

An Introduction to Geometric Algebra

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History

Geometric algebra is the Clifford algebra of a finite dimensional vector space over real scalars cast in a form most appropriate for physics and engineering. This was done by David Hestenes (Arizona State University) in the 1960's. From this start he developed the geometric calculus whose fundamental theorem includes the generalized Stokes theorem, the residue theorem, and new integral theorems not realized before. Hestenes likes to say he was motivated by the fact that physicists and engineers did not know how to multiply vectors.

Researchers at Arizona State and Cambridge have applied these developments to classical mechanics, quantum mechanics, general relativity (gauge theory of gravity), projective geometry, conformal geometry, etc.

Axioms of Geometric Algebra

Let $\mathcal{V}(p, q)$ be a finite dimensional vector space of signature (p, q) over \mathfrak{R} . Then $\forall a, b, c \in \mathcal{V}$ there exists a geometric product with the properties -

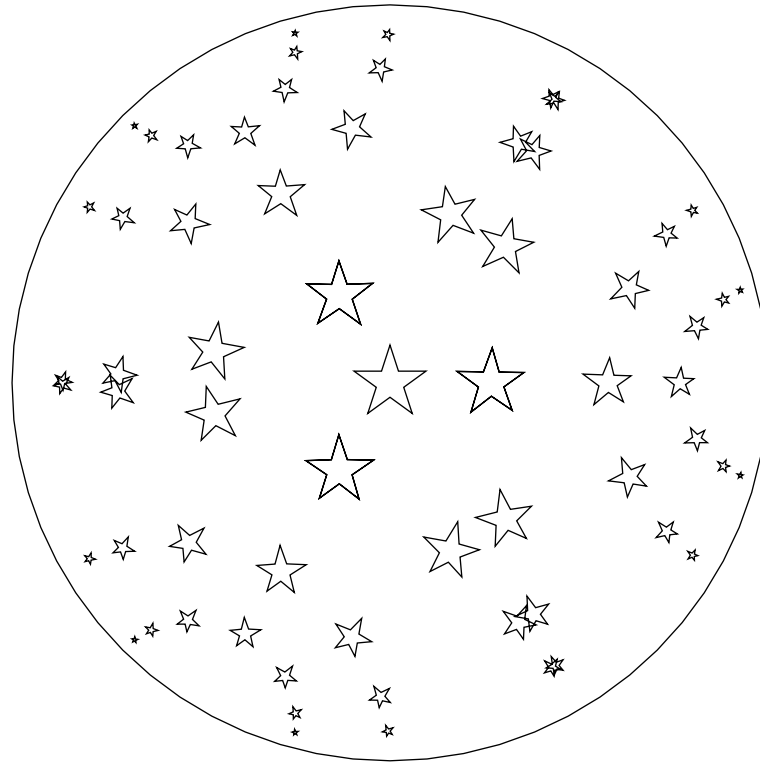
$$\begin{aligned}(ab)c &= a(bc) \\ a(b+c) &= ab+ac \\ (a+b)c &= ac+bc \\ aa &\in \mathfrak{R}\end{aligned}$$

If $a^2 \neq 0$ then $a^{-1} = \frac{1}{a^2}a$.

Why Learn This Stuff?

The geometric product of two (or more) vectors produces something “new” like the $\sqrt{-1}$ with respect to real numbers or vectors with respect to scalars. It must be studied in terms of its effect on vectors and in terms of its symmetries. It is worth the effort. Anything that makes understanding rotations in a N dimensional space simple is worth the effort! Also, if one proceeds on to geometric calculus many diverse areas in mathematics are unified and many areas of physics and engineering are greatly simplified.

Hyperbolic Translation



Inner, \cdot , and outer, \wedge , product of two vectors and their basic properties

$$a \cdot b = \frac{1}{2} (ab + ba) \quad (1)$$

$$a \wedge b = \frac{1}{2} (ab - ba) \quad (2)$$

$$ab = a \cdot b + a \wedge b \quad (3)$$

$$a \wedge b = -b \wedge a \quad (4)$$

$$\begin{aligned} c^2 &= (a + b)^2 \\ c^2 &= a^2 + ab + ba + b^2 \\ 2a \cdot b &= c^2 - a^2 - b^2 \end{aligned} \quad (5)$$

$$a \cdot b \in \mathfrak{R}$$

$$a \cdot b = |a| |b| \cos(\theta) \text{ if } a^2, b^2 > 0 \quad (6)$$

Orthogonal vectors are defined by $a \cdot b = 0$.

For orthogonal vectors $a \wedge b = ab$.

Now compute $(a \wedge b)^2$

$$(a \wedge b)^2 = -(a \wedge b)(b \wedge a) \quad (7)$$

$$= -(ab - a \cdot b)(ba - a \cdot b) \quad (8)$$

$$= -\left(abba - (a \cdot b)(ab + ba) + (a \cdot b)^2\right) \quad (9)$$

$$= -\left(a^2b^2 - (a \cdot b)^2\right) \quad (10)$$

$$= -a^2b^2(1 - \cos^2(\theta)) \quad (11)$$

$$= -a^2b^2 \sin^2(\theta) \quad (12)$$

Thus in a Euclidian space, $a^2, b^2 > 0$, $(a \wedge b)^2 \leq 0$ and $a \wedge b$ is proportional to $\sin(\theta)$. If e_{\parallel} and e_{\perp} are any two orthonormal unit vectors in a Euclidian space then $(e_{\parallel}e_{\perp})^2 = -1$. Who needs the $\sqrt{-1}$?

