Cosmological Consequences of a Flat-Space Theory of Gravity*

Anthony Lasenby[a], Chris Doran[b†] and Stephen Gull[a]
[a]MRAO, Cavendish Laboratory, Madingley Road, Cambridge CB3 0HE, UK
[b]DAMTP, Silver Street, Cambridge, CB3 9EW, UK

June 23, 1993

Abstract

In the preceding paper [1] we described some aspects of a new theory of gravity, and found radially-symmetric static solutions to the free-field equations. Here we apply the theory to the universe on the largest scales by investigating the consequences of spatial homogeneity. A guiding principle for our theory is that spacetime itself does not play an active role in physics. This means that spacetime cannot be thought of as expanding, and we show also that material test particles do not expand away from each other. Nevertheless, we do predict the observed galactic redshifts. Furthermore, we find that the only cosmological models compatible with complete spatial homogeneity are those at critical density.

1 Introduction

In the preceding paper [1] (henceforth Paper I) we outlined a theory of gravity based on the requirement of invariance under local, active Poincaré transformations. Gravitational interactions were described by gauge fields in a flat spacetime, and the theory contained only first-order derivatives of these fields. Here, we apply our theory to the universe on the largest scales. The guiding principle is spatial homogeneity, and we find that this requirement is more restrictive in our theory than in general relativity (GR). In particular, a testable consequence of our theory is that the density parameter Ω should be 1; that is, the universe is at critical density. This is quite different from general relativity, which in principle allows for any value of Ω. That we can make this prediction might seem surprising, since it is generally thought that first-order theories of gravity (such as ours) have identical outcomes to their second-order counterparts if the effects of quantum spin are neglected. Here we do indeed neglect the effects of spin, but nevertheless obtain several results not accessible to standard GR.

Some of the philosophy behind our approach was discussed in Paper I, and a fuller presentation, together with the detailed mathematics, will be given in a forthcoming paper [2].

---

*To appear in 'Third International Conference on Clifford Algebras and their Applications in Mathematical Physics', Deinze, 1993, eds. F. Brackx, R. Delanghe and H. Serras.
†Supported by a SERC studentship.

An essential point to stress, however, is that in our approach spacetime is an entirely *passive* participant in physics. That is, we reject the notion of a *dynamical* spacetime, and deal with gravitational effects solely in terms of forces in flat spacetime. We regard this rejection of curved spacetime as representing a very conservative approach to gravity, although some of its consequences may seem unconventional. In this paper we consider the cosmological implications of working within this fixed Minkowski space which has no evolution in time and no variation in space. In particular, since spacetime itself cannot be expanding, the observed galactic redshifts must have a different origin. Perhaps even more surprisingly, we show that the cosmological fluid which models the material content of the universe is not expanding either – its energy density is simply decaying with time!

We start by exploring the consequences of imposing spatial homogeneity on the $\overline{h}$ and Ω functions introduced in Paper I. The only preferred direction imposed is one in time. The resulting theory is then compared with standard Friedmann-Robertson-Walker (FRW) models. We confirm that our model reproduces some basic experimental facts, in particular the observed galactic redshifts. This entails discussing the motion of test particles, both with and without mass. A major prediction of this theory is that the universe is at critical density. We discuss whether this prediction is likely to be testable in the near future. We end by discussing whether any of our conclusions may be altered when multiparticle and spin effects are incorporated.

The conventions, notations and results of Paper I are assumed throughout, and references to equations in Paper I are made in the form (I.n). The standard notations of cosmology are also adopted. These can be found in most textbooks on the subject (see [3, 4, 5] for example).

2 A Simple Homogeneous Model

In Paper I we found a class of radially-symmetric static solutions to the free-field equations. These provide a model of a system in which the only preferred direction is the spatial vector e_r, pointing towards some chosen origin. The analogous problem for cosmological models is to find solutions of the field equations in which the only preferred direction is one in *time* rather than space. Following the example of Paper I, we take a trial form for the $\overline{h}$ function as

$$\overline{h}(a) = f(t) a\cdot e_t e_t + \alpha(t) a\wedge e_t e_t, \tag{1}$$

in which f and α are functions of time only. The form of the Ω function must also be consistent with spatial homogeneity, so that

$$\Omega(a) = A(t) a\wedge e_t. \tag{2}$$

The first of the field equations (I.28) can be written, in the absence of spin, as

$$\mathcal{D}\wedge\overline{h}(a) = 0, \quad \text{for all constant } a. \tag{3}$$

