(0)$ to be maximized (apogee) it is required that

$$\left.\frac{dr}{ds}\right|_{s=0} = 0 \tag{45}$$

this implies $\left\langle R\right\rangle_0 = 0$ and

$$U(s) = \sqrt{r_0}\left(\cos\left(\Omega s\right) + \left\langle R\right\rangle_2 \sin\left(\Omega s\right)\right) \tag{46}$$

With some manipulation we can also show that the angular momentum is

$$L = \mu \boldsymbol{x} \wedge \dot{\boldsymbol{x}} = \mu \left(U \frac{dU^\dagger}{ds} - \frac{dU}{ds} U^\dagger \right) \tag{47}$$

which results in

$$\langle R \rangle_2 = -\frac{L}{2\Omega \mu r_0} = -\tilde{L} \tag{48}$$

and

$$U(s) = \sqrt{r_0} \left(\cos\left(\Omega s\right) - \tilde{L} \sin\left(\Omega s\right) \right) \tag{49}$$

$$r(s) = U(s)\, U^\dagger(s) = \frac{r_0}{2} \left(\left(1 + \left| \tilde{L}^2 \right| \right) + \left(1 - \left| \tilde{L}^2 \right| \right) \cos\left(2\Omega s\right) \right) \tag{50}$$

and

$$t = \int_0^s r(s')\,ds' = \frac{r_0}{2}\int_0^s \left(\left(1+\left|\tilde{L}^2\right|\right) + \left(1-\left|\tilde{L}^2\right|\right)\cos\left(2\Omega s'\right)\right) ds' \tag{51}$$

$$= \frac{r_0}{2}\left(\left(1+\left|\tilde{L}^2\right|\right) s + \frac{1-\left|\tilde{L}^2\right|}{2\Omega}\sin\left(2\Omega s\right)\right) \tag{52}$$

Finally

$$\boldsymbol{x}(s) = U(s)\,\mathbf{e}_1 U^\dagger(s) \tag{53}$$

$$= r_0\left(\cos\left(\Omega s\right) - \tilde{L}\sin\left(\Omega s\right)\right)\mathbf{e}_1\left(\cos\left(\Omega s\right) + \tilde{L}\sin\left(\Omega s\right)\right) \tag{54}$$

but $\tilde{L} = \left|\tilde{L}\right| \mathbf{e}_1\mathbf{e}_2$ where $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0$ and $\left(\mathbf{e}_1\mathbf{e}_2\right)^2 = -1.$ So

$$\frac{\boldsymbol{x}}{r_0} = \frac{1}{2}\left(\left(1 - \left|\tilde{L}^2\right|\right) + \left(1 + \left|\tilde{L}^2\right|\right)\cos\left(2\Omega s\right)\right)\mathbf{e}_1 + \left|\tilde{L}\right|\sin\left(2\Omega s\right)\mathbf{e}_2 \tag{55}$$

Now define

$$\theta = 2\Omega s \quad \tau = \frac{4t\Omega}{r_0} \tag{56}$$

so that

$$\tau(\theta) = \left(1 + \left|\tilde{L}^2\right|\right)\theta + \left(1 - \left|\tilde{L}^2\right|\right)\sin\left(\theta\right) \tag{57}$$

and

$$\frac{\boldsymbol{x}}{r_0} = \frac{1}{2}\left(\left(1 - \left|\tilde{L}^2\right|\right) + \left(1 + \left|\tilde{L}^2\right|\right)\cos\left(\theta\right)\right)\mathbf{e}_1 + \left|\tilde{L}\right|\sin\left(\theta\right)\mathbf{e}_2 \tag{58}$$

$$= \frac{x^1(\theta)}{r_0}\mathbf{e}_1 + \frac{x^2(\theta)}{r_0}\mathbf{e}_2 \tag{59}$$

the area swept by the radius vector between angles θ_1 and θ_2 is

$$\begin{aligned}
A(\theta_1, \theta_2) &= \frac{1}{2}\int_{\theta_1}^{\theta_2} d\theta \left(x^1 \frac{dx^2}{d\theta} - \frac{dx^2}{d\theta} x^2 \right) && (60) \\
&= \frac{r_0^2 \left|\tilde{L}\right|}{4} \int_{\theta_1}^{\theta_2} d\theta \left(\left(1 + \left|\tilde{L}^2\right|\right) + \left(1 - \left|\tilde{L}^2\right|\right) \cos\left(\theta\right) \right) && (61) \\
&= \frac{r_0^2 \left|\tilde{L}\right|}{4} \left[\left(1 + \left|\tilde{L}^2\right|\right) \theta + \left(1 - \left|\tilde{L}^2\right|\right) \sin\left(\theta\right) \right]_{\theta_1}^{\theta_2} && (62) \\
&= \frac{r_0^2 \left|\tilde{L}\right|}{4} \left(\tau(\theta_2) - \tau(\theta_1)\right) && (63)
\end{aligned}$$

Thus equal areas are swept out in equal times by the radius vector and Keplers second law is obeyed.