Outer, \wedge , product for r Vectors in terms of the geometric product

$$a_1 \wedge \dots \wedge a_r = \frac{1}{r!} \sum_{i_1, \dots, i_r} \varepsilon_{1\dots r}^{i_1 \dots i_r} a_{i_1} \dots a_{i_r} \quad (13)$$

$$\begin{aligned} a_1 \wedge \dots \wedge (a_j + b_j) \wedge \dots \wedge a_r &= \\ a_1 \wedge \dots \wedge a_j \wedge \dots \wedge a_r + a_1 \wedge \dots \wedge b_j \wedge \dots \wedge a_r & \quad (14) \end{aligned}$$

$$\begin{aligned} a_1 \wedge \dots \wedge a_j \wedge a_{j+1} \wedge \dots \wedge a_r &= \\ -a_1 \wedge \dots \wedge a_{j+1} \wedge a_j \wedge \dots \wedge a_r & \quad (15) \end{aligned}$$

Useful Relation's

1. For a set of r orthogonal vectors, e_1, \dots, e_r

$$e_1 \wedge \dots \wedge e_r = e_1 \dots e_r \quad (16)$$

2. For a set of r linearly independent vectors, a_1, \dots, a_r , there exists a set of r orthogonal vectors, e_1, \dots, e_r , such that

$$a_1 \wedge \dots \wedge a_r = e_1 \dots e_r \quad (17)$$

If the vectors, a_1, \dots, a_r , are not linearly independent then

$$a_1 \wedge \dots \wedge a_r = 0 \quad (18)$$

The product $a_1 \wedge \dots \wedge a_r$ is call a “blade” of rank r . The dimension of the vector space is the highest rank any blade can have.

Basis Blades

The geometric algebra of a vector space, $\mathcal{V}(p, q)$, is denoted $\mathcal{G}(p, q)$ or $\mathcal{G}(\mathcal{V})$ where (p, q) is the signature of the vector space (first p unit vectors square to $+1$ and next q unit vectors square to -1 , dimension of the space is $p + q$). Examples are:

p	q	Type of Space
3	0	3D Euclidian
1	3	Relativistic Space Time
4	1	3D Conformal Geometry

If the orthonormal basis set of the vector space is e_1, \dots, e_N , the basis of the geometric algebra (multivector space) is formed from the geometric products (since we have chosen an orthonormal basis) of the basis vectors. For grade r multivectors the basis blades are all the combinations of basis vectors products taken r at a time from the set of N vectors. Thus the number basis blades of r rank are $\binom{N}{r}$, the binomial expansion coefficient and the total dimension of the multivector space is the sum of $\binom{N}{r}$ over r which is 2^N . Thus the basis blades for $\mathcal{G}(3, 0)$ are:

	Grade		
0	1	2	3
1	e_1	e_1e_2	$e_1e_2e_3$
	e_2	e_1e_3	
	e_3	e_2e_3	

The multiplication table for the $\mathcal{G}(3, 0)$ basis blades is

	1	e_1	e_2	e_3	e_1e_2	e_1e_3	e_2e_3	$e_1e_2e_3$
1	1	e_1	e_2	e_3	e_1e_2	e_1e_3	e_2e_3	$e_1e_2e_3$
e_1	e_1	1	e_1e_2	e_1e_3	e_2	e_3	$e_1e_2e_3$	e_2e_3
e_2	e_2	$-e_1e_2$	1	e_2e_3	$-e_1$	$-e_1e_2e_3$	e_3	$-e_1e_3$
e_3	e_3	$-e_1e_3$	$-e_2e_3$	1	$e_1e_2e_3$	$-e_1$	$-e_2$	e_1e_2
e_1e_2	e_1e_2	$-e_2$	e_1	$e_1e_2e_3$	-1	$-e_2e_3$	e_1e_3	$-e_3$
e_1e_3	e_1e_3	$-e_3$	$-e_1e_2e_3$	e_1	e_2e_3	-1	$-e_1e_2$	e_2
e_2e_3	e_2e_3	$e_1e_2e_3$	$-e_3$	e_2	$-e_1e_3$	e_1e_2	-1	$-e_1$
$e_1e_2e_3$	$e_1e_2e_3$	e_2e_3	$-e_1e_3$	e_1e_2	$-e_3$	e_2	$-e_1$	-1

Note that the squares of all the grade 2 and 3 basis blades are -1 . The highest rank basis blade (in this case $e_1e_2e_3$) is usually denoted by I and is called the pseudoscalar.

The multiplication table for the $\mathcal{G}(1, 3)$ basis blades is (Part I)

	1	γ_0	γ_1	γ_2	γ_3	$\gamma_0\gamma_1$	$\gamma_0\gamma_2$	$\gamma_1\gamma_2$
1	1	γ_0	γ_1	γ_2	γ_3	$\gamma_0\gamma_1$	$\gamma_0\gamma_2$	$\gamma_1\gamma_2$
γ_0	γ_0	1	$\gamma_0\gamma_1$	$\gamma_0\gamma_2$	$\gamma_0\gamma_3$	γ_1	γ_2	$\gamma_0\gamma_1\gamma_2$
γ_1	γ_1	$-\gamma_0\gamma_1$	-1	$\gamma_1\gamma_2$	$\gamma_1\gamma_3$	γ_0	$-\gamma_0\gamma_1\gamma_2$	$-\gamma_2$
γ_2	γ_2	$-\gamma_0\gamma_2$	$-\gamma_1\gamma_2$	-1	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2$	γ_0	γ_1
γ_3	γ_3	$-\gamma_0\gamma_3$	$-\gamma_1\gamma_3$	$-\gamma_2\gamma_3$	-1	$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$
$\gamma_0\gamma_1$	$\gamma_0\gamma_1$	$-\gamma_1$	$-\gamma_0$	$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_3$	1	$-\gamma_1\gamma_2$	$-\gamma_0\gamma_2$
$\gamma_0\gamma_2$	$\gamma_0\gamma_2$	$-\gamma_2$	$-\gamma_0\gamma_1\gamma_2$	$-\gamma_0$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2$	1	$\gamma_0\gamma_1$
$\gamma_1\gamma_2$	$\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2$	γ_2	$-\gamma_1$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_2$	$-\gamma_0\gamma_1$	-1
$\gamma_0\gamma_3$	$\gamma_0\gamma_3$	$-\gamma_3$	$-\gamma_0\gamma_1\gamma_3$	$-\gamma_0\gamma_2\gamma_3$	$-\gamma_0$	$\gamma_1\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$
$\gamma_1\gamma_3$	$\gamma_1\gamma_3$	$\gamma_0\gamma_1\gamma_3$	γ_3	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_1$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_2\gamma_3$
$\gamma_2\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$	γ_3	$-\gamma_2$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_3$	$\gamma_1\gamma_3$
$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2$	$\gamma_1\gamma_2$	$\gamma_0\gamma_2$	$-\gamma_0\gamma_1$	$\gamma_0\gamma_1\gamma_2\gamma_3$	γ_2	$-\gamma_1$	$-\gamma_0$
$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_1\gamma_3$	$\gamma_1\gamma_3$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_1$	γ_3	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_2\gamma_3$
$\gamma_0\gamma_2\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_2$	$\gamma_1\gamma_2\gamma_3$	γ_3	$\gamma_0\gamma_1\gamma_3$
$\gamma_1\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_2\gamma_3$	$\gamma_1\gamma_3$	$-\gamma_1\gamma_2$	$\gamma_0\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_3$	$-\gamma_3$
$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_3$	$-\gamma_0\gamma_1\gamma_2$	$\gamma_2\gamma_3$	$-\gamma_1\gamma_3$	$-\gamma_0\gamma_3$

