

Oersted Medal Lecture 2002: Reforming the Mathematical Language of Physics

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The connection between physics teaching and research at its deepest level can be illuminated by Physics Education Research (PER). For students and scientists alike, what they know and learn about physics is profoundly shaped by the conceptual tools at their command. Physicists employ a miscellaneous assortment of mathematical tools in ways that contribute to a fragmentation of knowledge. We can do better! Research on the design and use of mathematical systems provides a guide for designing a *unified mathematical language for the whole of physics* that facilitates learning and enhances physical insight. This has produced a comprehensive language called *Geometric Algebra*, which I introduce with emphasis on how it *simplifies and integrates classical and quantum physics*. Introducing research-based reform into a conservative physics curriculum is a challenge for the emerging PER community. Join the fun!

I. Introduction

The relation between teaching and research has been a perennial theme in academia as well as the Oersted Lectures, with no apparent progress on resolving the issues. Physics Education Research (PER) puts the whole matter into new light, for PER makes teaching itself a subject of research. This shifts attention to the relation of education research to scientific research as the central issue.

To many, the research domain of PER is exclusively pedagogical. Course content is taken as given, so the research problem is how to teach it most effectively. This approach to PER has produced valuable insights and useful results. However, it ignores the possibility of improving pedagogy by reconstructing course content. Obviously, a deep knowledge of physics is needed to pull off anything more than cosmetic reconstruction. It is here, I contend, in addressing the nature and structure of scientific subject matter, that PER and scientific research overlap and enrich one another with complementary perspectives.

The main concern of my own PER has been to develop and validate a scientific *Theory of Instruction* to serve as a reliable guide to improving physics teaching. To say the least, many physicists are dubious about the possibility. Even the late Arnold Arons, patron saint of PER, addressed a recent AAPT session with a stern warning against *any* claims of educational theory. Against this backdrop of skepticism, I will outline for you a system of general principles that have guided my efforts in PER. With sufficient elaboration (much of which

already exists in the published literature), I believe that these principles provide quite an adequate Theory of Instruction.

Like any other scientific theory, a Theory of Instruction must be validated by testing its consequences. This has embroiled me more than I like in developing suitable instruments to assess student learning. With the help of these instruments, the Modeling Instruction Program has amassed a large body of empirical evidence that I believe supports my instructional theory. We cannot review that evidence here, but I hope to convince you with theoretical arguments.

A brief account of my Theory of Instruction sets the stage for the main subject of my lecture: a constructive critique of the mathematical language used in physics with an introduction to a unified language that has been developed over the last forty years to replace it. The generic name for that language is *Geometric Algebra* (GA). My purpose here is to explain how GA simplifies and clarifies the structure of physics, and thereby convince you of its immense implications for physics instruction at all grade levels. I expound it here in sufficient detail to be useful in instruction and research and to provide an entrée to the published literature.

After explaining the utter simplicity of the GA grammar in Section V, I explicate the following unique features of the mathematical language:

(1) GA seamlessly integrates the properties of vectors and complex numbers to enable a completely coordinate-free treatment of 2D physics.

(2) GA articulates seamlessly with standard vector algebra to enable easy contact with standard literature and mathematical methods.

(3) GA Reduces “grad, div, curl and all that” to a single *vector derivative* that, among other things, combines the standard set of four Maxwell equations into a single equation and provides new methods to solve it.

(4) The GA formulation of spinors facilitates the treatment of rotations and rotational dynamics in both classical and quantum mechanics without coordinates or matrices.

(5) GA provides fresh insights into the geometric structure of quantum mechanics with implications for its physical interpretation.

All of this generalizes smoothly to a completely coordinate-free language for spacetime physics and general relativity to be introduced in subsequent papers.

The development of GA has been a central theme of my own research in theoretical physics and mathematics. I confess that it has profoundly influenced my thinking about PER all along, though this is the first time that I have made it public. I have refrained from mentioning it before, because I feared that my ideas were too radical to be assimilated by most physicists. Today I am coming out of the closet, so to speak, because I feel that the PER community has reached a new level of maturity. My suggestions for reform are offered as a challenge to the physics community at large and to the PER community in particular. The challenge is to seriously consider the design and use of mathematics as an important subject for PER. No doubt many of you are wondering why, if GA is so wonderful – why have you not heard of it before? I address that question in the penultimate Section by discussing the reception of GA and similar reforms by the physics community and their bearing on prospects for incorporating GA

into the physics curriculum. This opens up deep issues about the assimilation of new ideas – issues that are ordinarily studied by historians, but, I maintain, are worthy subjects for PER as well.

II. Five Principles of Learning

I submit that the common denominator of teaching and research is learning – learning by students on the one hand – learning by scientists on the other. Psychologists distinguish several kinds of learning. Without getting into the subtleties in the concept of “concept,” I use the term “conceptual learning” for the type of learning that concerns us here, especially to distinguish it from “rote learning.” Here follows a brief discussion of five general principles of conceptual learning that I have incorporated into my instructional theory and applied repeatedly in the design of instruction:

1. Conceptual learning is a creative act.

This is the crux of the so-called *constructivist revolution* in education, most succinctly captured in Piaget’s maxim: “To understand is to invent!”¹ Its meaning is best conveyed by an example: For a student to learn Newtonian physics is a creative act comparable to Newton’s original invention. The main difference is that the student has stronger hints than Newton did. The conceptual transition from the student’s naive physics to the Newtonian system recapitulates one of the great scientific revolutions, rewriting the codebook of the student’s experience. This perspective has greatly increased my respect for the creative powers of individual students. It is antidote for the elitist view that creativity is the special gift of a few geniuses.

2. Conceptual learning is systemic.

This means that concepts derive their meaning from their place in a coherent conceptual system. For example, the Newtonian concept of force is a multidimensional concept that derives its meaning from the whole Newtonian system.² Consequently, instruction that promotes coordinated use of Newton’s Laws should be more effective than a piecemeal approach that concentrates on teaching each of Newton’s Laws separately.

3. Conceptual learning depends on context.

This includes social and intellectual context. It follows that a central problem in the design of instruction is to create a classroom environment that optimizes student opportunities for systemic learning of targeted concepts.³ The context for scientific research is equally important, and it is relevant to the organization and management of research teams and institutes.

4. The quality of learning is critically dependent on conceptual tools at the learner’s command.

The design of tools to optimize learning is therefore an important subject for PER. As every physical theory is grounded in mathematics, the design of math tools is especially important. Much more on that below.

5. Expert learning requires deliberate practice with critical feedback.

There is substantial evidence that practice does not significantly improve intellectual performance unless it is guided by critical feedback and *deliberate* attempts to improve.⁴ Students waste an enormous amount of time in rote study that does not satisfy this principle.

I believe that all five principles are essential to effective learning and instructional design, though they are seldom invoked explicitly, and many efforts at educational reform founder because of insufficient attention to one or more of them. The terms “concept” and “conceptual learning” are often tossed about quite cavalierly in courses with names like “Conceptual Physics.” In my experience, such courses fall far short of satisfying the above learning principles, so I am skeptical of claims that they are successful in teaching physics concepts without mathematics. The degree to which physics concepts are essentially mathematical is a deep problem for PER.

I should attach a warning to the First (Constructivist) Learning Principle. There are many brands of constructivism, differing in the theoretical context afforded to the constructivist principle. An extreme brand called “radical constructivism” asserts that constructed knowledge is peculiar to an individual’s experience, so it denies the possibility of objective knowledge. This has radicalized the constructivist revolution in many circles and drawn severe criticism from scientists.⁵ I see the crux of the issue in the fact that the constructivist principle does not specify *how* knowledge is constructed. When this gap is closed with the other learning principles and scientific standards for evidence and inference, we have a brand that I call *scientific constructivism*.

I see the five Learning Principles as equally applicable to the conduct of research and to the design of instruction. They support the popular goal of “teaching the student to think like a scientist.” However, they are still too vague for detailed instructional design. For that we need to know what counts as a scientific concept, a subject addressed in the next Section.

III. Modeling Theory

Modeling Theory is about the structure and acquisition of scientific knowledge. Its central tenet is that *scientific knowledge is created, first, by constructing and validating models to represent structure in real objects and processes, and second, by organizing models into theories structured by scientific laws*. In other words, Modeling Theory is a particular brand of scientific epistemology that posits *models* as basic units of scientific knowledge and modeling (the process of creating and validating *models*) as the basic means of knowledge acquisition.

I am not alone in my belief that models and modeling constitute the core of scientific knowledge and practice. The same theme is prominent in recent History and Philosophy of Science, especially in the work of Nancy Nercessian and Ronald Giere,⁶ and in some math education research.⁷ It is also proposed as a unifying theme for K-12 science education in the *National Science Education*

Standards and the *AAAS Project 2061*. The Modeling Instruction Project has led the way in incorporating it into K-12 curriculum and instruction.³

Though I first introduced the term “Modeling Theory” in connection with my Theory of Instruction,⁸ I have always conceived of it as equally applicable to scientific research. In fact, Modeling Theory has been the main mechanism for transferring what I know about research into designs for instruction.

In so far as Modeling Theory constitutes an adequate epistemology of science, it provides a reliable framework for *critique* of the physics curriculum and a *guide* for revising it. From this perspective, I see the standard curriculum as seriously deficient at all levels from grade school to graduate school. In particular, the models inherent in the subject matter are seldom clearly delineated. Textbooks (and students) regularly fail to distinguish between models and their implications.² This results in a cascade of student learning difficulties. However, we cannot dwell on that important problem here.

I have discussed Modeling Theory and its instructional implications at some length elsewhere,^{2,8-10} although there is still more to say. The brief account above suffices to set the stage for application to the main subject of this lecture. Modeling Theory tells us that the primary conceptual tools mentioned in the 4th Learning Principle are modeling tools. Accordingly, I have devoted considerable PER effort to classification, design, and use of *modeling tools* for instruction. Heretofore, emphasis has been on the various kinds of graphs and diagrams used in physics, including analysis of the information they encode and comparison with mathematical representations.^{9,10} All of this was motivated and informed by my research experience with mathematical modeling.

In the balance of this lecture, I draw my mathematics research into the PER domain as an example of how PER can and should be concerned with basic physics research.

IV. Mathematics for Modeling Physical Reality

Mathematics is taken for granted in the physics curriculum—a body of immutable truths to be assimilated and applied. The profound influence of mathematics on our conceptions of the physical world is never analyzed. The possibility that mathematical tools used today were invented to solve problems in the past and might not be well suited for current problems is never considered. I aim to convince you that these issues have immense implications for physics education and deserve to be the subject of concerted PER.

One does not have to go very deeply into the history of physics to discover the profound influence of mathematical invention. Two famous examples will suffice to make the point: The invention of analytic geometry and calculus was essential to Newton’s creation of classical mechanics.² The invention of tensor analysis was essential to Einstein’s creation of the General Theory of Relativity.

Note my use of the terms “invention” and “creation” where others might have used the term “discovery.” This conforms to the epistemological stance of Modeling Theory and Einstein himself, who asserted that scientific theories

“cannot be extracted from experience, but must be freely invented.”¹¹ Note also that Einstein’s assertion amounts to a form of *scientific constructivism* in accord with the Learning Principles in Section II.

The point I wish to make by citing these two examples is that without essential mathematical concepts the two theories would have been literally inconceivable. The mathematical modeling tools we employ at once *extend and limit* our ability to conceive the world. Limitations of mathematics are evident in the fact that the analytic geometry that provides the foundation for classical mechanics is insufficient for General Relativity. This should alert one to the possibility of other conceptual limits in the mathematics used by physicists.

Since Newton’s day a variety of different symbolic systems have been invented to address problems in different contexts. Figure 1 lists nine such systems in use by physicists today. Few physicists are proficient with all of them, but each system has advantages over the others in some application domain. For example, for applications to rotations, quaternions are demonstrably more efficient than the vectorial and matrix methods taught in standard physics courses. The difference hardly matters in the world of academic exercises, but in the aerospace industry, for instance, where rotations are bread and butter, engineers opt for quaternions.