For a sanity check let us consider circular motion. In that case

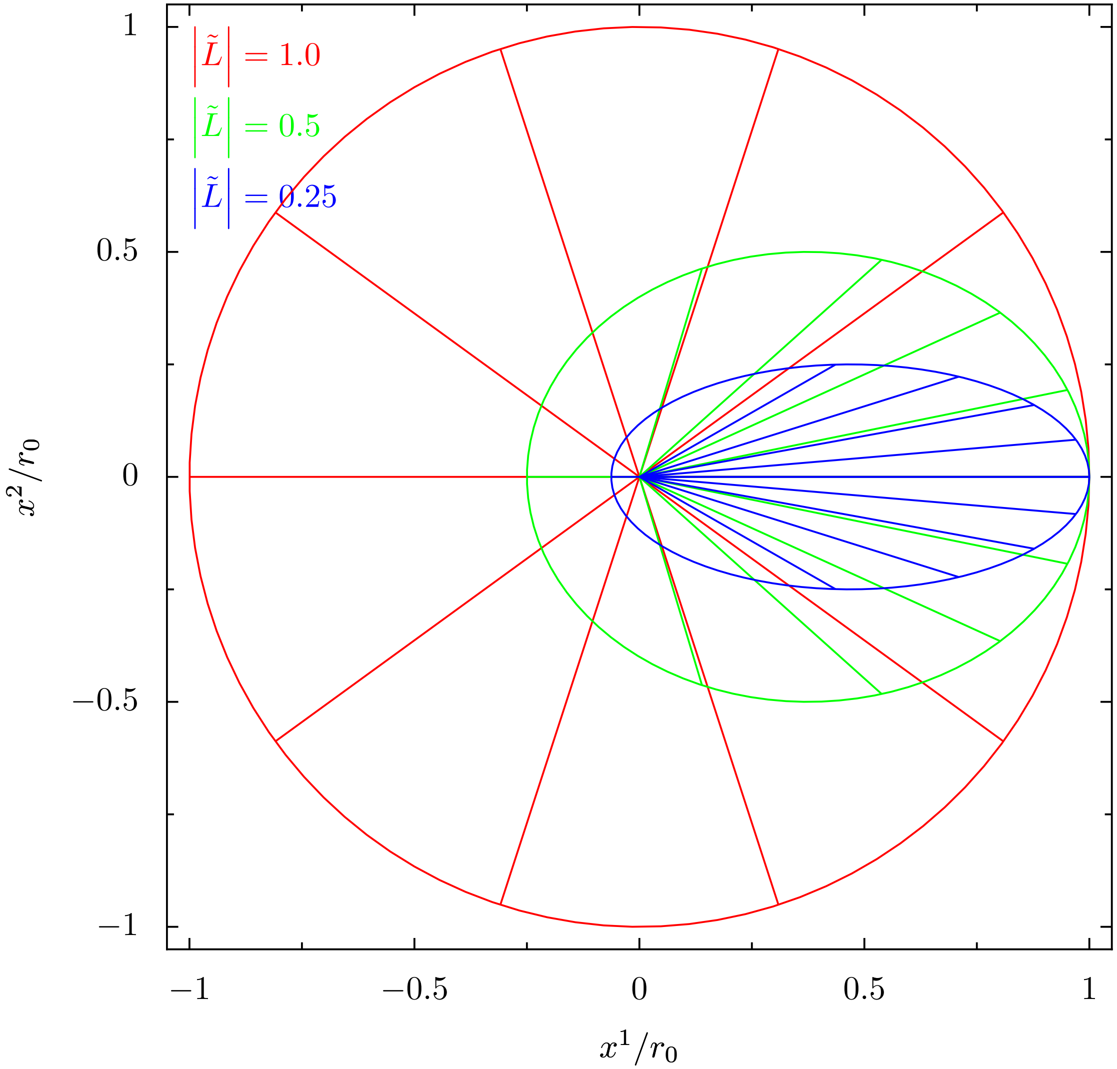
$$E = \frac{1}{2}\frac{k\mu}{r_0} \quad \Omega = \frac{1}{2}\sqrt{\frac{k}{r_0}} \quad |L| = \mu\sqrt{kr_0} \quad \omega = \sqrt{\frac{k}{r_0^3}} \tag{64}$$

so that when we substitute

$$\left|\tilde{L}\right| = 1 \quad 2\Omega s = \omega t \tag{65}$$

and

$$\frac{\boldsymbol{x}}{r_0} = \cos\left(\omega t\right)\mathbf{e}_1 + \sin\left(\omega t\right)\mathbf{e}_2 \tag{66}$$



Examples of Planetary Motion: Each segment is 0.1 period.