The multiplication table for the $\mathcal{G}(1, 3)$ basis blades is (Part II)

	$\gamma_0\gamma_3$	$\gamma_1\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$
1	$\gamma_0\gamma_3$	$\gamma_1\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$
γ_0	γ_3	$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2$	$\gamma_1\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$
γ_1	$-\gamma_0\gamma_1\gamma_3$	$-\gamma_3$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_2$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_2\gamma_3$	$\gamma_0\gamma_2\gamma_3$
γ_2	$-\gamma_0\gamma_2\gamma_3$	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_3$	$-\gamma_0\gamma_1$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_3$	$\gamma_1\gamma_3$	$-\gamma_0\gamma_1\gamma_3$
γ_3	γ_0	γ_1	γ_2	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_1$	$-\gamma_0\gamma_2$	$-\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2$
$\gamma_0\gamma_1$	$-\gamma_1\gamma_3$	$-\gamma_0\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$	γ_2	γ_3	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_2\gamma_3$	$\gamma_2\gamma_3$
$\gamma_0\gamma_2$	$-\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_3$	$-\gamma_1$	$\gamma_1\gamma_2\gamma_3$	γ_3	$\gamma_0\gamma_1\gamma_3$	$-\gamma_1\gamma_3$
$\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_2\gamma_3$	$-\gamma_1\gamma_3$	$-\gamma_0$	$\gamma_0\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_3$	$-\gamma_3$	$-\gamma_0\gamma_3$
$\gamma_0\gamma_3$	1	$\gamma_0\gamma_1$	$\gamma_0\gamma_2$	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_1$	$-\gamma_2$	$-\gamma_0\gamma_1\gamma_2$	$\gamma_1\gamma_2$
$\gamma_1\gamma_3$	$-\gamma_0\gamma_1$	-1	$\gamma_1\gamma_2$	$-\gamma_0\gamma_2\gamma_3$	$-\gamma_0$	$\gamma_0\gamma_1\gamma_2$	γ_2	$\gamma_0\gamma_2$
$\gamma_2\gamma_3$	$-\gamma_0\gamma_2$	$-\gamma_1\gamma_2$	-1	$\gamma_0\gamma_1\gamma_3$	$-\gamma_0\gamma_1\gamma_2$	$-\gamma_0$	$-\gamma_1$	$-\gamma_0\gamma_1$
$\gamma_0\gamma_1\gamma_2$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_3$	-1	$\gamma_2\gamma_3$	$-\gamma_1\gamma_3$	$-\gamma_0\gamma_3$	$-\gamma_3$
$\gamma_0\gamma_1\gamma_3$	$-\gamma_1$	$-\gamma_0$	$\gamma_0\gamma_1\gamma_2$	$-\gamma_2\gamma_3$	-1	$\gamma_1\gamma_2$	$\gamma_0\gamma_2$	γ_2
$\gamma_0\gamma_2\gamma_3$	$-\gamma_2$	$-\gamma_0\gamma_1\gamma_2$	$-\gamma_0$	$\gamma_1\gamma_3$	$-\gamma_1\gamma_2$	-1	$-\gamma_0\gamma_1$	$-\gamma_1$
$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2$	γ_2	$-\gamma_1$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_2$	$\gamma_0\gamma_1$	1	$-\gamma_0$
$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_1\gamma_2$	$\gamma_0\gamma_2$	$-\gamma_0\gamma_1$	γ_3	$-\gamma_2$	γ_1	γ_0	-1

Projection Operator

A multivector, the basic element of the geometric algebra, is made of a sum of scalars, vectors, blades. A multivector is homogenous (pure) if all the blades in it are of the same grade. The grade of a scalar is 0 and the grade of a vector is 1. The general multivector A is decomposed with the grade projection operator $\langle A \rangle_r$ as (N is dimension of the vector space):

$$A = \sum_{r=0}^N \langle A \rangle_r \quad (19)$$

As an example consider ab , the product of two vectors. Then

$$ab = \langle ab \rangle_0 + \langle ab \rangle_2 \quad (20)$$

Expansion of geometric product and generalization of \cdot and \wedge

If A_r and B_s are respectively grade r and s pure grade multivectors then

$$A_r B_s = \langle A_r B_s \rangle_{|r-s|} + \langle A_r B_s \rangle_{|r-s|+2} + \cdots + \langle A_r B_s \rangle_{\min(r+s, 2N-(r+s))} \quad (21)$$

$$A_r \cdot B_s = \langle A_r B_s \rangle_{|r-s|} \quad (22)$$

$$A_r \wedge B_s = \langle A_r B_s \rangle_{r+s} \quad (23)$$

Thus if $r + s > N$ then $A_r \wedge B_s = 0$, also note that these formulas are the most efficient way of calculating $A_r \cdot B_s$ and $A_r \wedge B_s$.