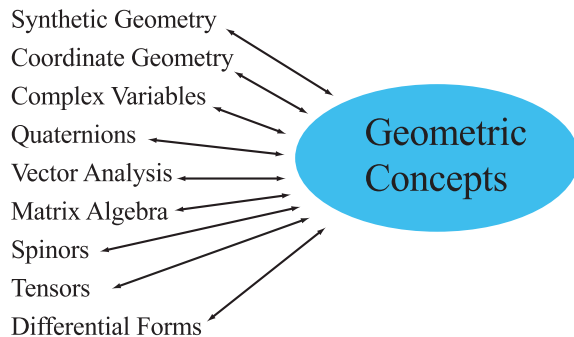


Fig. 1. Multiple mathematical systems contribute to the fragmentation of knowledge, though they have a common geometric nexus.

Each of the mathematical systems in Fig. 1 incorporates some aspect of geometry. Taken together, they constitute a highly redundant system of multiple representations for geometric concepts that are essential in physics. As a mathematical language for physics, this Babel of mathematical tongues has the following defects:

1. Limited access. The ideas, methods and results of theoretical physics are distributed broadly across these diverse mathematical systems. Since most physicists are proficient with only a few of the systems, their access to knowledge formulated in other systems is limited or denied. Of course, this language barrier is even greater for students.

2. Wasteful redundancy. In many cases, the same information is repre-

sented in several different systems, but one of them is invariably better suited than the others for a given application. For example, Goldstein's textbook on mechanics¹² gives three different ways to represent rotations: coordinate matrices, vectors and Pauli spin matrices. The costs in time and effort for translation between these representations are considerable.

3. Deficient integration. The collection of systems in Fig. 1 is not an integrated mathematical structure. This is especially awkward in problems that call for the special features of two or more systems. For, example, vector algebra and matrices are often awkwardly combined in rigid body mechanics, while Pauli matrices are used to express equivalent relations in quantum mechanics.

4. Hidden structure. Relations among physical concepts represented in different symbolic systems are difficult to recognize and exploit.

5. Reduced information density. The density of information about physics is reduced by distributing it over several different symbolic systems.

Evidently elimination of these defects will make physics easier to learn and apply. A clue as to how that might be done lies in recognizing that the various symbolic systems derive geometric interpretations from a common coherent core of geometric concepts. This suggests that one can create a unified mathematical language for physics by designing it to provide an optimal representation of geometric concepts. In fact, Hermann Grassmann recognized this possibility and took it a long way more than 150 years ago.¹³ However, his program to unify mathematics was forgotten and his mathematical ideas were dispersed, though many of them reappeared in the several systems of Fig. 1. A century later the program was reborn, with the harvest of a century of mathematics and physics to enrich it. This has been the central focus of my own scientific research.

Creating a unified mathematical language for physics is a problem in the design of mathematical systems. Here are some general criteria that I have applied to the design of Geometric Algebra as a solution to that problem:

1. Optimal algebraic encoding of the basic geometric concepts: magnitude, direction, sense (or orientation) and dimension.

2. Coordinate-free methods to formulate and solve basic equations of physics.

3. Optimal uniformity of method across classical, quantum and relativistic theories to make their common structures as explicit as possible.

4. Smooth articulation with widely used alternative systems (Fig. 1) to facilitate access and transfer of information.

5. Optimal computational efficiency. The unified system must be at least as efficient as any alternative system in every application.

Obviously, these design criteria ensure built-in benefits of the unified language. In implementing the criteria I deliberately sought out the best available mathematical ideas and conventions. I found that it was frequently necessary to modify the mathematics to simplify and clarify the physics.

This led me to coin the dictum: **Mathematics is too important to be left to the mathematicians!** I use it to flag the following *guiding principle for Modeling Theory*: **In the development of any scientific theory, a**

major task for theorists is to construct a mathematical language that optimizes expression of the key ideas and consequences of the theory.

Although existing mathematics should be consulted in this endeavor, it should not be incorporated without critically evaluating its suitability. I might add that the process also works in reverse. Modification of mathematics for the purposes of science serves as a stimulus for further development of mathematics. There are many examples of this effect in the history of physics.

Perhaps the most convincing evidence for validity of a new scientific theory is successful prediction of a surprising new phenomenon. Similarly, the most impressive benefits of Geometric Algebra arise from surprising new insights into the structure of physics.

The following Sections survey the elements of Geometric Algebra and its application to core components of the physics curriculum. Many details and derivations are omitted, as they are available elsewhere. The emphasis is on highlighting the unique advantages of Geometric Algebra as a unified mathematical language for physics.

V. Understanding Vectors

A recent study on the use of vectors by introductory physics students summarized the conclusions in two words: “*vector avoidance!*”¹⁴ This state of mind tends to propagate through the physics curriculum. In some 25 years of graduate physics teaching, I have noted that perhaps a third of the students seem incapable of reasoning with vectors as abstract elements of a linear space. Rather, they insist on conceiving a vector as a list of numbers or coordinates. I have come to regard this concept of vector as a kind of *conceptual virus*, because it impedes development of a more general and powerful concept of vector. I call it the *coordinate virus!*¹⁵

Once the coordinate virus has been identified, it becomes evident that the entire physics curriculum, including most of the textbooks, is infected with the virus. From my direct experience, I estimate that two thirds of the graduate students have serious infections, and half of those are so damaged by the virus that they will never recover. What can be done to control this scourge? I suggest that universal inoculation with Geometric Algebra could eventually eliminate the coordinate virus altogether.

I maintain that the origin of the problem lies not so much in pedagogy as in the mathematics. The fundamental geometric concept of a vector as a directed magnitude is not adequately represented in standard mathematics. The basic definitions of vector addition and scalar multiplication are essential to the vector concept but not sufficient. To complete the vector concept we need multiplication rules that enable us to compare directions and magnitudes of different vectors.

A. The Geometric Product

I take the standard concept of a real vector space for granted and define the *geometric product* \mathbf{ab} for vectors \mathbf{a} , \mathbf{b} , \mathbf{c} by the following rules:

$$(\mathbf{ab})\mathbf{c} = \mathbf{a}(\mathbf{bc}), \quad \text{associative} \quad (1)$$

$$\mathbf{a}(\mathbf{b} + \mathbf{c}) = \mathbf{ab} + \mathbf{ac}, \quad \text{left distributive} \quad (2)$$

$$(\mathbf{b} + \mathbf{c})\mathbf{a} = \mathbf{ba} + \mathbf{ca}, \quad \text{right distributive} \quad (3)$$

$$\mathbf{a}^2 = |\mathbf{a}|^2. \quad \text{contraction} \quad (4)$$

where $|\mathbf{a}|$ is a positive scalar called the *magnitude* of \mathbf{a} , and $|\mathbf{a}| = 0$ implies that $\mathbf{a} = 0$.

All of these rules should be familiar from ordinary scalar algebra. The main difference is *absence of a commutative rule*. Consequently, left and right distributive rules must be postulated separately. The *contraction rule* (4) is peculiar to geometric algebra and distinguishes it from all other associative algebras. But even this is familiar from ordinary scalar algebra as the relation of a signed number to its magnitude.

The rules for multiplying vectors are the basic grammar rules for GA, and they can be applied to vector spaces of any dimension. The power of GA derives from

- the *simplicity* of the grammar,
- the *geometric meaning of multiplication*,
- the way *geometry links the algebra to the physical world*.

My next task is to elucidate the geometric meaning of vector multiplication. From the geometric product \mathbf{ab} we can define two new products, a symmetric *inner product*

$$\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba}) = \mathbf{b} \cdot \mathbf{a}, \quad (5)$$

and an antisymmetric *outer product*

$$\mathbf{a} \wedge \mathbf{b} = \frac{1}{2}(\mathbf{ab} - \mathbf{ba}) = -\mathbf{b} \wedge \mathbf{a}. \quad (6)$$

Therefore, the geometric product has the *canonical decomposition*

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}. \quad (7)$$

From the contraction rule (4) it is easy to prove that $\mathbf{a} \cdot \mathbf{b}$ is scalar-valued, so it can be identified with the standard Euclidean inner product. The formal legitimacy and geometric import of adding scalars to bivectors as well as to vectors is discussed in Section VIA.

The geometric significance of the outer product $\mathbf{a} \wedge \mathbf{b}$ should also be familiar from the standard vector cross product $\mathbf{a} \times \mathbf{b}$. The quantity $\mathbf{a} \wedge \mathbf{b}$ is called a *bivector*, and it can be interpreted geometrically as an *oriented plane segment*, as shown in Fig. 2. It differs from $\mathbf{a} \times \mathbf{b}$ in being intrinsic to the plane containing \mathbf{a} and \mathbf{b} , independent of the dimension of any vector space in which the plane lies.

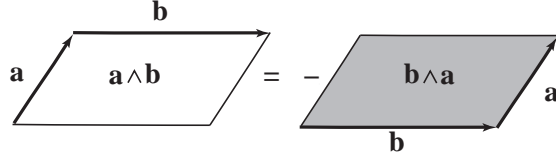


Fig. 2. Bivectors $\mathbf{a} \wedge \mathbf{b}$ and $\mathbf{b} \wedge \mathbf{a}$ represent plane segments of opposite orientation as specified by a “parallelogram rule” for drawing the segments.

From the geometric interpretations of the inner and outer products, we can infer an interpretation of the geometric product from extreme cases. For orthogonal vectors, we have from (5)

$$\mathbf{a} \cdot \mathbf{b} = 0 \quad \iff \quad \mathbf{ab} = -\mathbf{ba}. \quad (8)$$

On the other hand, collinear vectors determine a parallelogram with vanishing area (Fig. 2), so from (6) we have

$$\mathbf{a} \wedge \mathbf{b} = 0 \quad \iff \quad \mathbf{ab} = \mathbf{ba}. \quad (9)$$

Thus, the geometric product \mathbf{ab} provides a measure of the relative direction of the vectors. Commutativity means that the vectors are collinear. Anticommutativity means that they are orthogonal. Multiplication can be reduced to these extreme cases by introducing an orthonormal basis.

B. Basis and Bivectors

For an orthonormal set of vectors $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \dots\}$, the multiplicative properties can be summarized by putting (5) in the form

$$\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j = \frac{1}{2}(\boldsymbol{\sigma}_i \boldsymbol{\sigma}_j + \boldsymbol{\sigma}_j \boldsymbol{\sigma}_i) = \delta_{ij} \quad (10)$$

where δ_{ij} is the usual Kronecker delta. This relation applies to a Euclidean vector of any dimension, though for the moment we focus on the 2D case.

A unit bivector \mathbf{i} for the plane containing vectors $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ is determined by the product

$$\mathbf{i} = \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_1 \wedge \boldsymbol{\sigma}_2 = -\boldsymbol{\sigma}_2 \boldsymbol{\sigma}_1 \quad (11)$$

The suggestive symbol \mathbf{i} has been chosen because by squaring (11) we find that

$$\mathbf{i}^2 = -1 \quad (12)$$

Thus, \mathbf{i} is a truly geometric $\sqrt{-1}$. We shall see that there are others.

From (11) we also find that

$$\boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_1 \mathbf{i} = -\mathbf{i} \boldsymbol{\sigma}_1 \quad \text{and} \quad \boldsymbol{\sigma}_1 = \mathbf{i} \boldsymbol{\sigma}_2. \quad (13)$$

In words, multiplication by \mathbf{i} rotates the vectors through a right angle. It follows that \mathbf{i} rotates every vector in the plane in the same way. More generally, it follows that every unit bivector \mathbf{i} satisfies (12) and determines a unique plane in Euclidean space. Each \mathbf{i} has two complementary geometric interpretations: *It represents a unique oriented area for the plane, and, as an operator, it represents an oriented right angle rotation in the plane.*

C. Vectors and Complex Numbers

Assigning a geometric interpretation to the geometric product is more subtle than interpreting inner and outer products — so subtle, in fact, that the appropriate assignment has been generally overlooked to this day. The product of any pair of unit vectors \mathbf{a}, \mathbf{b} generates a new kind of entity U called a *rotor*, as expressed by the equation

$$U = \mathbf{a}\mathbf{b}. \quad (14)$$

The *relative direction* of the two vectors is completely characterized by the *directed arc* that relates them (Fig. 3), so we can interpret U as representing that arc. The name “rotor” is justified by the fact that U rotates \mathbf{a} and \mathbf{b} into each other, as shown by multiplying (14) by vectors to get

$$\mathbf{b} = \mathbf{a}U \quad \text{and} \quad \mathbf{a} = U\mathbf{b}. \quad (15)$$

Further insight is obtained by noting that

$$\mathbf{a} \cdot \mathbf{b} = \cos \theta \quad \text{and} \quad \mathbf{a} \wedge \mathbf{b} = \mathbf{i} \sin \theta, \quad (16)$$

where θ is the angle from \mathbf{a} to \mathbf{b} . Accordingly, with the angle dependence made explicit, the decomposition (7) enables us to write (14) in the form

$$U_\theta = \cos \theta + \mathbf{i} \sin \theta = e^{\mathbf{i}\theta}. \quad (17)$$

It follows that multiplication by U_θ , as in (15), will rotate any vector in the \mathbf{i} -plane through the angle θ . This tells us that we should *interpret U_θ as a directed arc of fixed length that can be rotated at will on the unit circle*, just as we interpret a vector \mathbf{a} as a directed line segment that can be translated at will without changing its length or direction (Fig. 4).