This implies the single equation

$$f\dot{\alpha} + \alpha^2 A = 0. \tag{4}$$

For the second of the field equations (I.29) we must couple to a source – the cosmological fluid. In this case (I.29) is modified, to become

$$G(a) \equiv R\underline{h}(a) - \tfrac{1}{2}a\mathcal{R} = 8\pi G T(a), \tag{5}$$

where $T(a)$ is the stress-energy tensor generated by the matter fields. For a 'dust' model (that is, one with no pressure), $T(a)$ takes the form

$$T(a) = \rho a \cdot e_t e_t, \tag{6}$$

where $\rho = \rho(t)$ is the matter density. Assuming such a model, we find that the remaining field equations give the pair of equations

$$3\alpha^2 A^2 = 8\pi G\rho, \tag{7}$$

$$\alpha^2 A^2 + 2\dot{A}\alpha f = 0. \tag{8}$$

Equations (4) and (8) immediately imply that A^2/α is a constant. If we make the further asumption that α is positive (so that the spatial part of $\overline{h}$ is simply-connected to the identity function) then we can write the solution in the form

$$A = -C\alpha^{1/2}, \tag{9}$$

$$f = C\alpha^{5/2}/\dot{\alpha}, \tag{10}$$

and

$$\alpha = \left(\frac{8\pi G\rho}{3C^2}\right)^{1/3}, \tag{11}$$

where C is an arbitrary constant of integration. From equation (10) we see that we must have $\dot{\alpha} \neq 0$, so that this model universe is necessarily dynamic.

The time history of α is determined completely by the time history of the density ρ – but what determines this density? It seems that we have an under-determined theory. At this stage a parallel analysis in general relativity has usually found that the scale factor R (related to α in our theory by $R = R_0/\alpha$) is proportional to $t^{2/3}$. In fact, the conventional approach has tacitly assumed $f = 1$ from the outset, which would imply that

$$\begin{aligned} \dot{\alpha} &= C\alpha^{5/2} \\ \Rightarrow \alpha &\propto t^{-2/3}. \end{aligned} \tag{12}$$

This assumption is made when the FRW metric is written as

$$ds^2 = \lambda dt^2 - \frac{R^2(t)}{c^2} \times \text{spatial part}, \tag{13}$$

and λ is taken equal to 1. This implies that $f^2 = 1$ in equation (1), which corresponds to the assumption that 'cosmic time' (dt) is the proper time (ds) measured by observers comoving with the Hubble flow of galaxies. (An observer can determine whether they are comoving by measuring a zero dipole component in the cosmic microwave background anisotropy — this provides a preferred time axis e_t which is the one we are using in our theory.)

In our theory, however, we cannot simply assume that $\overline{h}(e_t) = e_t$. Suppose, for example, that our universe is in a phase of 'expansion', so that $\dot{\alpha} < 0$. If we choose $f > 0$ it follows from (10) that the constant C must be negative. If the universe were subsequently to recontract, then f would have to *flip sign* since C is fixed. This argument is only schematic, since we have yet to establish the proper time history of α, but it suggests that time would indeed run backwards during a contracting phase. This 'solves' a long-standing problem (see [6] for