Using equations 17 and 21 we can prove that for a vector a and a

grade r multivector B_r

$$a \cdot B_r = \frac{1}{2} (aB_r - (-1)^r B_r a) \quad (24)$$

$$a \wedge B_r = \frac{1}{2} (aB_r + (-1)^r B_r a) \quad (25)$$

If equations 24 and 25 are true for a grade r blade they are also true for a grade r multivector (superposition of grade r blades). By equation 17 let $B_r = e_1 \dots e_r$ where the e 's are orthogonal and expand a

$$a = a_{\perp} + \sum_{j=1}^r \alpha_j e_j \quad (26)$$

where a_{\perp} is orthogonal to all the e 's. Then

$$aB_r = \sum_{j=1}^r (-1)^{j-1} \alpha_j e_1 \dots \check{e}_j \dots e_r + a_{\perp} e_1 \dots e_r$$

$$= a \cdot B_r + a \wedge B_r \quad (27)$$

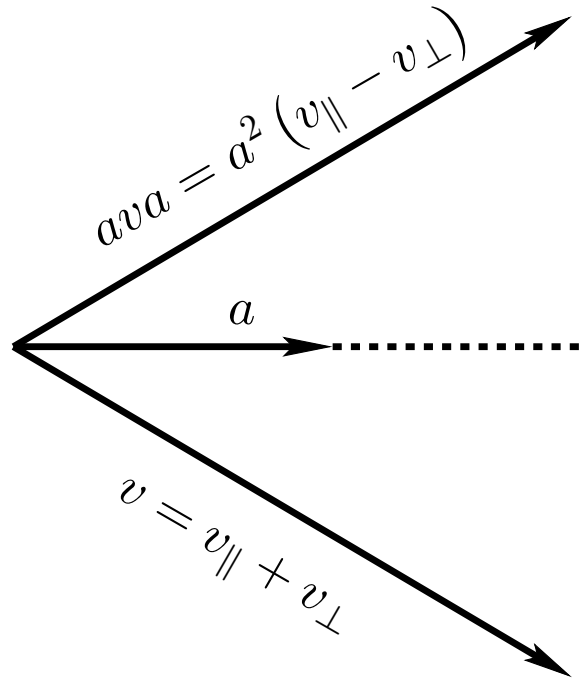
Now calculate

$$\begin{aligned}
B_r a &= \sum_{j=1}^r (-1)^{r-j} \alpha_j e_1 \cdots \check{e}_j \cdots e_r - (-1)^{r-1} a_{\perp} e_1 \cdots e_r \\
&= (-1)^{r-1} \left(\sum_{j=1}^r (-1)^{j-1} \alpha_j e_1 \cdots \check{e}_j \cdots e_r - a_{\perp} e_1 \cdots e_r \right) \\
&= (-1)^{r-1} (a \cdot B_r - a \wedge B_r) \quad (28)
\end{aligned}$$

Adding and subtracting equations 27 and 28 gives equations 24 and 25.

Reflections

We wish to show that $a, v \in \mathcal{V} \rightarrow ava \in \mathcal{V}$ and v is reflected about a if $a^2 = 1$.



1. Decompose $v = v_{\parallel} + v_{\perp}$ where v_{\parallel} is the part of v parallel to a and v_{\perp} is the part perpendicular to a .

2. $av = av_{\parallel} + av_{\perp} = v_{\parallel}a - v_{\perp}a$ since a and v_{\perp} are orthogonal.
3. $ava = a^2(v_{\parallel} - v_{\perp})$ is a vector since a^2 is a scalar.
4. ava is the reflection of v about the direction of a if $a^2 = 1$.
5. Thus $a_1 \dots a_r v a_r \dots a_1 \in \mathcal{V}$ and produces a composition of reflections of v if $a_1^2 = \dots = a_r^2 = 1$.

Rotations, Part 1

First define the reverse of a product of vectors. If $R = a_1 \dots a_s$ then the reverse is $R^\dagger = (a_1 \dots a_s)^\dagger = a_r \dots a_1$, the order of multiplication is reversed. Then let $R = ab$ so that

$$RR^\dagger = (ab)(ba) = ab^2a = a^2b^2 = R^\dagger R \quad (29)$$

Let $RR^\dagger = 1$ and calculate $(RvR^\dagger)^2$, where v is an arbitrary vector.

$$(RvR^\dagger)^2 = RvR^\dagger RvR^\dagger = Rv^2R^\dagger = v^2RR^\dagger = v^2 \quad (30)$$

Thus RvR^\dagger leaves the length of v unchanged.

Now we must also prove $Rv_1R^\dagger \cdot Rv_2R^\dagger = v_1 \cdot v_2$. Since Rv_1R^\dagger and Rv_2R^\dagger are both vectors we can use the definition of the dot product for two vectors

$$\begin{aligned}
Rv_1R^\dagger \cdot Rv_2R^\dagger &= \frac{1}{2} (Rv_1R^\dagger Rv_2R^\dagger + Rv_2R^\dagger Rv_1R^\dagger) \\
&= \frac{1}{2} (Rv_1v_2R^\dagger + Rv_2v_1R^\dagger) \\
&= \frac{1}{2}R(v_1v_2 + v_2v_1)R^\dagger \\
&= R(v_1 \cdot v_2)R^\dagger \\
&= v_1 \cdot v_2RR^\dagger \\
&= v_1 \cdot v_2
\end{aligned}$$

Thus the transformation RvR^\dagger preserves both length and angle and must be a rotation. The normal designation for R is a rotor.