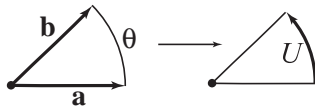


Fig. 3. A pair of unit vectors \mathbf{a}, \mathbf{b} determine a *directed arc* on the unit circle that represents their product $U = \mathbf{a}\mathbf{b}$. The length of the arc is (radian measure of) the angle θ between the vectors.

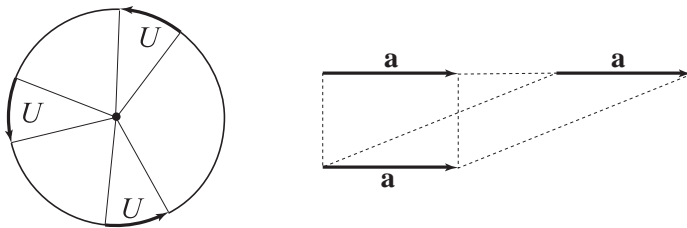


Fig. 4. All directed arcs with equivalent angles are represented by a single rotor U_θ , just as line segments with the same length and direction are represented by a single vector \mathbf{a} .

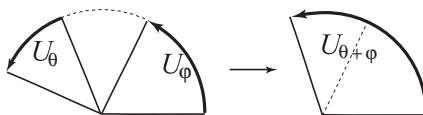


Fig. 5. The composition of 2D rotations is represented algebraically by the product of rotors and depicted geometrically by addition of directed arcs.

With rotors, the composition of 2D rotations is expressed by the rotor product

$$U_\theta U_\varphi = U_{\theta+\varphi} \quad (18)$$

and depicted geometrically in Fig. 5 as addition of directed arcs.

The generalization of all this should be obvious. We can always interpret the product \mathbf{ab} algebraically as a *complex number*

$$z = \lambda U = \lambda e^{i\theta} = \mathbf{ab}, \quad (19)$$

with *modulus* $|z| = \lambda = |\mathbf{a}||\mathbf{b}|$. And we can *interpret* z geometrically as a directed arc on a circle of radius $|z|$ (Fig. 6). It might be surprising that this geometric interpretation never appears in standard books on complex variables. Be that as it may, the value of the interpretation is greatly enhanced by its use in geometric algebra.

The connection to vectors via (19) removes a lot of the mystery from complex numbers and facilitates their application to physics. For example, comparison of (19) to (7) shows at once that the real and imaginary parts of a complex number are equivalent to inner and outer products of vectors. The complex conjugate of (19) is

$$z^\dagger = \lambda U^\dagger = \lambda e^{-i\theta} = \mathbf{ba}, \quad (20)$$

which shows that it is equivalent to reversing order in the geometric product. This can be used to compute the modulus of z in the usual way:

$$|z|^2 = z z^\dagger = \lambda^2 = \mathbf{baab} = \mathbf{a}^2 \mathbf{b}^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 \quad (21)$$

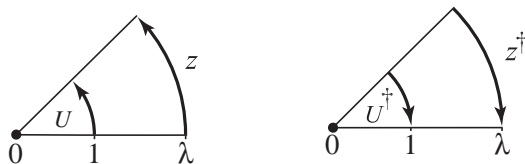


Fig. 6. A complex number $z = \lambda U$ with modulus λ and angle θ can be interpreted as a directed arc on a circle of radius λ . Its conjugate $z^\dagger = \lambda U^\dagger$ represents an arc with opposite orientation

Anyone who has worked with complex numbers in applications knows that it is usually best to avoid decomposing them into real and imaginary parts. Likewise, in GA applications it is usually best practice to work directly with the geometric product instead of separating it into inner and outer products.

GA gives complex numbers new powers to operate directly on vectors. For example, from (19) and (20) we get

$$\mathbf{b} = \mathbf{a}^{-1} z = z^\dagger \mathbf{a}^{-1}, \quad (22)$$

where the multiplicative inverse of vector \mathbf{a} is given by

$$\mathbf{a}^{-1} = \frac{1}{\mathbf{a}} = \frac{\mathbf{a}}{\mathbf{a}^2} = \frac{\mathbf{a}}{|\mathbf{a}|^2}. \quad (23)$$

Thus, z rotates and rescales \mathbf{a} to get \mathbf{b} . This makes it possible to construct and manipulate vectorial transformations and functions without introducing a basis or matrices.

This is a good point to pause and note some instructive implications of what we have established so far. Every physicist knows that complex numbers, especially equations (17) and (18), are ideal for dealing with plane trigonometry and 2D rotations. However, students in introductory physics are denied access to this powerful tool, evidently because it has a reputation for being conceptually difficult, and class time would be lost by introducing it. GA removes these barriers to use of complex numbers by linking them to vectors and giving them a clear geometric meaning.

GA also makes it possible to formulate and solve 2D physics problems in terms of vectors without introducing coordinates. Conventional vector algebra cannot do this, in part because the vector cross product is defined only in 3D. That is the main reason why coordinate methods dominate introductory physics. The available math tools are too weak to do otherwise. GA changes all that!

For example, most of the mechanics problems in introductory physics are 2D problems. Coordinate-free GA solutions for the standard problems are worked out in my mechanics book.¹⁶ Although the treatment there is for a more advanced course, it can easily be adapted to the introductory level. The essential GA concepts for that level have already been presented in this section.

Will comprehensive use of GA significantly enhance student learning in introductory physics? We have noted theoretical reasons for believing that it will. To check this out in practice is a job for PER. However, mathematical reform at the introductory level makes little sense unless it is extended to the whole physics curriculum. The following sections provide strong justification for doing just that. We shall see how simplifications at the introductory level get amplified to greater simplifications and surprising insights at the advanced level.

VI. Classical Physics with Geometric Algebra

This section surveys the fundamentals of GA as a mathematical framework for classical physics and demonstrates some of its unique advantages. Detailed applications can be found in the references.

A. Geometric Algebra for Physical Space

The arena for classical physics is a 3D Euclidean *vector space* \mathcal{P}^3 , which serves as a model for “Physical Space.” By multiplication and addition the vectors generate a geometric algebra $\mathcal{G}_3 = \mathcal{G}(\mathcal{P}^3)$. In particular, a basis for the whole algebra can be generated from a *standard frame* $\{\sigma_1, \sigma_2, \sigma_3\}$, a righthanded set of orthonormal vectors.

With multiplication specified by (10), the standard frame generates a unique trivector (3-vector) or *pseudoscalar*

$$i = \sigma_1 \sigma_2 \sigma_3, \quad (24)$$

and a *bivector* (2-vector) basis

$$\sigma_1 \sigma_2 = i \sigma_3, \quad \sigma_2 \sigma_3 = i \sigma_1, \quad \sigma_3 \sigma_1 = i \sigma_2. \quad (25)$$

Geometric interpretations for the pseudoscalar and bivector basis elements are depicted in Figs. 7 and 8.

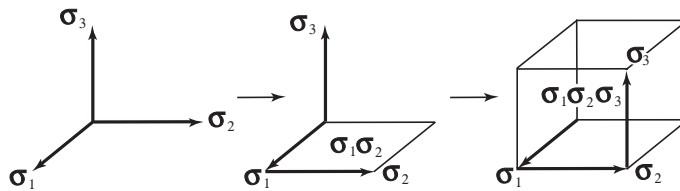


Fig. 7. Unit pseudoscalar i represents an *oriented unit volume*. The volume is said to be righthanded, because i can be generated from a righthanded vector basis by the ordered product $\sigma_1 \sigma_2 \sigma_3 = i$.

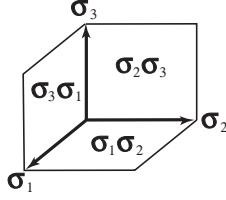


Fig. 8. Unit bivectors representing a basis of directed areas in planes with orthogonal intersections

The pseudoscalar i has special properties that facilitate applications as well as articulation with standard vector algebra. It follows from (24) that

$$i^2 = -1, \quad (26)$$

and it follows from (25) that every bivector \mathbf{B} in \mathcal{G}_3 is the *dual* of a vector \mathbf{b} as expressed by

$$\mathbf{B} = i\mathbf{b} = \mathbf{b}i. \quad (27)$$

Thus, the geometric *duality* operation is simply expressed as multiplication by the pseudoscalar i . This enables us to write the outer product defined by (6) in the form

$$\mathbf{a} \wedge \mathbf{b} = i\mathbf{a} \times \mathbf{b}. \quad (28)$$

Thus, the conventional *vector cross product* $\mathbf{a} \times \mathbf{b}$ is implicitly defined as the dual of the outer product. Consequently, the fundamental decomposition of the geometric product (7) can be put in the form

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + i\mathbf{a} \times \mathbf{b}. \quad (29)$$

This is the definitive relation among vector products that we need for *smooth articulation* between geometric algebra and standard vector algebra, as is demonstrated with many examples in my mechanics book.¹⁶

The elements in any geometric algebra are called *multivectors*. The special properties of i enable us to write any multivector M in \mathcal{G}_3 in the *expanded form*

$$M = \alpha + \mathbf{a} + i\mathbf{b} + i\beta, \quad (30)$$

where α and β are scalars and \mathbf{a} and \mathbf{b} are vectors. The main value of this form is that it reduces multiplication of multivectors in \mathcal{G}_3 to multiplication of vectors given by (29). Note that the four terms in (30) are linearly independent, so scalar, vector, bivector and pseudoscalar parts combine separately under multivector addition, though they are mixed by multiplication. Thus, the Geometric Algebra \mathcal{G}_3 is a linear space of dimension $1 + 3 + 3 + 1 = 2^3 = 8$.

The expansion (30) has the formal algebraic structure of a “complex scalar” $\alpha + i\beta$ added to a “complex vector” $\mathbf{a} + i\mathbf{b}$, but any physical interpretation

attributed to this structure hinges on the geometric meaning of i . The most important example is the expression of the *electromagnetic field* F in terms of an electric vector field \mathbf{E} and a magnetic vector field B :

$$F = \mathbf{E} + i\mathbf{B}. \quad (31)$$

Geometrically, this is a decomposition of F into vector and bivector parts. In standard vector algebra \mathbf{E} is said to be a *polar vector* while \mathbf{B} is an *axial vector*, the two kinds of vector being distinguished by a difference in sign under space inversion. GA reveals that an axial vector is just a bivector represented by its dual, so the magnetic field in (31) is fully represented by the complete bivector $i\mathbf{B}$, rather than \mathbf{B} alone. Thus GA makes the awkward distinction between polar and axial vectors unnecessary. The vectors \mathbf{E} and \mathbf{B} in (31) have the same behavior under space inversion, but an additional sign change comes from space inversion of the pseudoscalar.