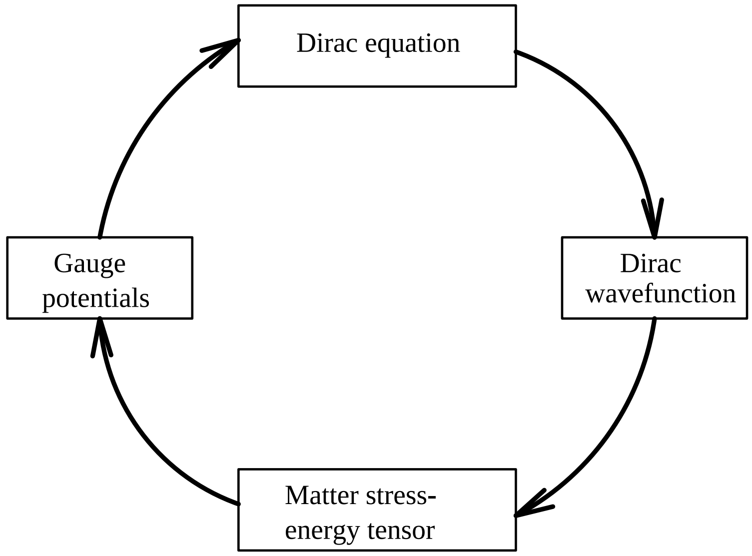

Figure 1: *The self-consistency loop for self-gravitating matter.*

a popular account), a conclusion which is only made possible by keeping track of the sign of f. In a metric theory such as GR this sign would have been lost, since the theory can only access the variable f^2.

A significant problem remains; that is, how to determine the time dependence of the energy density ρ (or equivalently α)? To find this we need to take account of a matter field satisfying its own field equations. This coupling is carried out explicitly in [2], where the matter is taken initially to satisfy the free-particle Dirac equation (expressed in real spacetime algebra form [7]). Gravitational fields are introduced by gauging local Poincaré transformations of the spinor field. In the resulting theory, matter generates gravitational fields (at least, at the level of classical fields) which themselves feed back into the matter equations. In principle, therefore, one would like to find self-consistent solutions of the 'loop' of equations illustrated in Figure 1. This has not yet proved possible, essentially because of the complication of quantum spin. Spin generates 'torsion', which alters the form of the first of the field equations (3).

On the other hand, it is possible to 'close the loop' of Figure 1 if the effects of spin can be ignored. This would be the case if the spin contributions to T average to zero, or if the matter sector consisted of spin-zero sources. Our approach requires that spin-zero sources be built from pairs of spin-1/2 particles, thus the second possibility would involve a multiparticle theory of gravity – a goal towards which we are currently working. Assuming that the spin can be ignored, we find that the Dirac wavefunction is of the form

$$\psi = e^{-mti\sigma_3 - \xi(t)}, \tag{14}$$

and that the Dirac equation fixes f to be $+1$. The form of the stress-energy tensor implies that

$$\rho = me^{-2\xi(t)}, \tag{15}$$

(ignoring the spin contribution) and the remaining content of the Dirac equation reduces to the conservation of the current $mJ = \rho e_t$. In other words, we must have

$$\mathcal{D}\cdot J = 0$$

$$\Rightarrow \nabla\cdot(\overline{h}(J)\det h^{-1}) = 0, \tag{16}$$

and this holds provided that ρ/α^3 is a constant – which is indeed consistent with equation (11). The full solution now takes the form

$$\alpha = \left(\frac{t}{t_0}\right)^{2/3} \tag{17}$$

and

$$\rho = \rho_0 \alpha^3 = \rho_0 \left(\frac{t_0}{t}\right)^2, \tag{18}$$

where $Ct_0 = -2/3$, and $\rho_0 = 3C^2/(8\pi G)$ is the density at the present epoch ($t = t_0$).

The Hubble parameter $H(t)$ is defined by

$$H(t) \equiv \frac{\dot{R}}{R}, \tag{19}$$

where R is the scale factor ($= R_0/\alpha$), and we find that $H(t)$ is equal to $2/(3t)$. The Hubble constant, $H(t_0)$, is therefore given by $2/(3t_0)$, and we now recognise that

$$C = -\text{Hubble constant} = -H_0. \tag{20}$$

The present energy density is therefore

$$\rho_0 = \frac{3H_0^2}{8\pi G}, \tag{21}$$

which is equal to the *critical density*. In conventional terms this is the maximum density that allows continued expansion, and this model universe will continue expanding forever. It is usual to talk in terms of the dimensionless parameter Ω, which is defined as (actual density)/($3{H_0}^2/8\pi G$) [5]. This model therefore predicts an Ω of 1.