Relativity

The fundamental concept of special relativity that we must encode with geometric algebra is

$$(ct)^2 - r^2 = (ct')^2 - (r')^2 = 0 \tag{67}$$

where (t, r) and (t', r') are the relative time and positions of a photon in two coincident frames moving at a constant relative velocity to one another. The geometric algebra that satisfies this is either $\mathcal{G}(1,3)$ or $\mathcal{G}(3,1)$. We shall use $\mathcal{G}(1,3)$ (this is used by particle physicists while general relativitists usually use the other). An orthonormal basis (in the general sense that the square of a basis vector is ± 1) is $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ where (roman indices run from 1 to 3, greek from 0 to 3)

$$\gamma_0^2 = 1 \quad \gamma_0 \cdot \gamma_i = 0 \quad \gamma_i \cdot \gamma_j = -\delta_{ij} \tag{68}$$

all the relations are summarised by

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} = 2\,\mathrm{diag}(+ - --) \tag{69}$$

A spacetime event is encoded by the 4-vector

$$x = x^\mu \gamma_\mu = ct\gamma_0 + x^i \gamma_i \tag{70}$$

From now on let $c = 1$ and note that since the 4-space has a mixed signature x^2 can be positive, negative, or zero. Note that in the following multiplication table

$$\begin{array}{lll} (\gamma_i\gamma_j)^2 = -1 & (\gamma_i\gamma_0)^2 = 1 & \\ I = \gamma_0\gamma_1\gamma_2\gamma_3 & I^2 = -1 & I = I^\dagger \end{array} \tag{71}$$

The basis blades for $\mathcal{G}\left(1,3\right)$ are:

Grade				
0	1	2	3	4
1	γ_0	$\gamma_0\gamma_1$	$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2\gamma_3$
	γ_1	$\gamma_0\gamma_2$	$\gamma_0\gamma_1\gamma_3$	
	γ_2	$\gamma_1\gamma_1$	$\gamma_0\gamma_2\gamma_3$	
	γ_3	$\gamma_0\gamma_3$	$\gamma_1\gamma_2\gamma_3$	
		$\gamma_1\gamma_3$		
		$\gamma_2\gamma_3$		

The multiplication table for the $\mathcal{G}\left(1,3\right)$ basis blades is (Part I)

	1	γ_0	γ_1	γ_2	γ_3	$\gamma_0\gamma_1$	$\gamma_0\gamma_2$	$\gamma_1\gamma_2$
1	1	γ_0	γ_1	γ_2	γ_3	$\gamma_0\gamma_1$	$\gamma_0\gamma_2$	$\gamma_1\gamma_2$
γ_0	γ_0	1	$\gamma_0\gamma_1$	$\gamma_0\gamma_2$	$\gamma_0\gamma_3$	γ_1	γ_2	$\gamma_0\gamma_1\gamma_2$
γ_1	γ_1	$-\gamma_0\gamma_1$	-1	$\gamma_1\gamma_2$	$\gamma_1\gamma_3$	γ_0	$-\gamma_0\gamma_1\gamma_2$	$-\gamma_2$
γ_2	γ_2	$-\gamma_0\gamma_2$	$-\gamma_1\gamma_2$	-1	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2$	γ_0	γ_1
γ_3	γ_3	$-\gamma_0\gamma_3$	$-\gamma_1\gamma_3$	$-\gamma_2\gamma_3$	-1	$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$
$\gamma_0\gamma_1$	$\gamma_0\gamma_1$	$-\gamma_1$	$-\gamma_0$	$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_3$	1	$-\gamma_1\gamma_2$	$-\gamma_0\gamma_2$
$\gamma_0\gamma_2$	$\gamma_0\gamma_2$	$-\gamma_2$	$-\gamma_0\gamma_1\gamma_2$	$-\gamma_0$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2$	1	$\gamma_0\gamma_1$
$\gamma_1\gamma_2$	$\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2$	γ_2	$-\gamma_1$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_2$	$-\gamma_0\gamma_1$	-1
$\gamma_0\gamma_3$	$\gamma_0\gamma_3$	$-\gamma_3$	$-\gamma_0\gamma_1\gamma_3$	$-\gamma_0\gamma_2\gamma_3$	$-\gamma_0$	$\gamma_1\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$
$\gamma_1\gamma_3$	$\gamma_1\gamma_3$	$\gamma_0\gamma_1\gamma_3$	γ_3	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_1$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_2\gamma_3$
$\gamma_2\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$	γ_3	$-\gamma_2$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_3$	$\gamma_1\gamma_3$
$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2$	$\gamma_1\gamma_2$	$\gamma_0\gamma_2$	$-\gamma_0\gamma_1$	$\gamma_0\gamma_1\gamma_2\gamma_3$	γ_2	$-\gamma_1$	$-\gamma_0$
$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_1\gamma_3$	$\gamma_1\gamma_3$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_1$	γ_3	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_2\gamma_3$
$\gamma_0\gamma_2\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_2$	$\gamma_1\gamma_2\gamma_3$	γ_3	$\gamma_0\gamma_1\gamma_3$
$\gamma_1\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_2\gamma_3$	$\gamma_1\gamma_3$	$-\gamma_1\gamma_2$	$\gamma_0\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_3$	$-\gamma_3$
$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_3$	$-\gamma_0\gamma_1\gamma_2$	$\gamma_2\gamma_3$	$-\gamma_1\gamma_3$	$-\gamma_0\gamma_3$