Rotations, Part 2

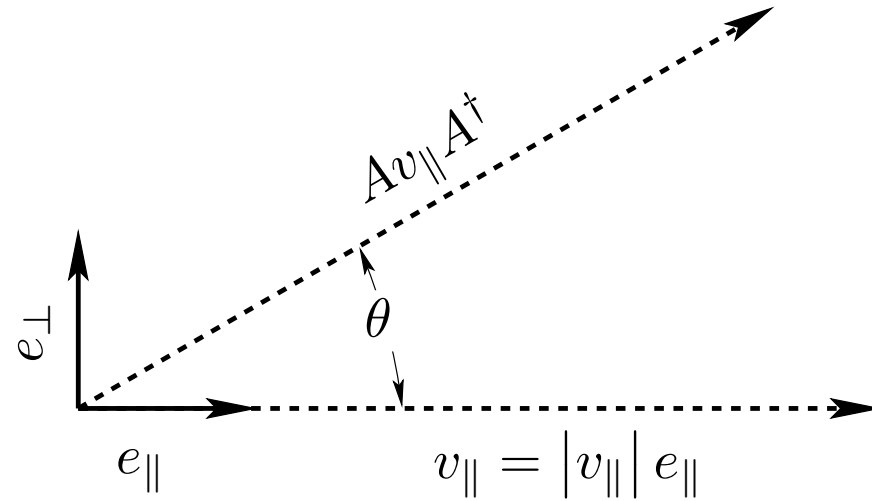
The general rotation can be represented by $R = e^{\frac{\theta}{2}u}$ where u is a unit bivector in the plane of the rotation and θ is the rotation angle in the plane. The two possible non-degenerate cases are $u^2 = \pm 1$

$$e^{\frac{\theta}{2}u} = \left\{ \begin{array}{ll} \text{(Euclidian plane)} & u^2 = -1 : \cos\left(\frac{\theta}{2}\right) + u \sin\left(\frac{\theta}{2}\right) \\ \text{(Minkowski plane)} & u^2 = 1 : \cosh\left(\frac{\theta}{2}\right) + u \sinh\left(\frac{\theta}{2}\right) \end{array} \right\} \quad (31)$$

Decompose $v = v_{\parallel} + (v - v_{\parallel})$ where v_{\parallel} is the projection of v into the plane defined by u . Note the $v - v_{\parallel}$ is orthogonal to all vectors in the u plane. Now let $u = e_{\perp}e_{\parallel}$ where e_{\parallel} is parallel to v_{\parallel} and of course e_{\perp} is in the plane u and orthogonal to e_{\parallel} . $v - v_{\parallel}$ anticommutes with e_{\parallel} and e_{\perp} and v_{\parallel} anticommutes with e_{\perp} (it is left to the viewer to show $RR^{\dagger} = 1$).

Euclidian Case

For the case of $u^2 = -1$



$$RvR^{\dagger} = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) + e_{\perp}e_{\parallel} \sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) + e_{\parallel}e_{\perp} \sin\left(\frac{\theta}{2}\right) \end{pmatrix} (v_{\parallel} + (v - v_{\parallel})) \times \quad (32)$$

Since $v - v_{\parallel}$ anticommutes with e_{\parallel} and e_{\perp} it commutes with R and

$$RvR^{\dagger} = Rv_{\parallel}R^{\dagger} + (v - v_{\parallel}) \quad (33)$$

So that we only have to evaluate

$$Rv_{\parallel}R^{\dagger} = \left(\cos\left(\frac{\theta}{2}\right) + e_{\perp}e_{\parallel}\sin\left(\frac{\theta}{2}\right) \right) v_{\parallel} \left(\cos\left(\frac{\theta}{2}\right) + e_{\parallel}e_{\perp}\sin\left(\frac{\theta}{2}\right) \right) \quad (34)$$

Since $v_{\parallel} = |v_{\parallel}| e_{\parallel}$

$$Rv_{\parallel}R^{\dagger} = |v_{\parallel}| (\cos(\theta) e_{\parallel} + \sin(\theta) e_{\perp}) \quad (35)$$

and the component of v in the u plane is rotated correctly.

Minkowski Case

For the case of $u^2 = 1$ there are two possibilities, $v_{\parallel}^2 > 0$ or $v_{\parallel}^2 < 0$. In the first case $e_{\parallel}^2 = 1$ and $e_{\perp}^2 = -1$. In the second case $e_{\parallel}^2 = -1$ and $e_{\perp}^2 = 1$. Again $v - v_{\parallel}$ is not affected by the rotation so that we need only evaluate

$$Rv_{\parallel}R^{\dagger} = \begin{pmatrix} \cosh\left(\frac{\theta}{2}\right) + e_{\perp}e_{\parallel}\sinh\left(\frac{\theta}{2}\right) \\ \cosh\left(\frac{\theta}{2}\right) + e_{\parallel}e_{\perp}\sinh\left(\frac{\theta}{2}\right) \end{pmatrix} v_{\parallel} \times \quad (36)$$

Note that in this case $|v_{\parallel}| = \sqrt{|v_{\parallel}^2|}$ and

$$Rv_{\parallel}R^{\dagger} = \left\{ \begin{array}{l} v_{\parallel}^2 > 0 : |v_{\parallel}| (\cosh(\theta) e_{\parallel} + \sinh(\theta) e_{\perp}) \\ v_{\parallel}^2 < 0 : |v_{\parallel}| (\cosh(\theta) e_{\parallel} - \sinh(\theta) e_{\perp}) \end{array} \right\} \quad (37)$$

Spinors

The general definition of a spinor is a even (all odd grade components are zero) multivector, ψ , such that $\psi v \psi^\dagger \in \mathcal{V} \forall v \in \mathcal{V}$. The most important case of spinors is for $\mathcal{G}(1, 3)$ (relativistic quantum mechanics). We can write the general spinor for $\mathcal{G}(1, 3)$ as

$$\psi = \alpha + \beta u + \gamma I \quad (38)$$

where u is a unit bivector (note that for $\mathcal{G}(1, 3)$ $I^\dagger = I$ and v commutes with I since the grade of I is even). Then

$$\begin{aligned} \psi v \psi^\dagger &= (\alpha + \beta u + \gamma I) v (\alpha + \beta u^\dagger + \gamma I^\dagger) \\ &= (\alpha + \gamma I)^2 v + \beta^2 u v u^\dagger \\ &= (\alpha^2 + \gamma^2 I^2 + \alpha \gamma I) v + \beta^2 u v u^\dagger \end{aligned} \quad (39)$$