In our current approach, however, does it make any sense to talk about expansion? Conventionally, it is spacetime itself that is held to be expanding, but for our theory this is impossible – the flat background Minkowski spacetime is unchanging. One might then suppose that it is the particles within the space that are expanding away from each other. We investigate this by examining the motion of test particles in this universe. Suppose that we launch a particle with

$$v = \cosh u \, e_t + \sinh u \, \hat{n}, \tag{22}$$

where $\hat{n}$ is a unit spatial vector. The geodesic equation $\mathcal{D}_v v = 0$ simplifies to

$$\dot{v} = \sinh u \, \alpha^{3/2} C (\sinh u \, e_t + \cosh u \, \hat{n}), \tag{23}$$

and this implies a *deceleration* in a straight line, since $C < 0$. It can easily be shown that the de Broglie wavelength, λ, satisfies $\lambda \propto t^{2/3}$, and the particle therefore gets *redshifted*. A similar analysis for massless particles reveals the same relationship between wavelength and time, so photons are redshifted in the same way. We note, however, that a test particle released at rest *remains* at rest ($u = 0$, where u is the rapidity), rather than expanding away radially from some fixed point. We are therefore led to some startling conclusions:

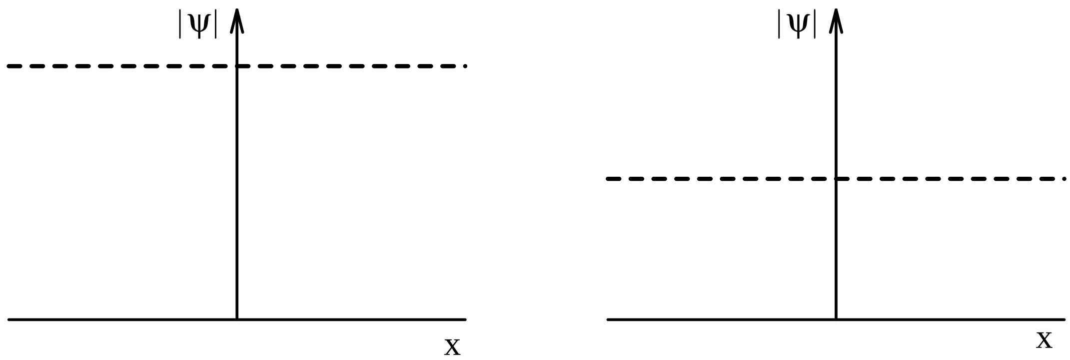

Figure 2: *The amplitude of the wavefunction ψ, plotted against one spatial coordinate. At left, an early time; right, a later time.*

- the universe is *not* expanding;
- the redshift is not Doppler in origin, but is due to a loss of energy to the gravitational field.

However, the universe we are considering is certainly dynamic, so what is it that is changing with time? In fact it is the *energy density* that changes. The amplitude of the wavefunction $|\psi|$ is decreasing with time, as shown in Figure 2, and therefore the stress-energy tensor derived from ψ also decreases in value, reflecting the fact that the matter field is losing energy to the gravitational field.

3 Models with $\Omega \neq 1$

This completes our analysis of the simplest model for the universe, which has turned out to be one at critical density. We now investigate the existence of $\Omega \neq 1$ (non-critical density) models. In order to simulate standard FRW $\Omega \neq 1$ models we have to recast the acceleration bivectors as

$$\begin{array}{ll} \Omega_t = 0 & \Omega_\theta = (be_r + Be_t)e_\theta/r \\ \Omega_r = Ae_re_t & \Omega_\phi = (be_r + Be_t)e_\phi/r, \end{array} \tag{24}$$

where $b = \sqrt{1-Kr^2} - 1$ and K measures the spatial curvature. The form of $\overline{h}$ also changes to include terms with an explicit r-dependence. This is conventionally thought to be unproblematic, since the Einstein tensor remains spatially homogeneous. But a series of worrying points now emerges when one repeats the analysis of Section 2:

1. The 'b' terms imply a preferred spatial direction, in the same manner as the corresponding terms for radially-symmetric solutions [1].
2. We find that the implied Dirac wavefunction ψ must have a spatial variation given by

$$\psi \propto \frac{1}{1+\sqrt{1-Kr^2}}, \tag{25}$$

so that it is no longer homogeneous.