To facilitate algebraic manipulations, it is convenient to introduce a special symbol for the operation (called *reversion*) of reversing the order of multiplication. The *reverse* of the geometric product is defined by

$$(\mathbf{ab})^\dagger = \mathbf{ba}. \quad (32)$$

We noted in (20) that this is equivalent to complex conjugation in 2D. From (24) we find that the reverse of the pseudoscalar is

$$i^\dagger = -i. \quad (33)$$

Hence the reverse of an arbitrary multivector in the expanded form (30) is

$$M^\dagger = \alpha + \mathbf{a} - i\mathbf{b} - i\beta, \quad (34)$$

The convenience of this operation is illustrated by applying it to the electromagnetic field F in (31) and using (29) to get

$$\frac{1}{2}FF^\dagger = \frac{1}{2}(\mathbf{E} + i\mathbf{B})(\mathbf{E} - i\mathbf{B}) = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) + \mathbf{E} \times \mathbf{B}, \quad (35)$$

which is recognized as an expression for the energy and momentum density of the field. Note how this differs from the field invariant

$$F^2 = (\mathbf{E} + i\mathbf{B})^2 = \mathbf{E}^2 - \mathbf{B}^2 + 2i(\mathbf{E} \cdot \mathbf{B}), \quad (36)$$

which is useful for classifying EM fields into different types.

You have probably noticed that the expanded multivector form (30) violates one of the basic *math strictures* that is drilled into our students, namely, that “it is meaningless to add scalars to vectors,” not to mention bivectors and pseudoscalars. On the contrary, GA tells us that such addition is not only geometrically meaningful, it is essential to simplify and unify the language of physics, as can be seen in many examples that follow.

Shall we say that this stricture against addition of scalars to vectors is a misconception or conceptual virus that infects the entire physics community?

At least it is a design flaw in standard vector algebra that has been almost universally overlooked. As we have just seen, elimination of the flaw enables us to combine electric and magnetic fields into a single electromagnetic field. And we shall see below how *it enables us to construct spinors from vectors* (contrary to the received wisdom that spinors are more basic than vectors)!

B. Reflections and Rotations

Rotations play an essential role in the conceptual foundations of physics as well as in many applications, so our mathematics should be designed to handle them as efficiently as possible. We have noted that conventional treatments employ an awkward mixture of vector, matrix and spinor or quaternion methods. My purpose here is to show how GA provides a unified, coordinate-free treatment of rotations and reflections that leaves nothing to be desired.

The main result is that any *orthogonal transformation* \underline{U} can be expressed in the *canonical form*¹⁶

$$\underline{U}\mathbf{x} = \pm U\mathbf{x}U^\dagger, \quad (37)$$

where U is a *unimodular* multivector called a *versor*, and the sign is the *parity* of U , positive for a rotation or negative for a reflection. The condition

$$U^\dagger U = 1. \quad (38)$$

defines unimodularity. The underbar notation serves to distinguish the linear operator \underline{U} from the versor U that generates it. The great advantage of (37) is that it reduces the study of linear operators to algebraic properties of their versors. This is best understood from specific examples.

The simplest example is *reflection* in a plane with unit normal \mathbf{a} (Fig. 9),

$$\mathbf{x}' = -\mathbf{a}\mathbf{x}\mathbf{a} = -\mathbf{a}(\mathbf{x}_\perp + \mathbf{x}_\parallel)\mathbf{a} = \mathbf{x}_\perp - \mathbf{x}_\parallel. \quad (39)$$

To show how this function works, the vector \mathbf{x} has been decomposed on the right into a parallel component $\mathbf{x}_\parallel = (\mathbf{x} \cdot \mathbf{a})\mathbf{a}$ that commutes with \mathbf{a} and an orthogonal component $\mathbf{x}_\perp = (\mathbf{x} \wedge \mathbf{a})\mathbf{a}$ that anticommutes with \mathbf{a} . As can be seen below, it is seldom necessary or even advisable to make this decomposition in applications. *The essential point is that the normal vector defining the direction of a plane also represents a reflection in the plane* when interpreted as a versor. A simpler representation for reflections is inconceivable, so it must be the optimal representation for reflections in every application, as shown in some important applications below. Incidentally, the term *versor* was coined in the 19th century for an operator that can *re-verse* a direction. Likewise, the term is used here to indicate a geometric operational interpretation for a multivector.

The reflection (39) is not only the simplest example of an orthogonal transformation, but all orthogonal transformations can be generated by reflections of this kind. The main result is expressed by the following theorem: *The product of two reflections is a rotation through twice the angle between the normals of*

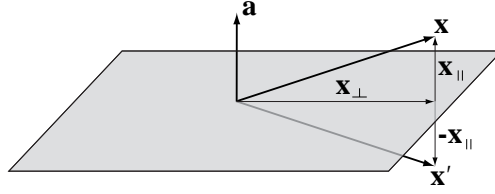


Fig. 9. Reflection in a plane.

the reflecting planes. This important theorem seldom appears in standard textbooks, primarily, I presume, because its expression in conventional formalism is so awkward as to render it impractical. However, it is an easy consequence of a second reflection applied to (39). Thus, for a plane with unit normal \mathbf{b} , we have

$$\mathbf{x}'' = -\mathbf{b}\mathbf{x}'\mathbf{b} = \mathbf{b}\mathbf{x}\mathbf{a}\mathbf{b} = U\mathbf{x}U^\dagger, \quad (40)$$

where a new symbol has been introduced for the versor product $U = \mathbf{b}\mathbf{a}$. The theorem is obvious from the geometric construction in Fig. 10. For an algebraic proof that the result does not depend on the reflecting planes, we use (17) to write

$$U = \mathbf{b}\mathbf{a} = \cos \frac{1}{2}\theta + \mathbf{i} \sin \frac{1}{2}\theta = e^{\frac{1}{2}\mathbf{i}\theta}, \quad (41)$$

where, anticipating the result from Fig. 9, we denote the angle between \mathbf{a} and \mathbf{b} by $\frac{1}{2}\theta$ and the unit bivector for the $\mathbf{b} \wedge \mathbf{a}$ -plane by \mathbf{i} . Next, we decompose \mathbf{x} into a component \mathbf{x}_\perp orthogonal to the \mathbf{i} -plane and a component \mathbf{x}_\parallel in the plane. Note that, respectively, the two components commute (anticommute) with \mathbf{i} , so

$$\mathbf{x}_\perp U^\dagger = U^\dagger \mathbf{x}_\perp, \quad \mathbf{x}_\parallel U^\dagger = U \mathbf{x}_\parallel. \quad (42)$$

Inserting this into (40) with $\mathbf{x} = \mathbf{x}_\parallel + \mathbf{x}_\perp$, we obtain

$$\mathbf{x}'' = U\mathbf{x}U^\dagger = \mathbf{x}_\perp + U^2\mathbf{x}_\parallel. \quad (43)$$

These equations show how the two-sided multiplication by the versor U picks out the component of \mathbf{x} to be rotated, so we see that one-sided multiplication works only in 2D. As we learned from our discussion of 2D rotations, the versor $U^2 = e^{\mathbf{i}\theta}$ rotates \mathbf{x}_\perp through angle θ , in agreement with the half-angle choice in (41).

The great advantage of the canonical form (37) for an orthogonal transformation is that it reduces the composition of orthogonal transformations to versor multiplication. Thus, composition expressed by the operator equation

$$\underline{U}_2 \underline{U}_1 = \underline{U}_3 \quad (44)$$

is reduced to the product of corresponding versors

$$U_2 U_1 = U_3. \quad (45)$$

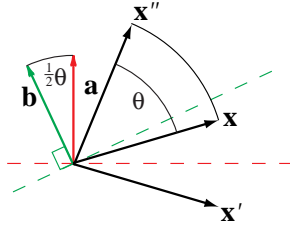


Fig. 10. Rotation as double reflection, depicted in the plane containing unit normals \mathbf{a} , \mathbf{b} of the reflecting planes.

The orthogonal transformations form a mathematical group with (44) as the group composition law. The trouble with (44) is that abstract operator algebra does not provide a way to compute \underline{U}_3 from given \underline{U}_1 and \underline{U}_2 . The usual solution to this problem is to represent the operators by matrices and compute by matrix multiplication. A much simpler solution is to represent the operators by versors and compute with the geometric product. We have already seen how the product of reflections represented by $U_1 = \mathbf{a}$ and $U_2 = \mathbf{b}$ produces a rotation $U_3 = \mathbf{ba}$. Matrix algebra does not provide such a transparent result.

As is well known, the *rotation group* is a subgroup of the *orthogonal group*. This is expressed by the fact that rotations are represented by unimodular versors of even parity, for which the term *rotor* was introduced earlier. The composition of 2D rotations is described by the rotor equation (18) and depicted in Fig. 5. Its generalization to composition of 3D rotations in different planes is described algebraically by (45) and depicted geometrically in Fig. 11. This deserves some explanation.

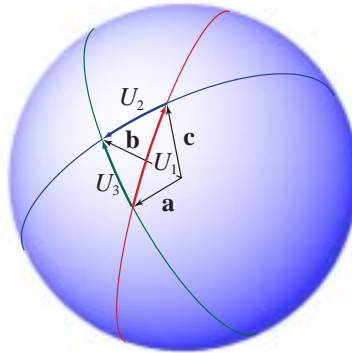


Fig. 11. Addition of directed arcs in 3D depicting the product of rotors. Vectors \mathbf{a} , \mathbf{b} , \mathbf{c} all originate at the center of the sphere.

In 3D a rotor is depicted as a directed arc confined to a great circle on the unit sphere. The product of rotors U_1 and U_2 is depicted in Fig. 11 by connecting the corresponding arcs at a point \mathbf{c} where the two great circles intersect. This determines points $\mathbf{a} = \mathbf{c}U_1$ and $\mathbf{b} = U_2\mathbf{c}$, so the rotors can be expressed as products with a common factor,

$$U_1 = \mathbf{c}\mathbf{a}, \quad U_2 = \mathbf{b}\mathbf{c}. \quad (46)$$

Hence (44) gives us

$$U_3 = U_2U_1 = (\mathbf{b}\mathbf{c})(\mathbf{c}\mathbf{a}) = \mathbf{b}\mathbf{a}, \quad (47)$$

with the corresponding arc for U_3 depicted in Fig. 11. It should not be forgotten that the arcs in Fig. 11 depict half-angles of the rotations. The non-commutativity of rotations is illustrated in Fig. 12, which depicts the construction of arcs for both U_1U_2 and U_2U_1 .

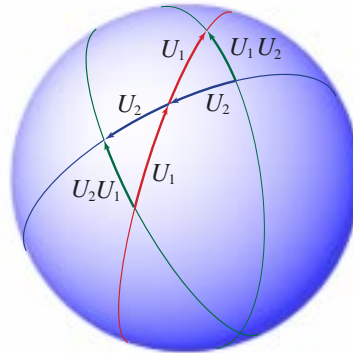


Fig. 12. Noncommutativity of rotations depicted in the construction of directed arcs representing rotor products

Those of you who are familiar with quaternions will have recognized that they are algebraically equivalent to rotors, so we might as well regard the two as one and the same. Advantages of the quaternion theory of rotations have been known for the better part of two centuries, but to this day only a small number of specialists have been able to exploit them. Geometric algebra makes them available to everyone by embedding quaternions in a more comprehensive mathematical system. More than that, *GA makes a number of significant improvements in quaternion theory — the most important being the integration of reflections with rotations* as described above. To make this point more explicit and emphatic, I describe two important practical applications where the generation of rotations by reflections is essential.

Multiple reflections. Consider a light wave (or ray) initially propagating with direction \mathbf{k} and reflecting off a sequence of plane surfaces with unit normals

$\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ (Fig. 12). By multiple applications of (39) we find that it emerges with direction

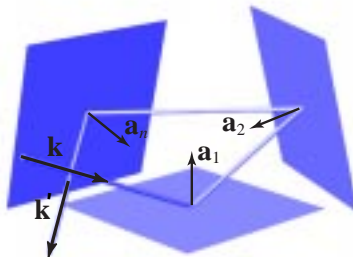


Fig. 13. Multiple reflections

$$\mathbf{k}' = (-1)^n \mathbf{a}_n \dots \mathbf{a}_2 \mathbf{a}_1 \mathbf{k} \mathbf{a}_1 \mathbf{a}_2 \dots \mathbf{a}_n \quad (48)$$

The net reflection is completely characterized by a single unimodular multivector $U = \mathbf{a}_n \dots \mathbf{a}_2 \mathbf{a}_1$, which, according to (30), can be reduced to the form $U = \mathbf{a} + i\beta$ if n is an odd integer, or $U = \alpha + i\mathbf{b}$ if n is even. This is one way that GA facilitates modeling of the interaction of light with optical devices.