Thus $\psi v \psi^\dagger \in \mathcal{V} \rightarrow \alpha\gamma = 0$. Either $\alpha = 0$ or $\gamma = 0$. If $\gamma = 0$ the the spinor can be expressed as ($\rho > 0$)

$$\psi = \sqrt{\rho} e^{u \frac{\theta}{2}} \quad (40)$$

so that $\psi \psi^\dagger = \rho$ and ψ rotates v and dialates it by a factor of ρ . If $\alpha = 0$ then

$$\psi = \sqrt{\rho} I e^{I u \frac{\theta}{2}} = \left\{ \begin{array}{l} u^2 = -1 : \quad \sqrt{\rho} I \left(\cosh \left(\frac{\theta}{2} \right) + I u \sinh \left(\frac{\theta}{2} \right) \right) \\ u^2 = 1 : \quad \sqrt{\rho} I \left(\cos \left(\frac{\theta}{2} \right) + I u \sin \left(\frac{\theta}{2} \right) \right) \end{array} \right\} \quad (41)$$

In this case $\psi \psi^\dagger = -\rho$ and ψ rotates v , dialates it by a factor of ρ , and reflects it. The case of a spinor for $\mathcal{G}(p, q)$ in general is a bit more complicated.

Quaternions

Any multivector $A \in \mathcal{G}(3,0)$ may be written as

$$A = \alpha + a + B + \beta I \quad (42)$$

where $\alpha, \beta \in \mathfrak{R}$, $a \in \mathcal{V}(3,0)$, B is a bivector, and I is the unit pseudoscalar. The quaternions are the multivectors of even grades

$$A = \alpha + B \quad (43)$$

B can be represented as

$$B = \alpha \mathbf{i} + \beta \mathbf{j} + \gamma \mathbf{k} \quad (44)$$

where $\mathbf{i} = e_2e_3$, $\mathbf{j} = e_1e_3$, and $\mathbf{k} = e_1e_2$, and

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \quad (45)$$

The quaternions form a subalgebra of $\mathcal{G}(3, 0)$ since the geometric product of any two quaternions is also a quaternion since the geometric product of two even grade multivector components is a even grade multivector. For example the product of two grade 2 multivectors can only consist of grades 0, 2, and 4, but in $\mathcal{G}(3, 0)$ we can only have grades 0 and 2 since the highest possible grade is 3.

Linear Transformations

Let f be a linear transformation $f : \mathcal{V} \rightarrow \mathcal{V}$ with $f(\alpha a + \beta b) = \alpha f(a) + \beta f(b) \forall a, b \in \mathcal{V}$ and $\alpha, \beta \in \mathfrak{R}$. Then define the action of f on a blade of the geometric algebra by

$$f(a_1 \wedge \dots \wedge a_r) = f(a_1) \wedge \dots \wedge f(a_r) \quad (46)$$

and the action of f on any two $A, B \in \mathcal{G}(\mathcal{V})$ by

$$f(\alpha A + \beta B) = \alpha f(A) + \beta f(B) \quad (47)$$

Since any multivector A can be expanded as a sum of blades $f(A)$ is defined. This has many consequences. Consider the following definition for the determinant of f , $\det(f)$.

$$f(I) = \det(f) I \quad (48)$$

First show that this definition is equivalent to the standard definition of the determinant (again e_1, \dots, e_N is an orthonormal basis for \mathcal{V}).

$$f(e_r) = \sum_{s=1}^N a_{rs} e_s \quad (49)$$

Then

$$\begin{aligned} f(I) &= \left(\sum_{s_1=1}^N a_{1s_1} e_{s_1} \right) \wedge \dots \wedge \left(\sum_{s_N=1}^N a_{Ns_N} e_{s_N} \right) \\ &= \sum_{s_1, \dots, s_N} a_{1s_1} \dots a_{Ns_N} e_{s_1} \dots e_{s_N} \end{aligned} \quad (50)$$

But

$$e_{s_1} \dots e_{s_N} = \varepsilon_{1\dots N}^{s_1 \dots s_N} e_1 \dots e_N \quad (51)$$

so that

$$f(I) = \sum_{s_1, \dots, s_N} \varepsilon_{1\dots N}^{s_1 \dots s_N} a_{1s_1} \dots a_{Ns_N} I \quad (52)$$

or

$$\det(f) = \sum_{s_1, \dots, s_N} \varepsilon_{1\dots N}^{s_1 \dots s_N} a_{1s_1} \dots a_{Ns_N} \quad (53)$$

which is the standard definition. Now compute the determinant of the product of the linear transformations f and g

$$\begin{aligned} \det(fg) I &= fg(I) \\ &= f(g(I)) \\ &= f(\det(g) I) \\ &= \det(g) f(I) \\ &= \det(g) \det(f) I \end{aligned} \quad (54)$$

or

$$\det (fg) = \det (f) \det (g) \quad (55)$$

Do you have any idea of how miserable that is to prove from the standard definition of determinant?

Another linear algebraic relation in geometric algebra is

$$f^{-1} (A) = \frac{\underline{I} \underline{f} (I^{-1} A)}{\det (f)} \quad \forall A \in \mathcal{G} (\mathcal{V}) \quad (56)$$