The multiplication table for the $\mathcal{G}\,(1,3)$ basis blades is (Part II)

	$\gamma_0\gamma_3$	$\gamma_1\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$
1	$\gamma_0\gamma_3$	$\gamma_1\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$
γ_0	γ_3	$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2$	$\gamma_1\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$
γ_1	$-\gamma_0\gamma_1\gamma_3$	$-\gamma_3$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_2$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_2\gamma_3$	$\gamma_0\gamma_2\gamma_3$
γ_2	$-\gamma_0\gamma_2\gamma_3$	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_3$	$-\gamma_0\gamma_1$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_3$	$\gamma_1\gamma_3$	$-\gamma_0\gamma_1\gamma_3$
γ_3	γ_0	γ_1	γ_2	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_1$	$-\gamma_0\gamma_2$	$-\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2$
$\gamma_0\gamma_1$	$-\gamma_1\gamma_3$	$-\gamma_0\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$	γ_2	γ_3	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_2\gamma_3$	$\gamma_2\gamma_3$
$\gamma_0\gamma_2$	$-\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_3$	$-\gamma_1$	$\gamma_1\gamma_2\gamma_3$	γ_3	$\gamma_0\gamma_1\gamma_3$	$-\gamma_1\gamma_3$
$\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_2\gamma_3$	$-\gamma_1\gamma_3$	$-\gamma_0$	$\gamma_0\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_3$	$-\gamma_3$	$-\gamma_0\gamma_3$
$\gamma_0\gamma_3$	1	$\gamma_0\gamma_1$	$\gamma_0\gamma_2$	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_1$	$-\gamma_2$	$-\gamma_0\gamma_1\gamma_2$	$\gamma_1\gamma_2$
$\gamma_1\gamma_3$	$-\gamma_0\gamma_1$	-1	$\gamma_1\gamma_2$	$-\gamma_0\gamma_2\gamma_3$	$-\gamma_0$	$\gamma_0\gamma_1\gamma_2$	γ_2	$\gamma_0\gamma_2$
$\gamma_2\gamma_3$	$-\gamma_0\gamma_2$	$-\gamma_1\gamma_2$	-1	$\gamma_0\gamma_1\gamma_3$	$-\gamma_0\gamma_1\gamma_2$	$-\gamma_0$	$-\gamma_1$	$-\gamma_0\gamma_1$
$\gamma_0\gamma_1\gamma_2$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_3$	-1	$\gamma_2\gamma_3$	$-\gamma_1\gamma_3$	$-\gamma_0\gamma_3$	$-\gamma_3$
$\gamma_0\gamma_1\gamma_3$	$-\gamma_1$	$-\gamma_0$	$\gamma_0\gamma_1\gamma_2$	$-\gamma_2\gamma_3$	-1	$\gamma_1\gamma_2$	$\gamma_0\gamma_2$	γ_2
$\gamma_0\gamma_2\gamma_3$	$-\gamma_2$	$-\gamma_0\gamma_1\gamma_2$	$-\gamma_0$	$\gamma_1\gamma_3$	$-\gamma_1\gamma_2$	-1	$-\gamma_0\gamma_1$	$-\gamma_1$
$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2$	γ_2	$-\gamma_1$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_2$	$\gamma_0\gamma_1$	1	$-\gamma_0$
$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_1\gamma_2$	$\gamma_0\gamma_2$	$-\gamma_0\gamma_1$	γ_3	$-\gamma_2$	γ_1	γ_0	-1

Observers, Trajectories, and Frames

Let $fctx\lambda$ be a parameterized curve in spacetime where λ is a monotonically increasing parameter. The tangent to the curve is given by

$$x' = \frac{dx}{d\lambda} \tag{72}$$

If the parameter is changed from λ to τ then

$$\frac{dx}{d\tau} = \frac{d\lambda}{d\tau}\frac{dx}{d\lambda} \tag{73}$$

$$\left(\frac{dx}{d\tau}\right)^2 = \left(\frac{d\lambda}{d\tau}\right)^2 \left(\frac{dx}{d\lambda}\right)^2 \tag{74}$$

since $\left(\frac{d\lambda}{d\tau}\right)^2 \geq 0$ the sign of $\left(\frac{dx}{d\lambda}\right)^2$ is an invarient function of the path and does not depend upon the parameterization. and the sign of $\frac{dx}{d\lambda}$ is an intrisic property of the spacetime curve.