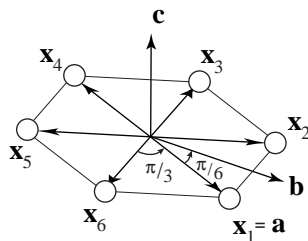


Fig. 14. Symmetry vectors for the benzene molecule.

Point Symmetry Groups. Molecules and crystals can be classified by their symmetries under reflections and rotations in planes through a fixed point. Despite the increasing importance of this subject in the modern molecular age, it is addressed only in advanced specialty courses after lengthy mathematical preparation. GA changes the ground rules drastically, making it possible to investigate point symmetries *without any special preparation* as soon as reflections have been introduced. Students can investigate interesting point groups as exercises in learning the properties of versors. From there it is only a small step to characterize all the point groups systematically.

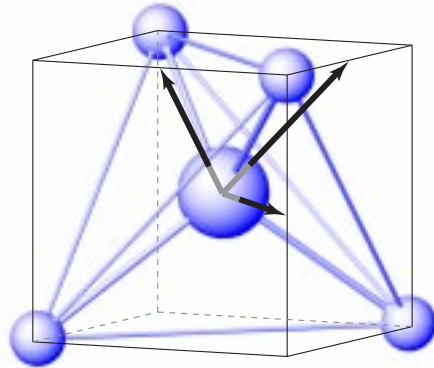


Fig. 15. Symmetry vectors for the methane molecule.

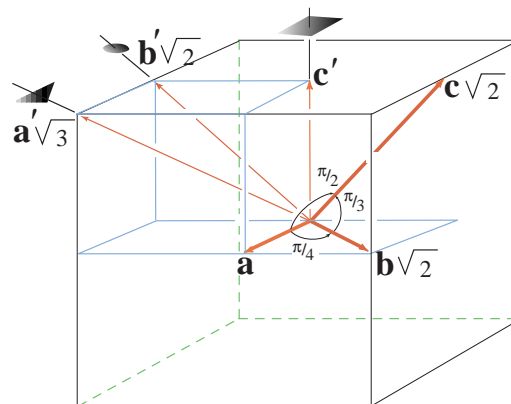


Fig. 16. Symmetry vectors for crystals with cubic symmetry. Vectors \mathbf{a} , \mathbf{b} , \mathbf{c} generate reflection symmetries. Vectors \mathbf{a}' , \mathbf{b}' , \mathbf{c}' specify axes of rotation symmetries.

Remarkably, every point symmetry group is generated multiplicatively by some combination of three unit vectors \mathbf{a} , \mathbf{b} , \mathbf{c} satisfying the versor conditions

$$(\mathbf{ab})^p = (\mathbf{bc})^q = (\mathbf{ca})^r = -1, \quad (49)$$

where $\{p, q, r\}$ is a set of three integers that characterize the symmetry group. These conditions describe p -fold, q -fold and r -fold rotation symmetries. For example, the set is $\{6, 2, 2\}$ for the planar *benzene molecule* (Fig. 14), and $(\mathbf{ab})^6 = -1$ represents a sixfold rotation that brings all atoms back to their original positions.

The *methane molecule* (Fig. 15) has tetrahedral symmetry characterized by $\{3, 3, 3\}$, which specifies the 3-fold symmetry of the tetrahedron faces. This particular symmetry cannot be extended to a space-filling crystal, for which it can be shown that at least one of the symmetries must be 2-fold.

There are precisely 32 *crystallographic point groups* distinguished by a small set of allowed values of p and q in $\{p, q, 2\}$. For example, generating vectors for the case $\{4, 3, 2\}$ of crystals with cubic symmetry are shown in Fig. 16. A complete analysis of all the point groups is given elsewhere,¹⁷ along with an extension of GA techniques to handle the 230 space groups.

The point symmetry groups of molecules and crystals are increasing in importance as we enter the age of nanoscience and molecular biology. Yet the topic remains relegated to specialized courses, no doubt because the standard treatment is so specialized. However, we have just seen that the GA approach to reflections and rotations brings with it an easy treatment of the point groups at no extra cost.

This is a good place to summarize with a list of the advantages of the GA approach to rotations, including some to be explained in subsequent Sections:

1. Coordinate-free formulation and computation.
2. Simple algebraic composition.
3. Geometric depiction of rotors as directed arcs.
4. Rotor products depicted as addition of directed arcs.
5. Integration of rotations and reflections in a single method.
6. Efficient parameterizations (see ref.¹⁶ for details).
7. Smooth articulation with matrix methods.
8. Rotational kinematics without matrices.

Moreover, the approach generalizes directly to Lorentz transformations, as will be demonstrated in a subsequent paper.

C. Frames and Rotation Kinematics

Any orthonormal righthanded frame $\{\mathbf{e}_k, k = 1, 2, 3\}$ can be obtained from our standard frame $\{\boldsymbol{\sigma}_k\}$ by a rotation in the canonical form

$$\mathbf{e}_k = U \boldsymbol{\sigma}_k U^\dagger. \quad (50)$$

Alternatively, the frames can be related by a rotation matrix

$$\mathbf{e}_k = \alpha_{kj} \boldsymbol{\sigma}_j. \quad (51)$$

These two sets of equations can be solved for the matrix elements as a function of U , with the result

$$\alpha_{kj} = \mathbf{e}_k \cdot \boldsymbol{\sigma}_j = \langle U \boldsymbol{\sigma}_k U^\dagger \boldsymbol{\sigma}_j \rangle, \quad (52)$$

where $\langle \dots \rangle$ means *scalar part*. Alternatively, they can be solved for the rotor as a function of the frames or the matrix.¹⁶ One simply forms the quaternion

$$\psi = 1 + \mathbf{e}_k \boldsymbol{\sigma}_k = 1 + \alpha_{kj} \boldsymbol{\sigma}_j \boldsymbol{\sigma}_k \quad (53)$$

and normalizes to get

$$U = \frac{\psi}{(\psi\psi^\dagger)^{\frac{1}{2}}}. \quad (54)$$

This makes it easy to move back and forth between matrix and rotor representations of a rotation. We have already seen that the rotor is much to be preferred for both algebraic computation and geometric interpretation.

Let the frame $\{\mathbf{e}_k\}$ represent a set of directions fixed in a rigid body, perhaps aligned with the principal axes of the inertia tensor. For a moving body the $\mathbf{e}_k = \mathbf{e}_k(t)$ are functions of time, and (50) reduces the description of the rotational motion to a time dependent rotor $U = U(t)$. By differentiating the constraint $UU^\dagger = 1$, it is easy to show that the derivative of U can be put in the form

$$\frac{dU}{dt} = \frac{1}{2} \boldsymbol{\Omega} U, \quad (55)$$

where

$$\boldsymbol{\Omega} = -i \boldsymbol{\omega} \quad (56)$$

is the rotational (angular) velocity bivector. By differentiating (50) and using (55), (56), we derive the familiar equations

$$\frac{d\mathbf{e}_k}{dt} = \boldsymbol{\omega} \times \mathbf{e}_k \quad (57)$$

employed in the standard vectorial treatment of rigid body kinematics.

The point of all this is that GA reduces the set of three vectorial equations (57) to the single rotor equation (55), which is easier to solve and analyze for given $\boldsymbol{\Omega} = \boldsymbol{\Omega}(t)$. Specific solutions for problems in rigid body mechanics are discussed elsewhere.¹⁶ However, the main reason for introducing the classical rotor equation of motion in this lecture is to show its equivalence to equations in quantum mechanics given below.

D. Maxwell's Equation

We have seen how electric and magnetic field vectors can be combined into a single multivector field

$$F(\mathbf{x}, t) = \mathbf{E}(\mathbf{x}, t) + i\mathbf{B}(\mathbf{x}, t) \quad (58)$$

representing the complete electromagnetic field. Standard vector algebra forces one to consider electric and magnetic parts separately, and it requires four field equations to describe their coordinated action. GA enables us to put Humpty Dumpty together and describe the complete electromagnetic field by a single equation. But first we need to learn how to differentiate with respect to the position vector \mathbf{x} .

We can define the derivative $\nabla = \partial_{\mathbf{x}}$ with respect to the vector \mathbf{x} most quickly by appealing to your familiarity with the standard concepts of divergence and curl. Then, since ∇ must be a vector operator, we can use (29) to define the vector derivative by

$$\nabla \mathbf{E} = \nabla \cdot \mathbf{E} + i \nabla \times \mathbf{E} = \nabla \cdot \mathbf{E} + \nabla \wedge \mathbf{E}. \quad (59)$$

This shows the divergence and curl as components of a single vector derivative. Both components are needed to determine the field. For example, for the field due to a static charge density $\rho = \rho(\mathbf{x})$, the field equation is

$$\nabla \mathbf{E} = \rho. \quad (60)$$

The advantage of this form over the usual separate equations for divergence and curl is that ∇ can be inverted to solve for

$$\mathbf{E} = \nabla^{-1} \rho. \quad (61)$$

Of course ∇^{-1} is an integral operator determined by a Green's function, but GA provides new insight into such operators. For example, for a source ρ with 2D symmetry in a localized 2D region \mathcal{R} with boundary $\partial\mathcal{R}$, the \mathbf{E} field is planar and ∇^{-1} can be given the explicit form^{18,19}

$$\mathbf{E}(\mathbf{x}) = \frac{1}{2\pi} \int_{\mathcal{R}} |d^2\mathbf{x}'| \frac{1}{\mathbf{x}' - \mathbf{x}} \rho(\mathbf{x}') + \frac{1}{2\pi i} \oint_{\partial\mathcal{R}} \frac{1}{\mathbf{x}' - \mathbf{x}} d\mathbf{x}' \mathbf{E}(\mathbf{x}'), \quad (62)$$

where \mathbf{i} is the unit bivector for the plane. In the absence of sources, the first integral on the right vanishes and the field within \mathcal{R} is given entirely by a line integral of its value over the boundary. The resulting equation is precisely equivalent to the celebrated *Cauchy integral formula*, as is easily shown by changing to the complex variable $z = \mathbf{x}\mathbf{a}$, where \mathbf{a} is a fixed unit vector in the plane that defines the "real axis" for z . Thus GA automatically incorporates the full power of complex variable theory into electromagnetic theory. Indeed, formula (62) generalizes the Cauchy integral to include sources and the generalization can be extended to 3D with arbitrary sources.^{18,19} But this is not the place to discuss such matters.

An electromagnetic field $F = F(\mathbf{x}, t)$ with charge density $\rho = \rho(\mathbf{x}, t)$ and charge current $\mathbf{J} = J(\mathbf{x}, t)$ as sources is determined by *Maxwell's Equation*

$$\left(\frac{1}{c}\partial_t + \nabla\right)F = \rho - \frac{1}{c}\mathbf{J}. \quad (63)$$

To show that this is equivalent to the standard set of four equations, we employ (58), (59) and (30) to separate, respectively, its scalar, vector, bivector, and pseudoscalar parts:

$$\nabla \cdot \mathbf{E} = \rho, \quad (64)$$

$$\frac{1}{c}\partial_t\mathbf{E} - \nabla \times \mathbf{B} = -\frac{1}{c}\mathbf{J}, \quad (65)$$

$$i\frac{1}{c}\partial_t\mathbf{B} + \nabla \times \mathbf{E} = 0, \quad (66)$$

$$i\nabla \cdot \mathbf{B} = 0 \quad (67)$$

Here we see the standard set of Maxwell's equations as four geometrically distinct parts of one equation. Note that this separation into several parts is similar to separately equating real and imaginary parts in an equation for complex variables.