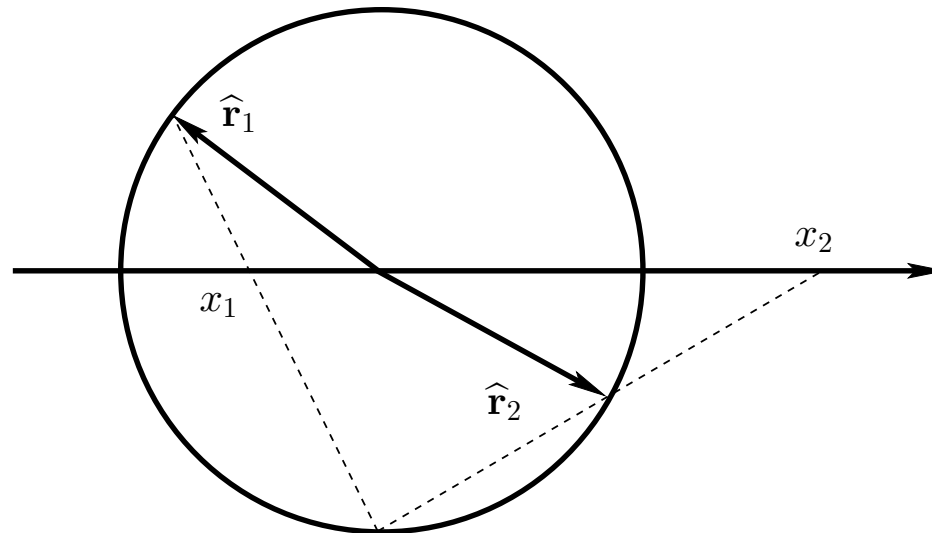
where \underline{f} is the adjoint transformation defined by $a \cdot \underline{f} (b) = b \cdot f (a) \quad \forall a, b \in \mathcal{V}$ and you have an explicit formula for the inverse of a linear transformation!

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Conformal Geometry

In Conformal Geometry the vector space $\mathcal{V}(p, q)$ is enlarged to $\mathcal{V}(p + 1, q + 1)$ with a mapping function $F : \mathcal{V}(p, q) \rightarrow \mathcal{V}(p + 1, q + 1)$ such that $F(x)^2 = 0 \forall x \in \mathcal{V}(p, q)$. $F(x)$ is a null vector. The motivation for this is that translations, rotations, dilations, and inversions in $\mathcal{V}(p, q)$ can all be encoded as rotations in $\mathcal{V}(p + 1, q + 1)$. The geometric algebra of $\mathcal{G}(p + 1, q + 1)$ makes handling the rotations very simple. The starting point of conformal geometric is the stereographic projection of the line as show:



The points x_1 and x_2 are mapped into the unit vectors $\hat{\mathbf{r}}_1$ and $\hat{\mathbf{r}}_2$. A little algebra shows that for a general point x on the line

$$\hat{\mathbf{r}} = \frac{2x}{1+x^2}e_1 + \frac{1-x^2}{1+x^2}e_2 \quad (57)$$

But this representation is not homogenous since $\hat{\mathbf{r}}^2 \neq 0$. Rename e_2 to e to distinguish it from the basis vector(s) of the space being described (in this case the line) and scale and add the vector \bar{e} to equation 57 to get

$$X = 2xe_1 + (1-x^2)e + (1+x^2)\bar{e} \quad (58)$$

where

$$e^2 = 1, \quad \bar{e}^2 = -1, \quad e \cdot \bar{e} = 0$$

and $X^2 = 0$ and the vector space $\mathcal{V}(1,0)$ has been extended to $\mathcal{V}(2,1)$. To put equation 58 into standard form define

$$\begin{aligned}
n &= e + \bar{e}, & \bar{n} &= e - \bar{e} \\
n^2 &= \bar{n}^2 = 0, & n \cdot \bar{n} &= 2 \\
X &= 2xe_1 + x^2n - \bar{n}
\end{aligned}$$

Then the general mapping $F : \mathcal{V}(p, q) \rightarrow \mathcal{V}(p + 1, q + 1)$ is given by

$$F(x) = x^2n + 2x - \bar{n}, \quad \forall x \in \mathcal{V}(p, q) \quad (59)$$

Note that

$$\begin{aligned}
F(x) \cdot F(y) &= (x^2n + 2x - \bar{n}) \cdot (y^2n + 2y - \bar{n}) \\
&= -2x^2 - 2y^2 + 4x \cdot y \\
&= -2(x - y)^2
\end{aligned} \quad (60)$$

so that the inner product in $\mathcal{V}(p + 1, q + 1)$ encodes the distance between two points in $\mathcal{V}(p, q)$. Note that $F(x) \cdot F(y) = 0$ implies that $x = y$, such are the wonders of null vectors.

Conformal Transformations

Conformal transformations consist of translations, rotations, dilations, and inversions in $\mathcal{V}(p, q)$. In the conformal space $\mathcal{V}(p+1, q+1)$ all these transformations can be represented by rotations and reflections.

1. Translations: $a \in \mathcal{V}(p, q)$ is the translation vector then

$$F(x') = e^{\frac{na}{2}} F(x) e^{-\frac{na}{2}} \text{ where } x' = x + a \quad (61)$$

2. Rotations: B is a unit bivector in $\mathcal{G}(p, q)$ and ϕ the rotation angle then

$$F(x') = e^{\frac{B\phi}{2}} F(x) e^{-\frac{B\phi}{2}} \quad (62)$$

3. Dilations: $\alpha \in \mathfrak{R}$ then $x' = e^{-\alpha} x$

$$F(x') = e^{\frac{\alpha e \bar{e}}{2}} F(x) e^{-\frac{\alpha e \bar{e}}{2}} \quad (63)$$

4. Inversion: $x' = \frac{x}{x^2}$

$$F(x') = eF(x)e \quad (64)$$

Composition of transformations is as expected. If $T(a) = e^{\frac{na}{2}}$, $R(B, \phi) = e^{\frac{B\phi}{2}}$, and $D(\alpha) = e^{\frac{\alpha e \bar{e}}{2}}$ and we wish to translate, rotate, dilate, and invert (in that order) the composite transformation f is

$$f = eD(\alpha)R(B, \phi)T(a) \quad (65)$$

and

$$F(x') = fF(x)f^\dagger \quad (66)$$

where

$$f^\dagger = T(a)^\dagger R(B, \phi)^\dagger D(\alpha)^\dagger e^\dagger, \text{ Note that } e^\dagger = e \quad (67)$$

Geometric Primitives

Geometric primitives are formed from the repeated application of the exterior product to null vectors in $\mathcal{V}(p+1, q+1)$ (conformal space).