Now consider those curves such that $\left(\frac{dx}{d\lambda}\right)^2 > 0$ in the parameter interval and define

$$\Delta\tau = \int_{\lambda_1}^{\lambda_2} \left(\frac{dx}{d\lambda} \cdot \frac{dx}{d\lambda}\right)^{\frac{1}{2}} d\lambda \tag{75}$$

and note that if we reparametrize from λ to λ' we get

$$\Delta\tau = \int_{\lambda'_1}^{\lambda'_2} \left(\frac{d\lambda'}{d\lambda}\frac{dx}{d\lambda'} \cdot \frac{dx}{d\lambda'}\frac{d\lambda'}{d\lambda}\right)^{\frac{1}{2}} \left|\frac{d\lambda}{d\lambda'}\right| d\lambda' \tag{76}$$

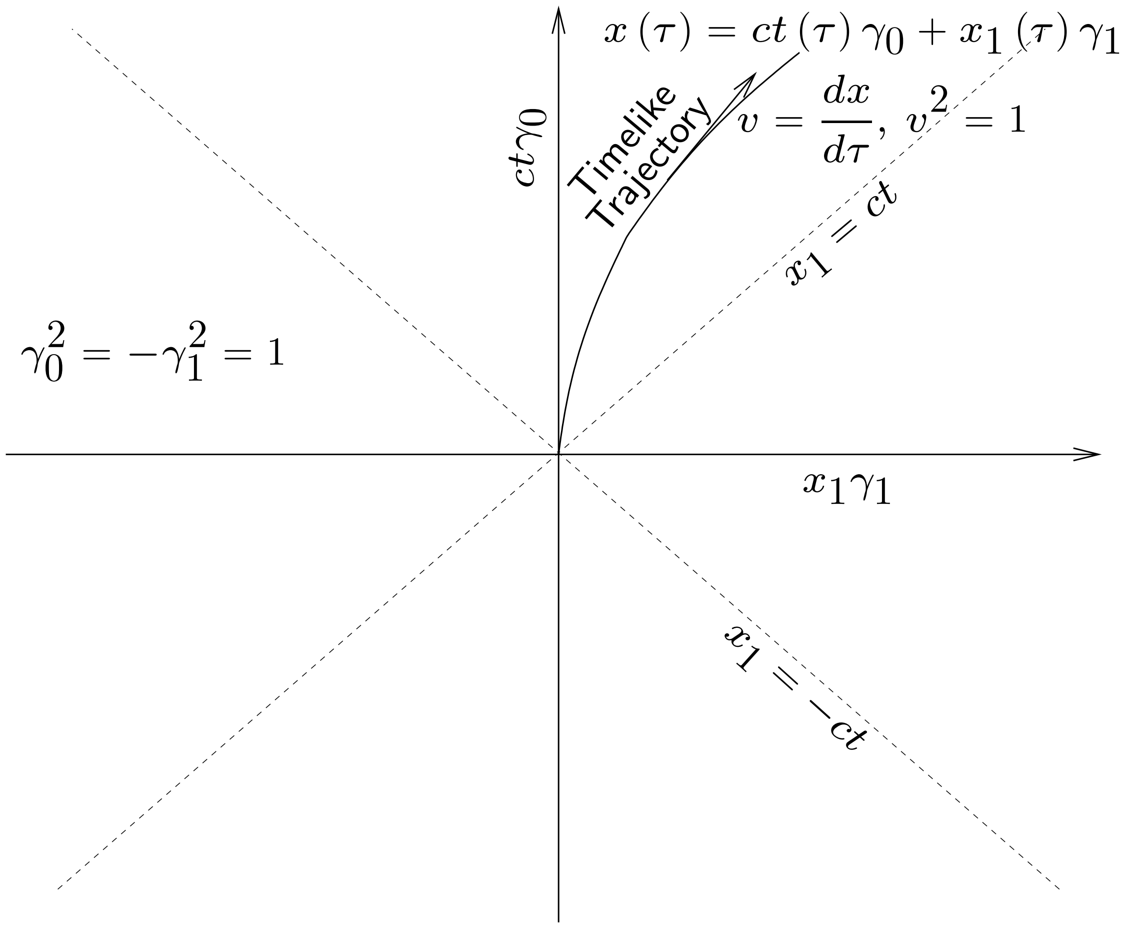
$$\Delta\tau = \int_{\lambda_1'}^{\lambda_2'} \left(\frac{dx}{d\lambda'}\cdot\frac{dx}{d\lambda'}\right)^{\frac{1}{2}} \left|\frac{d\lambda'}{d\lambda}\right| \left|\frac{d\lambda}{d\lambda'}\right| d\lambda' \tag{77}$$

$$= \int_{\lambda_1'}^{\lambda_2'} \left(\frac{dx}{d\lambda'}\cdot\frac{dx}{d\lambda'}\right)^{\frac{1}{2}} d\lambda' \tag{78}$$

so that $\Delta\tau$ is an invarient property of the timelike curve and not dependent upon the parametrization. Now parametrize x in terms of τ which is called the proper time of the spacetime path. Then we have from the definition of $\Delta\tau$ that

$$v^2 = \left(\frac{dx}{d\tau}\right)^2 = 1 \tag{79}$$

where $v = \dfrac{dx}{d\tau}$ is the 4-velocity of the spacetime trajectory.