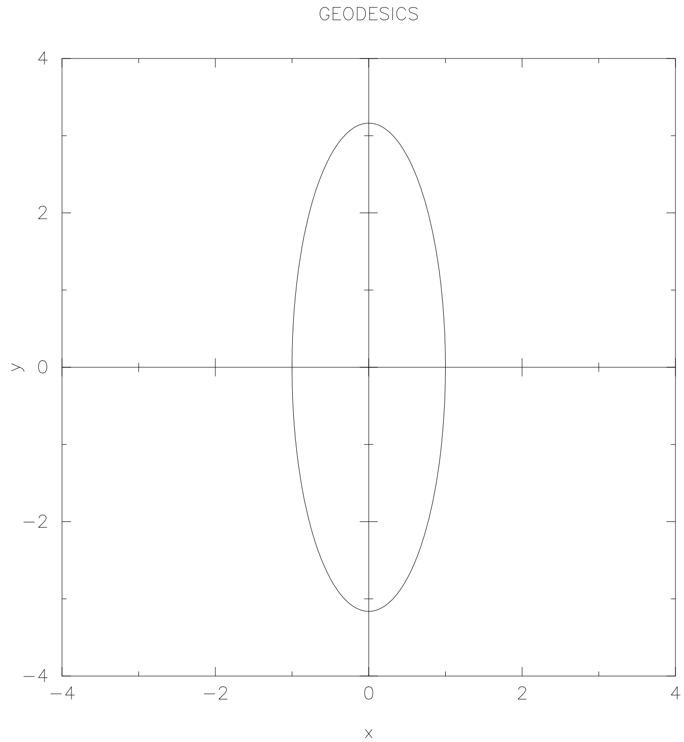

Figure 3: A particle projected parallel to the y-axis in a universe with $K = 0.1$ (corresponding to $\Omega > 1$) and $C = -0.5$.

3. The stress-energy tensor derived from the Dirac field is also no longer spatially homogeneous, so we cannot close the loop of Figure 1 and find a self-consistent solution.

We must therefore conclude that $\Omega \neq 1$ FRW models do not work, since it does not seem possible to find *any* self-consistent solution to the coupled field equations.

Further support for this conclusion arises when considering the motion of a test particle in an $\Omega \neq 1$ universe. Test particles no longer move in a straight line, but orbit the origin, as shown in Figure 3. In conventional approaches to cosmology the non-radial motions of a test particle are seldom plotted, but if they were, the interpretation of Figure 3 would be that one is looking down at the North Pole of a sphere, on which the particle is executing great-circle (*i.e.* geodesic) motion. This motion is projected into an ellipse when one dimension is supressed. This missing dimension is neither the z- nor t-axes, which are indeed missing in our figure, but is the extra dimension of an embedding picture in which the 3 dimensions of ordinary space become the surface of a 3-sphere (*c.f.* box 27.2 of reference [4]). In a flat spacetime this trick of reinterpretation is simply not available to us; instead, Figure 3 is that of a test particle revealing by its motion that a preferred direction e_r has been introduced.

The above points all lead us to the conclusion that $\Omega \neq 1$ FRW universes are not admisssible, since they break spatial homogeneity. The universe is therefore required to be at $\Omega = 1$ (critical density). This fits in well with some current observations, which give $\Omega \approx 0.85 \pm 0.2$ from results on large-scale streaming motions, and $\Omega = 1$ from measurements of the angular diameter of sub-kiloparsec-scale jets in a sample of radio sources [8] (see [9] for a recent discussion). These results notwithstanding, the value of Ω remains a matter of great debate and controversy, and it will probably require several further years of observations to settle the matter.

We have considered several more tests and predictions of our model and find that in general, once one has accepted that the density is at the critical value, there is a fairly straightforward mapping between (a) a conventional viewpoint in which the material particles expand away from each other and the energy density decreases because the comoving volume is increasing, and (b) our current viewpoint, in which the particles are fixed, and the energy density of the cosmological fluid decreases due to interaction with the gravitational field. In particular, the prediction for angular size versus redshift remains the same as in the conventional (GR) $\Omega = 1$ model. Discussions of the physics of the 'big bang' are also largely unaltered, since the relevant factor is the energy density as a function of time. Our theory will therefore agree with the current predictions of big-bang cosmology, for example calculations of nucleosynthesis rates in the early universe. More details of these comparisons and predictions will be presented in a forthcoming paper.