1. Points: If A and B are null vectors in the conformal space then the solutions to

$$G \wedge X = A \wedge B \wedge X = 0, \quad X^2 = 0 \text{ are exactly } A \text{ and } B \quad (68)$$

2. Circles/Lines: If A , B , and C are null vectors in the conformal space then the solutions to

$$L \wedge X = A \wedge B \wedge C \wedge X = 0, \quad X^2 = 0 \text{ are circles} \quad (69)$$

with points a , b , and c on the circle. If $C = n$ the circle becomes a straight line (n is the point at infinity).

The circle parameters are

$$\rho^2 = -\frac{L^2}{(L \wedge n)^2} \text{ radius, } X_0 = LnL \text{ center.} \quad (70)$$

3. Spheres/Planes: If A , B , C , and D are null vectors in the conformal space then the solutions to

$$P \wedge X = A \wedge B \wedge C \wedge D \wedge X = 0, \quad X^2 = 0 \text{ are spheres} \quad (71)$$

with points a , b , c , and d on the sphere. If $D = n$ the sphere becomes a plane (n is the point at infinity). The sphere parameters are

$$\rho^2 = \frac{P^2}{(P \wedge n)^2} \text{ radius, } X_0 = PnP \text{ center.} \quad (72)$$

Intersections

The dual operation for pure grade multivectors (denoted by $*$) in the conformal space is defined by

$$C^* = IC \quad (73)$$

where I is the pseudo-scalar for the conformal space. The intersection of two lines and/or two circles defined by tri-vectors L_1 and L_2 (2D geometry) is given by the bi-vector

$$B = (L_1^* \wedge L_2^*)^* . \quad (74)$$

B represents zero, one, or two points depending upon the sign of B^2 . Likewise if one has a line/circle represents by the trivector L and a

plane/sphere represented by the quad-vector P (3D geometry) the points of intersection are given by the bi-vector B

$$B = (L^* \wedge P^*) \quad (75)$$

Finally (for 3D space) the intersection of two planes/spheres denoted by quad-vectors P_1 and P_2 is given by the tri-vector L (line or circle)

$$L = (P_1^* \wedge P_2^*) \quad (76)$$

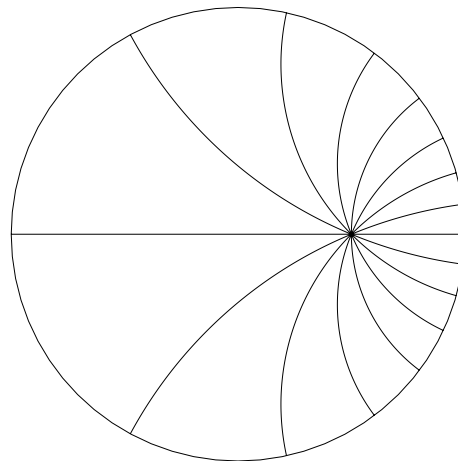
The intersection of lines/circles in 3D space does not have such simple formulas and must be approached indirectly by analyzing the intersection of planes defined by circles and their intersection with each other or with lines.

Non Euclidian Geometry

The unit circle in 2D space, $\mathcal{G}(3,1)$, is defined by the tri-vector Ie . The angle of intersection of the unit circle with a general line L that is perpendicular to the unit circle is given by

$$(Ie) \cdot L = I(e \wedge L) = 0. \quad (77)$$

This implies $L = A \wedge B \wedge e$.



With typical L 's (D-lines) as shown.

We now need a non-euclidian distance function. The first step is to determine a rotor that maps D-lines into D-lines. For this we require that the rotor $R = e^{\frac{\alpha \hat{U}}{2}}$ map $e = ReR^\dagger$ where $U = |U| \hat{U}$ is a bi-vector. e must be an invariant point of the transformation.

Thus we need

$$e = \left(c \left(\frac{\alpha}{2} \right) + s \left(\frac{\alpha}{2} \right) \hat{U} \right) e \left(c \left(\frac{\alpha}{2} \right) - s \left(\frac{\alpha}{2} \right) \hat{U} \right) \quad (78)$$

If $U^2 > 0$ $c()$ = cosh() and $s()$ = sinh() or if $B^2 < 0$ $c()$ = cos() and $s()$ = sin(). Equation 78 is true if $eU = Ue$. If we let $U = Le$ we have $Le = eL$ which implies $Ue = eU = L$ thus

$$U = Le = (A \wedge B \wedge e) e \text{ and } \hat{U} = \frac{U}{|U|} \quad (79)$$

Note that $U^2 = L^2$. After some laborious algebra we find that

$$\begin{aligned} L^2 &= (A \cdot B)^2 - 2(A \cdot B)(A \cdot e)(B \cdot e) \\ &= 4(a - b)^2 \left((a - b)^2 + (1 - a^2)(1 - b^2) \right) \end{aligned} \quad (80)$$

So that $L^2 > 0 \forall a^2, b^2 < 1$ or $\forall a^2, b^2 > 1$ and we must use the hyperbolic functions in expanding $e^{\frac{\alpha \hat{U}}{2}}$. Now we must determine what α corresponds to the transport of A to B along the D-line connecting A and B . With our definition of $R(\alpha)$ we have $e = R(\alpha)eR(\alpha)^\dagger$. Now consider the equation

$$X(\alpha) = R(\alpha)AR(\alpha)^\dagger \quad (81)$$

We have $X(0) = A$ and must determine if there is an α such that $B = X(\alpha)$. Or equivalently is there a solution to $B \cdot R(\alpha)AR(\alpha)^\dagger = 0$ since B is a null vector.

After a lot of algebra we determine that

$$\begin{aligned}
 d(A, B) &= 2 \sinh^{-1} \left(\sqrt{-\frac{A \cdot B}{2(A \cdot e)(B \cdot e)}} \right) \\
 &= 2 \sinh^{-1} \left(\sqrt{\frac{|a - b|^2}{(1 - a^2)(1 - b^2)}} \right). \tag{82}
 \end{aligned}$$

For three points A , B , and C on the same D-line we have

$$d(A, C) = d(A, B) + d(B, C) \tag{83}$$

since

$$R(d(A, C)) = R(d(A, B)) R(d(B, C)). \tag{84}$$