A spacetime frame $\{\mathbf{e}_\mu\}$ is constructed at each point $x(\tau)$ of a spacetime path in the following manner:

- Set $\mathbf{e}_0 = v(\tau)$
- Construct an orthonormal triad $\{\mathbf{e}_i\}$ such that $\mathbf{e}_\mu \cdot \mathbf{e}_\nu = \eta_{\mu\nu}$
- Assume that any event in spacetime can be given a set of coordinates $x^\mu = \mathbf{e}^\mu \cdot x$.
- Take the point on the spacetime curve as the spatial origin of the frame vectors $\{\mathbf{e}_i\}$.

Note that in our system $\mathbf{e}^0 = \mathbf{e}_0$ and $\mathbf{e}^i = -\mathbf{e}_i$.

Relative Vectors

Using the frame $\{\mathbf{e}_\mu\}$ we have constructed with $\mathbf{e}_0 = v$ we can express an event x in this frame as

$$x = t\mathbf{e}_0 + x^i\mathbf{e}_i \tag{80}$$

and

$$t = x \cdot \mathbf{e}^0 = x \cdot \mathbf{e}_0 = x \cdot v \tag{81}$$

and spatial coordinates

$$x^i = x \cdot \mathbf{e}^i = -x \cdot \mathbf{e}_i \tag{82}$$

Let the event be a point on the worldline of an object at rest in our frame. The spatial vector to this object is then

$$x^i\mathbf{e}_i = x \cdot \mathbf{e}^\mu\mathbf{e}_\mu - x \cdot \mathbf{e}^0\mathbf{e}_0 = xvv - (x \cdot v)\, v = (x \wedge v)\, v \tag{83}$$

Defining the relative vector $\boldsymbol{x}$ as

$$\boldsymbol{x} = x \wedge v \tag{84}$$

Then

$$xv = x \cdot v + x \wedge v = t + \boldsymbol{x} \tag{85}$$

but

$$x^2 = xvvx = (x \cdot v + x \wedge v)\,(x \cdot v + v \wedge x) \tag{86}$$
$$= (t + \boldsymbol{x})\,(t - \boldsymbol{x}) = t^2 - \boldsymbol{x}^2 = t^2 - \left(x^i\right)^2 \tag{87}$$

giving the correct invarient quantity. Thus one can define the relative spatial basis vectors (which are actually bivectors)

$$\boldsymbol{\sigma}_i = \mathbf{e}_i \wedge v = \mathbf{e}_i \mathbf{e}_0 = \gamma_i \gamma_0 \tag{88}$$

This definition give the term relative vector additional meaning since the vector space spanned by the $\boldsymbol{\sigma}_i$'s inherits the geometric algebra $\mathcal{G}\left(3,0\right)$ of a Euclidian three space due to the multiplicitive properties of the γ_μ's.

$$\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j + \boldsymbol{\sigma}_j\boldsymbol{\sigma}_i = 2\delta_{ij} \tag{89}$$

$$\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = \gamma_0\gamma_1\gamma_2\gamma_3 = I \tag{90}$$

$$\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j - \boldsymbol{\sigma}_j\boldsymbol{\sigma}_i = 2\epsilon_{ijk}I\boldsymbol{\sigma}_k \tag{91}$$

Thus

$$\boldsymbol{x} = x^i\boldsymbol{\sigma}_i \tag{92}$$

Relative Velocity

Let an observer with constant 4-velocity v measure the relative velocity of a particle with a 4-velocity $u(\tau) = \dot{x}(\tau)$ with $u^2 = 1$. Then

$$uv = \frac{d}{d\tau}(x(\tau)\, v) = \frac{d}{d\tau}(t + \boldsymbol{x}) \tag{93}$$

and

$$\frac{dt}{d\tau} = u \cdot v, \quad \frac{d\boldsymbol{x}}{d\tau} = u \wedge v \tag{94}$$

The relative velocity $\boldsymbol{u}$ measuered in the v frame is

$$\boldsymbol{s} = \frac{d\boldsymbol{x}}{dt} = \frac{d\boldsymbol{x}}{d\tau}\frac{d\tau}{dt} = \frac{u \wedge v}{u \cdot v} \tag{95}$$

but

$$\begin{aligned} (u \wedge v)^2 &= -(u \wedge v)(v \wedge u) && (96) \\ &= -\frac{1}{4}(uv - vu)(vu - uv) && (97) \\ &= -\frac{1}{4}(uvvu - uvuv - vuvu + vuuv) && (98) \\ &= -\frac{1}{4}(4 - (uv + vu)(uv + vu)) && (99) \\ &= (u \cdot v)^2 - 1 && (100) \end{aligned}$$

and

$$\left(\frac{u \wedge v}{u \cdot v}\right)^2 = 1 - \frac{1}{(u \cdot v)^2} < 1 \quad (101)$$

but

$$u \cdot v = (u^\mu \mathbf{e}_\mu) \cdot \mathbf{e}_0 = u^0 \quad (102)$$

and

$$u^\mu u^\mu = 1 \rightarrow \left(u^0\right)^2 \geq 1 \rightarrow (u \cdot v)^2 \geq 1 \tag{103}$$