4 Discussion and Conclusions

In connection with self-consistent solutions to the coupled Einstein-Dirac equations we should mention the work of Isham & Nelson [10], who studied the quantization of a Dirac field in FRW universes. They solved for a classical Dirac field, and found that they could not obtain a self-consistent solution to the GR field equations, with the Dirac field providing the stress-energy tensor, except for the case where $\Omega = 1$. This reinforces the conclusions of Section 3, but we should note that Isham & Nelson claimed that the system was underdetermined even after the inclusion of the Dirac equation, and that a choice of time coordinate (equivalent to a free choice of f) still remained. Our analysis disagrees with this. In addition, they did not have available to them the further arguments presented in Section 3, which we believe provide additional, compelling reasons for excluding non-critical density models.

For the reasons discussed in the Introduction, it may seem surprising that our current theory is able to make such a strong prediction, at variance with GR. A point that increases the plausibility of our conclusions is that the theory of gravity discussed here is first-order in derivatives, and with complete spatial homogeneity contains only one adjustable free parameter when integrated in time. In our case this parameter was taken as the Hubble constant, $-C$. Alternatively, we could have characterized the free parameter by the present density, ρ_0, in which case H_0 would have been fixed. This is to be contrasted with GR, which is explicitly a second-order theory, so both the 'velocity' (H_0) and 'acceleration' ($q_0 = \frac{1}{2}\Omega$) must be specified in terms of their values at the current epoch.

We can summarise the conclusions arising from the approach in this paper as follows:

- Spacetime is not curved.
- The universe is not expanding.
- The redshift is a loss of energy to the gravitational field, not a Doppler effect.
- The universe is at critical density, and so its energy density will continue decreasing forever.

- $\Omega \neq 1$ FRW models are *not* spatially homogeneous.

A further conclusion, not fully addressed in the text, is that proper time would run backwards for observers in any 'recontraction' phase of the universe. Of course, the arguments presented in this paper 'censor out' such a phase by requiring that $\Omega = 1$.

These conclusions have been presented in unqualified form so far, so we end with some necessary provisos. Firstly, we have not considered the whole class of spatially-homogeneous models. It is possible that a Bianchi model will work in the 'self-consistent' sense of Figure 1, although it seems unlikely that it could meet the other objections raised to non-spatially-flat FRW models. Secondly, the spin-torsion sector of our theory has not yet been properly considered. To address this fully, we must provide a spinless source for the stress-energy tensor which is compatible with the rest of the theory. This requires a *multiparticle* Dirac theory, incorporating gravitational interactions. Although this is an ambitious undertaking, we are confident that the flat-space, gauge-force approach will work here as well, and could provide a fruitful approach to a full theory of quantum gravity.

Acknowledgements

The authors thank the organisers for arranging such an enjoyable conference. We also thank Anton Garrett for his careful reading of these manuscripts, and his numerous suggestions for improvements.

References

[1] C.J.L. Doran, A.N. Lasenby, and S.F. Gull. Gravity as a gauge theory in the spacetime algebra. *These proceedings.*

[2] A.N. Lasenby, C.J.L. Doran, and S.F. Gull. Gravity, gauge theory and geometric algebra. In Preparation, 1993.

[3] S. Weinberg. *Gravitation and Cosmology.* John Wiley and Sons, 1972.

[4] C.W. Misner, K.S. Thorne, and J.A. Wheeler. *Gravitation.* W.H. Freeman and Company, 1973.

[5] M. Berry. *Principles of Cosmology and Gravitation.* Cambridge University Press, 1976.

[6] S.W. Hawking. *A Brief History of Time.* Bantam Press, 1988.

[7] C.J.L. Doran, A.N. Lasenby, and S.F. Gull. States and operators in the spacetime algebra. To appear in: *Foundations of Physics.*, 1993.

[8] K.I. Kellermann. The cosmological deceleration parameter estimated from the angular size/redshift relation for compact radio sources. *Nature*, 361:134, 1993.

[9] P. Scheuer. Weighing the universe (News and Views). *Nature*, 361:112, 1993.

[10] C.J. Isham and J.E. Nelson. Quantization of a coupled Fermi field and Robertson-Walker metric. *Phys. Rev. D*, 10(10):3226, 1974.