Of course, it is preferable to solve and analyze Maxwell's Equation without decomposing it into parts. This is especially true for propagating wave solutions. For example, monochromatic plane waves in a vacuum have solutions with the familiar form

$$F(\mathbf{x}, t) = fe^{\pm i(\omega t - \mathbf{k} \cdot \mathbf{x})}. \quad (68)$$

What may not be so familiar to physicists is that

$$F^2 = f^2 = 0 \quad (69)$$

is required here, the i is the unit pseudoscalar and the *two signs in (68) represent the two states of circular polarization*. A more detailed GA treatment of plane waves was published in this journal three decades ago.²⁰

VII. Real Quantum Mechanics

Schroedinger's version of quantum mechanics requires that the state of an electron be represented by a complex wave function $\psi = \psi(\mathbf{x}, t)$, and Born added that the real bilinear function

$$\rho = \psi\psi^\dagger \quad (70)$$

should be interpreted as a probability density for finding the electron at point \mathbf{x} at time t . This mysterious relation between probability and a complex wave

function has stimulated a veritable orgy of philosophical speculation about the nature of matter and our knowledge of it. Curiously, virtually all philosophizing about the interpretation of quantum mechanics has been based on Schroedinger theory, despite the fact that electrons, like all other fermions, are known to have intrinsic spin. We shall see that that is a serious mistake, for *it is only in a theory with electron spin that one can see why the wave function is complex*. You may wonder why this fact is not common knowledge.

The reason is that the geometric meaning of the wave function lies buried in the standard matrix version of the Pauli theory. We shall exhume it by translating the matrix wave function Ψ into a *real spinor* ψ in GA where, as we have seen, every $\sqrt{-1}$ has a geometric meaning. We discover then that the $\sqrt{-1}$ in Schroedinger theory emerges in the real Pauli version as a bivector that is related to spin in an essential way. In other words, we see that geometry dictates that *spin is not a mere add-on in quantum mechanics*, but an essential feature of fermion wave functions.

By reformulating quantum mechanics in terms of real spinors, we establish GA as a common mathematical language for quantum mechanics and classical mechanics as formulated in preceding Sections. This simplifies and clarifies the relation of classical theory to quantum theory. In particular, we find that the spinor wave function operates as a rotor in essentially the same way as rotors in classical mechanics. This suggests that *the bilinear dependence of observables on the wave function is not unique to quantum mechanics — it is equally natural in classical mechanics for geometrical reasons*.

Though the relation of spin to the unit imaginary was first discovered in the Dirac theory,^{21,22} it is easiest to see in the Pauli theory. The resulting *real* spinor wave equation leads to the surprising conclusion that *spin was inadvertently incorporated into the original Schroedinger equation* in the guise of the distinctive factor $\sqrt{-1}\hbar$. Extension to the Dirac theory will be covered in a sequel to this paper.

A. Vectors vs. Matrices

No doubt some of you noticed in Section VB the similarity of the basis vectors σ_i to the *Pauli spin matrices*. Indeed, the *Pauli algebra* is a matrix representation of the geometric algebra \mathcal{G}_3 . I have co-opted the standard symbols σ_i for Pauli matrices to make the correspondence as obvious as possible. To emphasize that correspondence, I use the same symbols σ_i for the basis vectors and the Pauli matrices that represent them.

My purpose is to lay bare some serious misconceptions that complicate quantum mechanics and obscure its relation to classical mechanics. The most basic of these misconceptions is that the Pauli matrices are intrinsically related to spin. On the contrary, I claim that their physical significance is derived solely from their correspondence with orthogonal directions in space. The representation of σ_i by 2×2 matrices is irrelevant to physics. That being so, it should be possible to eliminate matrices altogether and make the geometric structure of quantum mechanics explicit through direct formulation in terms of GA. How to do that

is explained below. For the moment, we note the potential for this change in perspective to bring classical mechanics and quantum mechanics closer together.

Texts on quantum mechanics describe the three σ_i as a vector $\boldsymbol{\sigma}$ with matrices for components. They combine $\boldsymbol{\sigma}$ with an ordinary vector \mathbf{a} with scalar components a_i by writing

$$\boldsymbol{\sigma} \cdot \mathbf{a} = a_i \sigma_i. \quad (71)$$

(sum over repeated indices). Then they derive the identity

$$\boldsymbol{\sigma} \cdot \mathbf{a} \boldsymbol{\sigma} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b} I + i \boldsymbol{\sigma} \cdot (\mathbf{a} \times \mathbf{b}). \quad (72)$$

where I is the identity matrix. This formula relating vector algebra to Pauli algebra is a prime example of wasteful redundancy in standard physics, for we know that the two algebras have common geometric content. GA eliminates this redundancy entirely. Indeed, (72) is the matrix representation of the GA product formula (29), where it is seen not as a relation between different algebras but between different geometric products.

B. Pauli's Matrix Theory

To set the stage for deriving the GA version of nonrelativistic quantum mechanics, a brief review of Pauli's matrix theory is in order. To describe an electron with spin, Pauli generalized Schroedinger's complex wave function to a column matrix Ψ with two complex components representing "spin up" and "spin down" states. The two spin states with Ψ_{\pm} are eigenstates of a hermitian spin operator

$$\sigma_3 \Psi_{\pm} = \pm \Psi_{\pm}. \quad (73)$$

A complete set of hermitian spin operators is given by the *Pauli matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i' \\ i' & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (74)$$

where $i' = \sqrt{-1}$ is a scalar imaginary with no geometric significance, so (as we shall see) no physical significance!

To describe the interaction of electron spin with an external magnetic field \mathbf{B} , Pauli added an interaction term to Schroedinger's equation for his two component wave function, with the result

$$i' \hbar \partial_t \Psi = \underline{H}_S \Psi - \frac{e \hbar}{2mc} \boldsymbol{\sigma} \cdot \mathbf{B} \Psi \quad (75)$$

where \underline{H}_S is the Schroedinger hamiltonian and $\boldsymbol{\sigma} \cdot \mathbf{B}$ is as defined in (71). Pauli inserted the coefficient of the interaction term "by hand," assuming a gyromagnetic ratio $g = 2$ to agree with experiment. When Dirac derived it shortly thereafter, the result was regarded as confirmation of the Dirac equation and a consequence of relativity theory. However, it was realized much later that the

coefficient can be derived by mimicking Dirac’s argument in the nonrelativistic domain, so spin is not a “relativistic phenomenon” after all. The trick is to define a momentum operator

$$\boldsymbol{\sigma} \cdot \underline{\mathbf{p}} = \boldsymbol{\sigma} \cdot (-i\hbar\nabla - \frac{e}{c}\mathbf{A}) \quad (76)$$

and assume a hamiltonian of the form

$$\underline{H}_P = \frac{1}{2m}(\boldsymbol{\sigma} \cdot \underline{\mathbf{p}})^2 + V(\mathbf{x}). \quad (77)$$

When this is expanded with the help of the identity (72), it gives the result in (75). I do not know who originated this argument, but I learned it from Feynman, who gave me the impression that he had devised it himself. As we see below, the definition (76) of the momentum operator is less mysterious in GA, where it is apparent that the appearance of $\boldsymbol{\sigma}$ in it has nothing to do with spin. Indeed, the “trick” (76) is justified by GA, where $\boldsymbol{\sigma}_i$ appear naturally as basis for the vector \mathbf{A} and the vector derivative ∇ .

C. Real Pauli-Schroedinger Theory

This section proves that the Pauli wave function can be represented as a *real spinor* (or quaternion) satisfying a real wave equation in the geometric algebra $\mathcal{G}_3 = \mathcal{G}(\mathcal{P}^3)$ introduced in Section VI for classical physics. The resulting real version of Pauli-Schroedinger electron theory is isomorphic to the standard matrix version, but it has a more transparent geometric structure that elucidates and simplifies physical interpretation and relations to classical mechanics. The objective here is to develop the “real theory” to the point where it is ready to be used in applications with no further appeal to the matrix theory, except to make contact with the literature.

We can extract the “real” version of the Pauli theory from the matrix version in the following way.²³ Consistent with the representation of Pauli matrices (74), let u be a basis spinor for the “spin-up” eigenstate of $\boldsymbol{\sigma}_3$, so we can write

$$\boldsymbol{\sigma}_3 u = u, \quad \text{where} \quad u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (78)$$

Then(74) gives us

$$\boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 u = i' u. \quad (79)$$

Now write Ψ in the form

$$\Psi = \psi u, \quad (80)$$

where ψ is a polynomial in the Pauli matrices. The coefficients in this polynomial can be taken as real, for if there is a term with an imaginary coefficient, we can use (79) to replace i' by multiplication with $\boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2$ on the right side of the term.

Furthermore, the polynomial can be taken as an even multivector, for if any term is odd, then (78) allows us to make it even by multiplying it on the right by σ_3 . Therefore, we can assume with complete generality that ψ in (80) is a *real even multivector*. Now we can reinterpret the σ_k in ψ as vectors in GA instead of matrices. Thus, we have established a one-to-one correspondence between Pauli spinors Ψ and even multivectors ψ in GA.

Since the ungeometrical imaginary i' has been thereby eliminated, I refer to ψ as a *real spinor* (or quaternion). Note that ψ has four degrees of freedom, one for its scalar part and three for its bivector part, as does Ψ in its two complex matrix components. The big gain in replacing Ψ by ψ is that the latter has a more transparent geometric interpretation, as shown below.

Now, it is easy to extract a *real wave equation* for the *real spinor wave function* from the matrix equation (75). Thus we arrive at the *real Pauli-Schroedinger* (PS) *equation*

$$\partial_t \psi i\sigma_3 \hbar = \underline{H}_S \psi - \frac{e\hbar}{2mc} \mathbf{B} \psi \sigma_3, \quad (81)$$

where i is now the unit pseudoscalar so $i\sigma_3 = \sigma_1\sigma_2$, and $\sigma \cdot \mathbf{B}$ is replaced by \mathbf{B} for reasons explained above. I have inserted Schroedinger's name here to emphasize the fact (explained below) that in this real theory the Schroedinger wave function is the same real spinor, and (81) reduces to the Schroedinger equation by dropping the last term. In the same way, equations (76) and (77) can be re-expressed in the real theory, but we will not have need of them below.

The equivalence of (81) to the matrix equation (75) is easily proved by interpreting (81) as a matrix equation, multiplying on the right by the basis spinor u and using (78), (79) and (80). Note that the σ_3 on the right side of (81) is essential to make the last term have even parity, for the vector \mathbf{B} is odd while ψ is an even multivector. Thus all terms in (81) are even.

It should be noted that the condition imposed in the Pauli theory by assuming that the Pauli spin matrices are hermitian is equivalent to the definition (34) of reversion in GA. Thus we see that the standard association of "spin observables" with hermitian operators amounts to declaring that they represent vectors. Evidently, this is an assumption about geometry rather than spin — a critical geometric assumption that has been inadvertently incorporated into quantum mechanics. Spin must get into the theory some other way. This raises serious questions about the physical significance of the common association of physical observables with hermitian operators.

The explicit σ_3 in equation (81) may make it look less general than (75), but that is an illusion, because σ_3 is an arbitrarily chosen constant vector. It can be related to any other choice σ'_3 by a constant rotation

$$\sigma'_3 = C \sigma_3 C^\dagger, \quad (82)$$

Thus, multiplication of (81) on the right by C^\dagger gives

$$(\psi \sigma_3) C^\dagger = \psi C^\dagger C \sigma_3 C^\dagger = \psi' \sigma'_3, \quad (83)$$

where $\psi' = \psi C^\dagger$ is the wave function relative to the alternative quantization axis σ'_3 . The matrix analog of this transformation is a change in matrix representation for the column spinor Ψ .