Consider the case where relative velocities $\boldsymbol{u}$ and $\boldsymbol{v}$ are aligned. Then we have

$$u = u^0\mathbf{e}_0 + u^1\mathbf{e}_1 \tag{104}$$

$$v = v^0\mathbf{e}_0 + v^1\mathbf{e}_1 \tag{105}$$

and

$$\boldsymbol{s} = \frac{u \wedge v}{u \cdot v} = \frac{u^1v^0 - u^0v^1}{u^0v^0 - u^1v^1} \tag{106}$$

$$= \frac{\dfrac{u^1}{u^0} - \dfrac{v^1}{v^0}}{1 - \dfrac{u^1}{u^0}\dfrac{v^1}{v^0}} \tag{107}$$

The critical thing here is that the components are parameterized in terms of τ not t. To parameterize in terms of t for the moving frame we must divide each component by u^0 and v^0 respectively. Using our relative vector convention for $\boldsymbol{u}$ and $\boldsymbol{v}$ we have

$$\boldsymbol{s} = \frac{u \wedge v}{u \cdot v} = \frac{\boldsymbol{u} - \boldsymbol{v}}{1 - \boldsymbol{u} \cdot \boldsymbol{v}} \tag{108}$$

Now write the Lorentz factor γ as

$$\begin{aligned} \gamma^{-2} &= 1 - \boldsymbol{u}^2 & (109) \\ &= 1 + (u \cdot v)^{-2} \left((uv - u \cdot v)(vu - v \cdot u) \right) & (110) \\ &= (u \cdot v)^{-2} & (111) \end{aligned}$$

and

$$\gamma = u \cdot v \tag{112}$$

and

$$u = uvv = (u \cdot v + u \wedge v)\, v = \gamma\, (1 + \boldsymbol{u})\, v \tag{113}$$

Lorentz Transformation

We now have all the tools needed to derive the Lorentz transformation with Geometric Algebra. Consider a two dimensional time-like plane with with coordinates t and x and basis vectors $\boldsymbol{e}_t$ and $\boldsymbol{e}_x$ $\left(\boldsymbol{e}_t^2 = -\boldsymbol{e}_x^2 = 1\right)$. Then a general space-time vector in the plane is given by

$$x = t\boldsymbol{e}_t + x\boldsymbol{e}_x = t'\boldsymbol{e}'_t + x'\boldsymbol{e}'_x \tag{114}$$

where the basis vectors of the two coordinate systems are related by

$$\boldsymbol{e'}_{\{t,x\}} = R\boldsymbol{e}_{\{t,x\}}R^{\dagger} \tag{115}$$

and R is a Minkowski plane rotor since $\left(\boldsymbol{e}_t\boldsymbol{e}_x\right)^2 = 1$

$$R = e^{\dfrac{\alpha}{2}\boldsymbol{e}_t\boldsymbol{e}_x} = \sinh\left(\frac{\alpha}{2}\right) + \cosh\left(\frac{\alpha}{2}\right)\boldsymbol{e}_t\boldsymbol{e}_x \tag{116}$$

so that

$$R\boldsymbol{e}_tR^\dagger = \cosh(\alpha)\,\boldsymbol{e}_t + \sinh(\alpha)\,\boldsymbol{e}_x \tag{117}$$

and

$$R\boldsymbol{e}_xR^\dagger = \cosh(\alpha)\,\boldsymbol{e}_x + \sinh(\alpha)\,\boldsymbol{e}_t \tag{118}$$

Now consider the special case that the primed coordinate system is moving with velocity β in the direction of $\boldsymbol{e}_x$ and the two coordinate systems were coincident at time $t = 0$. Then $x = \beta t$ and $x' = 0$ so we may write

$$t\boldsymbol{e}_t + \beta t\boldsymbol{e}_x = t' R\boldsymbol{e}_tR^\dagger \tag{119}$$

$$\frac{t}{t'}\left(\boldsymbol{e}_t + \beta\boldsymbol{e}_x\right) = \cosh(\alpha)\,\boldsymbol{e}_t + \sinh(\alpha)\,\boldsymbol{e}_x \tag{120}$$

Equating components gives

$$\cosh(\alpha) = \frac{t}{t'} \tag{121}$$

$$\sinh(\alpha) = \frac{t}{t'}\beta \tag{122}$$

Solving for α and $\dfrac{t}{t'}$ in equations 121 and 122 gives

$$\tanh(\alpha) = \beta \tag{123}$$

$$\frac{t}{t'} = \gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{124}$$

Now consider the general case of x, t and x', t' giving

$$\begin{aligned} t\boldsymbol{e}_t + x\boldsymbol{e}_x &= t'R\boldsymbol{e}_tR^\dagger + x'R\boldsymbol{e}_xR^\dagger \qquad (125)\\ &= t'\gamma(\boldsymbol{e}_t + \beta\boldsymbol{e}_x) + x'\gamma(\boldsymbol{e}_x + \beta\boldsymbol{e}_t) \qquad (126)\end{aligned}$$