D. Interpretation of the Real Wave Function.

Having established equivalence to the standard matrix theory, from now on we can work with the real theory alone. Our main objective is to establish the physical meaning of the real wave function $\psi = \psi(\mathbf{x}, t)$. Adopting the Born probability assumption (70), we can write

$$\psi = \rho^{\frac{1}{2}} U, \quad \text{where} \quad UU^\dagger = 1. \quad (84)$$

This determines a frame of bilinear “local observables”

$$\psi \sigma_k \psi^\dagger = \rho \mathbf{e}_k, \quad (85)$$

where $\{\sigma_k\}$ is a standard frame and, of course,

$$\mathbf{e}_k = U \sigma_k U^\dagger. \quad (86)$$

These observables are invariant under the change in choice of standard frame $\psi \sigma_k \psi^\dagger = \psi' \sigma'_k \psi'^\dagger$ defined by (82). Thus, the wave function determines an invariant time-dependent field of frames $\{\mathbf{e}_k(\mathbf{x}, t)\}$ attached to each position \mathbf{x} . We show below that

$$\mathbf{s} = \frac{1}{2} \hbar \mathbf{e}_3 = \frac{1}{2} \hbar U \sigma_3 U^\dagger \quad (87)$$

can be interpreted as a *spin vector*, so $\rho \mathbf{s}$ is a *spin density*. Actually, angular momentum is a bivector quantity, so it is more correct to represent spin by the bivector

$$\mathbf{S} = i \mathbf{s} = \frac{\hbar}{2} U \mathbf{i} U^\dagger = \frac{\hbar}{2} \mathbf{e}_1 \mathbf{e}_2, \quad (88)$$

where

$$\mathbf{i} \equiv i \sigma_3 = \sigma_1 \sigma_2 \quad (89)$$

plays the role of $\sqrt{-1}$ in the real PS equation (81).

The hidden relation of spin to the imaginary i' in the matrix theory can be made more manifest by regarding $\mathbf{S} = S_{ij} \sigma_j \sigma_i$ as a spin operator on the wave function. Multiplying (88) on the right by (84) we get

$$\mathbf{S} \psi = \psi \mathbf{i} \frac{\hbar}{2}. \quad (90)$$

Regarding this as a matrix equation, we use (80) and (79) to get

$$\mathbf{S} \Psi = \frac{1}{2} i' \hbar \Psi. \quad (91)$$

Thus, $\frac{1}{2}i\hbar$ is the eigenvalue of the “spin operator” \mathbf{S} . Otherwise said, the factor $i\hbar$ in the Pauli matrix equation (75) is a representation of the spin bivector by its eigenvalue. The eigenvalue is imaginary because the spin tensor $S_{ij} = -S_{ji}$ is skewsymmetric. We can conclude, therefore, that *spin was originally introduced into quantum mechanics with the factor $i\hbar$ in the original Schroedinger equation.*

Spin components $s_k = \mathbf{s} \cdot \boldsymbol{\sigma}_k = \langle \mathbf{s} \boldsymbol{\sigma}_k \rangle$ are related to conventional matrix elements by

$$\rho s_k = \frac{\hbar}{2} \langle \boldsymbol{\sigma}_k \psi \boldsymbol{\sigma}_3 \psi^\dagger \rangle = \frac{\hbar}{2} \text{Tr}(\boldsymbol{\sigma}_k \Psi \Psi^\dagger) = \frac{\hbar}{2} \Psi^\dagger \boldsymbol{\sigma}_k \Psi, \quad (92)$$

where the scalar part $\langle M \rangle$ of a real multivector M corresponds to the Trace $\text{Tr}(M)$ of its matrix representation.

In conventional Pauli theory,³¹ the $\boldsymbol{\sigma}_k$ are regarded as “spin operators” with eigenvalues corresponding to results of spin measurements in orthogonal directions. However, (92) shows that in Real PS Theory the $\boldsymbol{\sigma}_k$ are operators only in the trivial sense of basis vectors that pick out components of the spin vector determined by the wave function. Conventional theory also interprets the non-vanishing commutator $[\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2] = \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_1$ as a measure of incompatibility in spin measurements. Whereas, according to (11) and (25), the real theory interprets

$$[\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2] = 2\boldsymbol{\sigma}_1 \wedge \boldsymbol{\sigma}_2 = 2i\boldsymbol{\sigma}_3 \quad (93)$$

as a geometric product with no particular relation to spin. This raises serious questions about the conventional view of spin in quantum mechanics. More questions are raised by further study of the real theory.²⁴

The role of spin in the PS equation (81) can be made explicit by using (87) to write the last term in the form

$$-\frac{e\hbar}{2mc} \mathbf{B} \psi \boldsymbol{\sigma}_3 = \frac{1}{2} \left(\frac{e}{mc} i \mathbf{B} \right) \psi i \hbar = -\frac{e}{mc} \mathbf{B} \mathbf{s} \psi. \quad (94)$$

Note that

$$\mathbf{B} \mathbf{s} = \mathbf{B} \cdot \mathbf{s} + i(\mathbf{B} \times \mathbf{s}) \quad (95)$$

splits into terms proportional to magnetic energy and torque.

A physical interpretation of the PS equation is supplied by Schroedinger’s assumption that the energy E of a stationary state is given by the eigenvalue equation

$$\partial_t \psi i \hbar = E \psi. \quad (96)$$

Pauli’s additional term changes this to

$$E = E_S - \frac{e}{mc} \mathbf{B} \cdot \mathbf{s}, \quad (97)$$

where E_S is the Schroedinger energy. For a stationary solution with $\mathbf{B} \times \mathbf{s} = 0$, \mathbf{s} must be parallel or antiparallel to \mathbf{B} and we have

$$\mathbf{B} \cdot \mathbf{s} = \pm \frac{\hbar}{2} |\mathbf{B}|. \quad (98)$$

This is the basis for declaring that spin is “two-valued.” However, when \mathbf{B} is variable the vectorial nature of \mathbf{s} becomes apparent.

We can generalize (96) to arbitrary states by interpreting

$$\rho E = \langle \partial_t \psi | i\sigma_3 \hbar \psi^\dagger \rangle. \quad (99)$$

as *energy density*. Here the energy $E = E(\mathbf{x}, t)$ can be a variable function, and we see that an energy eigenstate is defined by the assumption that E is uniform. Inserting (84) into (99), we discover that the density ρ drops out to give us

$$E = \langle \partial_t U | i\sigma_3 \hbar U^\dagger \rangle. \quad (100)$$

To interpret this expression we need to relate it to observables, in particular, the spin.

As we have seen before, the unimodularity condition on U implies that its derivative can be written in the form

$$\partial_t U = -\frac{1}{2} i\boldsymbol{\omega} U, \quad (101)$$

so that (86) gives us

$$\partial_t \mathbf{e}_k = \boldsymbol{\omega} \times \mathbf{e}_k, \quad (102)$$

in exact correspondence with the classical equations (55) and (57). In particular, (102) gives us the kinematic equation for spin precession

$$\partial_t \mathbf{s} = \boldsymbol{\omega} \times \mathbf{s}. \quad (103)$$

Inserting (101) into (100) and using (87) we get

$$E = \boldsymbol{\omega} \cdot \mathbf{s} = \frac{1}{2} \boldsymbol{\omega} \cdot 2\mathbf{s}. \quad (104)$$

This is identical to the classical expression for the rotational kinetic energy of a rigid body with angular momentum $2\mathbf{s}$.¹⁶ All this suggests that the rotor U describes continuous kinematics of electron motion rather than a probabilistic combination of spin-up and spin-down states as asserted in conventional Pauli theory.

The most surprising thing about the energy expression (104) is that it applies to any solution of the Schroedinger equation, where $\boldsymbol{\omega} \times \mathbf{s} = 0$. But, according to (102), \mathbf{e}_1 and \mathbf{e}_2 are spinning about the spin axis with angular velocity $\boldsymbol{\omega}$, and (104) *associates energy with the rotation rate*. The big question is, “What is the physical meaning of this spinning?” I have discussed one intriguing answer

at some length before,²⁴ and I will return to it in a subsequent paper within the context of the Dirac theory.

When the magnetic field $\mathbf{B} = \mathbf{B}(t)$ is a function of time alone, we can define a rotor D by assuming that it satisfies the equation

$$\frac{dD}{dt} = \frac{1}{2} \left(\frac{e}{mc} i\mathbf{B} \right) D. \quad (105)$$

and factoring the wave function into

$$\psi = \psi(\mathbf{x}, t) = D(t)\psi_S(\mathbf{x}, t). \quad (106)$$

Substituting this into the PS-equation (81) separates the factors so that D satisfies (105) and ψ_S satisfies Schroedinger's equation

$$\partial_t \psi_S i\hbar = \underline{H}_S \psi_S. \quad (107)$$

The rotor equation (105) exhibits a magnetic torque $(e/m)\mathbf{B}$ in perfect agreement with the classical model of magnetic resonance discussed in my book.¹⁶ This exact analogy with classical physics is a great help in interpreting magnetic resonance experiments, and it raises more questions about the interpretation of electron spin.

Note that the factor $\frac{1}{2}$ on the right side of (105) is the same factor that, following Pauli, was attributed to spin in (87) and (98). However, (105) and (101) suggest that the $\frac{1}{2}$ is more correctly associated with the rotational velocity in a rotor equation.

Although our real version of the Pauli theory gives new insight into the geometric properties of the wave function, spin and interaction with the magnetic field, it is mathematically isomorphic with the standard matrix version, so no new physical consequences are to be expected, and it is straightforward to translate results from one version to the other. More details about the real PS theory have been published in this journal before.²⁴

Standard techniques for analyzing angular momentum and solving the Pauli equation³¹ can be converted to GA,²⁵ where they yield some surprises and simplifications that deserve further study.

Without delving into the complexities of angular momentum analysis, it is obvious that the standard Schroedinger wave function is a solution the Schroedinger equation (107), but with the bivector $\mathbf{i} = i\sigma_3$ as unit imaginary, so there is no way to eliminate spin from the theory without eliminating complex numbers. It must be concluded, therefore, that standard Schroedinger theory does not describe electrons without spin, but rather electrons with constant spin (or, equivalently, electrons in a spin eigenstate).

The difference between Pauli and Schroedinger solutions of (107) is in the class of eigenfunctions allowed. The simple Schroedinger hamiltonian \underline{H}_S does not discriminate between them. A distinction is forced when the hamiltonian is generalized to include spin-orbit interactions,³¹ but this is not the place to go into details.

This discussion has been limited to single particle Pauli theory. But in closing, it should be mentioned that GA methods for spin representations of many particle systems is currently an area of active research,^{32,38} where GA gives fresh insights into entangled states, quantum computing and the like.

To summarize, let me highlight the most provocative conclusions from this section:

- The explicit $\sqrt{-1}$ in fermion wave functions represents a bivector specifying the direction of spin.
- $i\hbar = i\sigma_3\hbar$ represents spin in Schroedinger's equation. This implies that spin is not a simple add-on in quantum mechanics but an essential ingredient of the theory. That is likely to be true for all fermions and bosons that are composites of fermions.
- Pauli matrices represent vectors, not spin operators in quantum mechanics.
- Bilinear observables are geometric consequences of rotational kinematics, so they are as natural in classical mechanics as in quantum mechanics.
- The real spinor wave function is easier to interpret and solve than the matrix version.

VIII. Prospects for Curriculum Reform

Our physics curriculum has been largely shaped by the efforts of “lone wolf” textbook writers, and we owe them a debt of gratitude for their fine contributions. Nevertheless, the curriculum would be well served by scholarly critique and analysis to identify weaknesses and promote constructive innovation. I submit that this is worthy activity for the PER community. A community is needed to incubate innovations and promote them when they are ready for wide adoption. The prevailing *laissez-faire* process of textbook adoption has suppressed many fine contributions. It also fails to address problems of integration and coherence of the whole curriculum, as textbook reform tends to be confined to a single course. As a case in point, I submit the problem of integrating geometric algebra into the physics curriculum.

Assuming, for the sake of argument, that there is strong support for GA in the physics community, what will it take to integrate it into the curriculum? History tells us that it will take a long time if left to normal evolutionary processes. In a perceptive essay on “fashionable pursuits” in science,²⁸ Freeman Dyson argues that innovations in mathematical physics are almost always unfashionable when they are introduced, and he estimates that it takes fifty to a hundred years for them to achieve general recognition. It is clear that new mechanisms for curriculum reform will be needed to accelerate adoption of GA, or any other substantial reform for that matter. As an indicator of difficulties involved and prospects for success, I offer a brief account of my own experience with GA and discuss some hurdles to broad adoption.