Equating basis vector coefficients recovers the Lorentz transformation

$$\begin{gathered} t = \gamma(t' + \beta x') \\ x = \gamma(x' + \beta t') \end{gathered} \tag{127}$$

Maxwell's Equations

In a given frame the electric and magnetic field vectors are

$$\vec{E} = E^1\boldsymbol{\gamma_1} + E^2\boldsymbol{\gamma_2} + E^3\boldsymbol{\gamma_3}, \tag{128}$$

$$\vec{B} = B^1\boldsymbol{\gamma_1} + B^2\boldsymbol{\gamma_2} + B^3\boldsymbol{\gamma_3} \tag{129}$$

and the four current J is

$$J = J^0\boldsymbol{\gamma_0} + J^1\boldsymbol{\gamma_1} + J^2\boldsymbol{\gamma_2} + J^3\boldsymbol{\gamma_3} \tag{130}$$

with the electric and magnetic field bi-vectors (relative vectors) given by

$$E = \vec{E}\gamma_0 \quad \text{and} \quad B = \vec{B}\gamma_0 \tag{131}$$

$$\begin{aligned} F = E + IB &= -E^1\boldsymbol{\gamma_0}\boldsymbol{\gamma_1} - E^2\boldsymbol{\gamma_0}\boldsymbol{\gamma_2} - E^3\boldsymbol{\gamma_0}\boldsymbol{\gamma_3} \\ &\quad -B^1\boldsymbol{\gamma_2}\boldsymbol{\gamma_3} + B^2\boldsymbol{\gamma_1}\boldsymbol{\gamma_3} - B^3\boldsymbol{\gamma_1}\boldsymbol{\gamma_2} \end{aligned} \tag{132}$$

In this nomeclature all of the Maxwell equations become

$$\nabla F = J \tag{133}$$

$$\begin{aligned} \langle \nabla F \rangle_1 - J &= \left(\frac{\partial}{\partial x^1} E^1 + \frac{\partial}{\partial x^3} E^3 - J^0 + \frac{\partial}{\partial x^2} E^2 \right) \boldsymbol{\gamma_0} \\ &+ \left(-J^1 - \frac{\partial}{\partial x^0} E^1 - \frac{\partial}{\partial x^3} B^2 + \frac{\partial}{\partial x^2} B^3 \right) \boldsymbol{\gamma_1} \\ &+ \left(\frac{\partial}{\partial x^3} B^1 - \frac{\partial}{\partial x^0} E^2 - \frac{\partial}{\partial x^1} B^3 - J^2 \right) \boldsymbol{\gamma_2} \\ &+ \left(-J^3 + \frac{\partial}{\partial x^1} B^2 - \frac{\partial}{\partial x^2} B^1 - \frac{\partial}{\partial x^0} E^3 \right) \boldsymbol{\gamma_3} = 0 \end{aligned} \tag{134}$$

and

$$
\begin{aligned}
\langle \nabla F \rangle_3 \quad = \quad & \left(\frac{\partial}{\partial x^2} E^1 - \frac{\partial}{\partial x^1} E^2 - \frac{\partial}{\partial x^0} B^3 \right) \boldsymbol{\gamma_0} \wedge \boldsymbol{\gamma_1} \wedge \boldsymbol{\gamma_2} \\
& + \left(\frac{\partial}{\partial x^3} E^1 - \frac{\partial}{\partial x^1} E^3 + \frac{\partial}{\partial x^0} B^2 \right) \boldsymbol{\gamma_0} \wedge \boldsymbol{\gamma_1} \wedge \boldsymbol{\gamma_3} \\
& + \left(-\frac{\partial}{\partial x^0} B^1 + \frac{\partial}{\partial x^3} E^2 - \frac{\partial}{\partial x^2} E^3 \right) \boldsymbol{\gamma_0} \wedge \boldsymbol{\gamma_2} \wedge \boldsymbol{\gamma_3} \\
& + \left(\frac{\partial}{\partial x^1} B^1 + \frac{\partial}{\partial x^2} B^2 + \frac{\partial}{\partial x^3} B^3 \right) \boldsymbol{\gamma_1} \wedge \boldsymbol{\gamma_2} \wedge \boldsymbol{\gamma_3} = 0 \qquad (135)
\end{aligned}
$$

Equation 134 is equivalent to the $\nabla \cdot \boldsymbol{E}$ and $\nabla \times \boldsymbol{B}$ equations, while equation 135 is equivalent to the $\nabla \cdot \boldsymbol{B}$ and $\nabla \times \boldsymbol{E}$ equations.