A. History and reception of geometric algebra

Geometric algebra originated in the work of Hermann Grassmann (1809–1877) and William Kingdon Clifford (1845–1879). Clifford provided the axiomatic formulation for the geometric product in Section VA, so the system is called *Clifford algebra* in most of the mathematics literature. However, Clifford himself called it *geometric algebra*, and he acknowledged the primary contribution of Grassmann. He recognized its unique significance as an algebraic representation of geometric concepts as conceived by Grassmann. Unfortunately, his deep insight into the geometric foundations of algebra did not survive his early death. Mathematicians abstracted Clifford algebra from its geometric origins, and, for the better part of a century, it languished as a minor subdiscipline of mathematics — one more algebra among so many others.

Grassmann’s program to develop a universal geometric algebra for mathematics and physics was revived in 1966 with my book *Space Time Algebra*.²¹ Actually, it was many years later before I realized how much of my own research program for developing a universal mathematical language had been anticipated by Grassmann.¹³ Of course, I had an additional century of mathematics and physics to draw on, so geometric algebra today is vastly more developed with a scope that was inconceivable in Grassmann’s day.

My 1966 book introduced the critical innovations that made the unified language in this paper possible. It developed a coordinate-free formulation of spacetime physics that will be discussed in the sequel to this paper. After employing it exclusively in my research and teaching over the next few years, my own perspective on GA in physics was consolidated around the essential form and content in this lecture. I found GA so empowering that ever since I have not been satisfied with my understanding of any subject in physics until I have formulated it in terms of GA. Invariably I have found that GA simplifies and illuminates every subject.

My 1966 book had a serious limitation. Though it simplified the fundamental equations of physics, applications required access to the great repertoire of available mathematical methods, and translation into standard formulations often destroyed the advantages of geometric algebra. This motivated me to extend geometric algebra to a unified language for mathematics that incorporated all the methods needed in physics. Besides, I must confess to an abiding interest in pure mathematics for its own sake. I consider myself equal parts mathematician and physicist. This ambitious program to redesign mathematics was greatly stimulated by an amazing early success: the realization that GA enables a natural generalization and extension to arbitrary dimension of the central result of complex variable theory, Cauchy’s integral formula.¹⁸ This result has been independently discovered by several others, and mathematicians have developed it in recent years into a vigorous new branch of mathematics called *Clifford analysis*. Unfortunately, most practitioners in this field pay insufficient attention to the geometric meaning of their results that is needed for easy application to physics. Consequently, the evolution of geometric algebra into a comprehensive *Geometric Calculus* to serve as a unified language for both math-

ematics and physics has so far proceeded independently of Clifford analysis.²⁹ The development of Geometric Calculus culminated in the 1986 publication of a comprehensive mathematics monograph¹⁹ and has continued to this day. To my knowledge, no other mathematical system integrates such a broad range of mathematics, just about all the mathematics needed for physics.

For the first 20 years of my professional life my research on developing and applying geometric algebra proceeded in almost complete isolation, except for interaction with a few students. During that period I published regularly, but there were only a handful of citations to my papers and no positive feedback from colleagues. I learned to be very careful about when, where and how I presented my work to other physicists, because the reaction was invariably dismissive as soon as they detected deviation from standard practice or beliefs.²⁷

I did not even try to publish in *Physical Review*. Instead, I conducted a sociological experiment by submitting proposals to the NSF for support of my work on relativistic electron theory. After the first couple of tries I was convinced that it would never get funded, but I continued to submit 12 times in all as an experiment on how the system works. I found that there was always a split opinion on my proposal that typically fell into three groups. About one third dismissed me outright as a crank. About one third was intrigued and sometimes gave my proposal an Excellent rating. The other third was noncommittal, mainly because they were not sure they understood what I was talking about. The fate of the proposal was always decided by averaging the scores from the reviewers, despite the fact that the justifications for different scores were blatantly contradictory. Such is the logic of the funding process. I thought that my last proposal was particularly strong. But one reviewer torpedoed it with an exceptionally long and impressively documented negative opinion that overwhelmed the positive reviews. He was factually and logically wrong, however, and I could prove it. So I decided to appeal the decision. I wanted to challenge the NSF to resolve serious contradictions among reviews before making a funding decision. I learned that the first two stages of the appeal process were entirely procedural without considering merits of the case. In the third stage, the appeal had to be submitted through the Office of Research at my university. The functionaries in the Office worried that the appeal would irritate the NSF, so it was delayed until the deadline passed. Such is the politics of grant proposals.

The prospects for dissemination of GA were suddenly changed about 1986 by two new developments. The first was an “International Conference on Clifford Algebras and their Applications in Mathematical Physics” conceived and organized by Roy Chisholm.³⁰ I was invited as keynote speaker on the strength of my STA book. There I learned that my book had impressed a lot of people, though they did not see how to use it in their own work. I was fortunate to have it published in a series with many distinguished authors that was affordably priced and widely marketed. Chisholm’s conference precipitated the formation of an international community of researchers in mathematics and physics, and the sixth in the series of international conferences that he started was held in May 2002.

The second new development to enhance dissemination of GA was publication of my mechanics and mathematics books.^{16,19} Early in my career, I naively thought that if you give a good idea to competent mathematicians or physicists, they will work out its implications for themselves. I have learned since that most of them need the implications spelled out in utter detail. For example, I published the ideas in Section V in a 1971 *AJP* article²⁰ with the expectation that others would find them as compelling as I do. But there was no noticeable reaction until the implications for mechanics were fully worked out in my book.¹⁶ To be fair, the implications turned out to be a lot richer than I had anticipated. I learned that only when I taught the graduate mechanics course where I was forced to work out the details.

I measure the impact of my work, not by words of praise or approval, but by evidence of influence in the published work of others, not including papers where GA has been incompetently used. I regard the impact as strong when GA is employed in essentially the way that I would do it, and exceptional when GA is used in a way that surprises me. By those criteria, my work had little impact before the late 1980s, but its impact has accelerated during the 1990s as more and more able researchers adopt GA as their mathematical language of choice. During the last decade GA has started spreading to engineering and computer science with applications to robotics, computer vision, space flight navigation and control, collision detection, optical design and much more. In the last year alone, several volumes of applications have been published.³³⁻³⁵ This lively use of GA in the research domain has convinced me at last that it is time to press for GA in the physics curriculum.

B. GA in the physics curriculum.

At most universities it is easy to introduce GA into graduate courses, given the autonomy for structuring the course that professors enjoy. I was able to negotiate an agreement allowing me to do that immediately when I was hired in 1966. Since that time I have employed GA exclusively in graduate courses in mechanics, E&M and relativity over more than twenty years. I adopted the textbooks Goldstein for mechanics and Jackson for E&M and assigned most of the problems in them, because they set the standard for problems on the graduate comprehensive exams prepared by my colleagues. In my lectures, I often used GA as a framework for critique of the textbooks, and I emphasized how smoothly GA articulates with standard methods while demonstrating its advantages.

The response of the graduate students was the usual mixed bag. About a third were very excited about it; another third was opposed to learning anything that is not required for the comprehensive exam; the remaining third just went through the motions. Unbeknownst to my colleagues or the students, I tracked the exam performance of every student who went through my courses. Though we have a high failure rate on our comprehensive exams, no student who took my courses on mechanics or E&M failed that portion of the exam. At least my teaching with GA was not impeding their careers.

Many of the theoretically inclined students in my courses wanted to pursue doctoral research with me. Unfortunately, it takes something like three years to develop sufficient proficiency with GA to use it in research (unless one has an exceptionally strong background in mathematical physics). Since GA was not used in any courses besides mine, the students had to get the rest by independent study. This meant an exceptionally long induction time before starting research. Besides, I was acutely aware of the reception they were likely to get for a GA based dissertation. Consequently, I took on only a few of the most dedicated students. Then when I took my turn teaching introductory physics the supply of doctoral students dried up completely.

I have not ventured to teach GA in undergraduate courses. However, one young professor at a prestigious liberal arts college adopted my mechanics book for an upper division mechanics course and published two pedagogical papers on GA in the *AJP*³⁶ that editor Bob Romer put on the list of his favorite papers in 1993. His reward was denial of tenure. This is not an unfamiliar fate for untended faculty who attempt teaching innovation of any kind.

There is really no point in introducing GA into the core undergraduate curriculum unless it is employed in a sufficient number of courses to develop proficiency in the language. That requires collaboration of several faculty members, at least, and the blessing of the rest of the physics department — a near impossibility in most departments. However, we have already noted two good reasons why departments need not fear that introducing GA in the undergraduate curriculum will impede students in subsequent graduate studies where GA is not used. First, after a working proficiency with GA has been acquired, it is easy to translate to other formalisms. Second, such translation is a powerful stimulant to understanding.

The usual mechanism for curriculum reform through textbooks is as slow as it is perilous for the author. Any GA-based textbook is unlikely to be taken on by a major textbook publisher, no matter how superb it might be, because the prevailing demand is near zero. For that reason, I was forced to publish my mechanics textbook in a series for advanced monographs in physics. Since few such monographs are purchased by anyone but libraries, prices are necessarily high for the publisher to break even on most books, too high for the competitive textbook market. Moreover, the advertising does not reach the textbook market. Though my book has been a continual best seller in the series for well over a decade, it is still unknown to most teachers of mechanics in the U.S. To be suitable for the series, I had to design it as a multipurpose book, including a general introduction to GA and material of interest to researchers, as well as problem sets for students. It is not what I would have written to be a mechanics textbook alone. Most students need judicious guidance by the instructor to get through it.

Despite the perils, there are other authors writing GA based textbooks.^{37, 38} But history tells us that incremental change is the most we can expect from textbooks, while revolutionary change will be needed to give GA a powerful presence in the curriculum within, say, ten years. That will require a new mechanism for curriculum change. I suggest that PER can play that role with a

strong program of scholarly research and consensus building. I suggest further that GA is only one among many topics that need to be addressed. The physics curriculum has many other deficiencies that should be subjected to scholarly scrutiny.

C. GA in the mathematics curriculum

If GA is to be adopted throughout the physics curriculum, there must be concomitant reform of the mathematics curriculum for physics majors. Even today the math curriculum is not well matched to the needs of physics, so most physics departments have their own course in mathematical methods for physicists. Anyway, theoretical physicists are confident in their ability to teach any math that physicists need, so it will not be necessary to reform the mathematicians, who may be even more recalcitrant to reform than physicists.

My monograph on Geometric Algebra and Calculus as a unified language for mathematics¹⁹ is the most comprehensive reference on the subject, but it is not suitable as a textbook, except perhaps for a graduate course in mathematics. Besides there have been some important developments since it was first published. A GA textbook for physics students is soon to be published.³⁸ Of course, it will take more than one book to define a full curriculum.

As it does for physics, GA provides a framework for critique of the current math curriculum. I mention only courses that are mainstays of mathematical physics. A full critique of these courses requires much more space than we can afford here. By first introducing GA as the basic language, the course in *linear algebra* can be simplified and enriched.³⁹ For example, we have seen how GA facilitates the treatment of rotations and reflections. GA will then supplant matrix algebra as the basic computation system. Of course, matrix algebra is a very powerful and well-developed system, but it is best developed from GA rather than the other way around. Courses on *advanced calculus* and *multivariable calculus* with differential forms and differential geometry are unified and simplified by geometric calculus.²⁶ Likewise, GA unifies courses on *real and complex analysis*. Group theory can also be developed within the GA framework,^{17,40} but much work remains to incorporate the full range of methods and results used by physicists.

X. Challenge

Let me close with a challenge to PER and the physics community to critically examine the following claims supported by the argument in this paper:

- GA provides a unified language for the whole of physics that is conceptually and computationally superior to alternative mathematical systems in every application domain.
- GA can enhance student understanding and accelerate student learning of physics.

- GA is ready to incorporate into the physics curriculum.
- GA provides new insight into the structure and interpretation of quantum mechanics as well as its relation to quantum mechanics.
- Research on the design and use of mathematical tools is equally important for instruction and for theoretical physics.
- Reforming the mathematical language of physics is the single most essential step toward simplifying physics education at all levels from high school to graduate school.

Note. Most of my papers listed in the references are available on line. PER papers can be accessed from <<http://modeling.asu.edu>>. GA papers can be accessed from <<http://modelingnts.la.asu.edu>>. Many fine papers on GA applications in physics and engineering are available at the Cambridge website <<http://www.mrao.cam.ac.uk/~clifford/>>.

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