Gravity as a gauge theory in the spacetime algebra*

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Abstract

We outline a theory of gravitational interactions utilising the spacetime algebra – the geometric algebra of spacetime. The theory arises by demanding invariance under active Poincaré transformations. Making this symmetry local results in a first-order theory with 40 degrees of freedom. The matter-free field equations are presented, and are solved for radially-symmetric static fields. We discuss the behaviour of point particles under the fields described by these solutions, and compare and contrast the results with those of general relativity.

1 Introduction

For some time we have been convinced that geometric algebra is the best available mathematical tool for physics [1]. This is particularly true for relativistic physics, in which the spacetime algebra, or STA [2], offers great advantages over conventional 4-vector techniques. A major stumbling block to wider acceptance of the STA approach has been the inability to formulate general relativity satisfactorily. The spacetime algebra is, after all, the geometric (Clifford) algebra of *flat* spacetime. Points in this spacetime are represented as *vectors*, and the Clifford algebra is built on this vector space. It therefore seems at first sight to be impossible to formulate a theory of gravity within the STA, since the structure of general relativity is intrinsically related to the notion of spacetime curvature.

Hestenes & Sobczyk [3] have developed a beautiful theory of curved manifolds within geometric algebra by viewing the manifold as a surface embedded in a larger 'ambient' flat space. The properties of the manifold are determined solely by the pseudoscalar (the volume element of tangent space) as it moves through the embedding space, and all the standard (intrinsic) results of Riemannian geometry are recovered by the projection of multivector quantities onto the manifold. This approach is useful both for concrete calculations and

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for working abstractly, when results can be generated without specifying the dimension of the embedding space. Nevertheless, it is not clear that this approach is the one required for general relativity. Einstein's equations are entirely local and do not predict any global features of a manifold, so that they do not contain information about how the pseudoscalar moves through an embedding space. In formulating general relativity within the framework of geometric calculus we are therefore presented with two alternatives. We must either modify the Einstein equations so that they specify some extrinsic properties, or we must assume that the pseudoscalar is constant, in which case we are explicitly working in a flat spacetime.

In this paper we adopt the second of these two approaches, and formulate a theory of gravity in terms of multilinear functions defined in the algebra of flat spacetime. That this is possible may seem surprising at first, since we are traditionally taught that incorporating gravity into special relativity leads inexorably to the concept of a curved spacetime [4, 5]. Yet our theory does reproduce the *experimentally-verified* predictions of general relativity. At the very least, this means that physicists have a choice between formulating gravity in terms of spacetime curvature or in terms of forces in a flat spacetime. A detailed discussion of the issues involved in this choice is given in a forthcoming paper [6], in which it is argued that the existence of torsion generated by quantum spin strongly favours the flat-space approach.

In the present paper we provide an introduction to our theory, concentrating on the matter-free field equations. Our theory is a gauge theory based on the gauge group of active Poincaré transformations of spacetime fields. Geometric algebra is a coordinate-free language, so that the passive coordinate transformations of general relativity have no place. We discover a class of radially-symmetric static solutions and discuss the relationship between these and the Schwarzschild metric of general relativity. By considering the motion of a test particle, we recover the standard modification of the Newtonian radial acceleration of the particle under the influence of these radial fields. A horizon still exists, but particles can now cross the horizon in a finite external coordinate time. This is illustrated with some simple diagrams. We conclude by discussing some implications for black-hole physics.

We follow throughout the conventions of [1, 7, 8, 9]; thus (Clifford) vectors are written in lower case Roman (a) or Greek (γ_μ) and general multivectors in upper case Roman (A) or Greek (ψ). The symbol $\langle A\rangle_r$ denotes the projection onto the grade-r components of A, and the scalar (grade-0) part is written as $\langle A\rangle$. We define the interior, exterior, scalar and commutator products as follows:

$$\begin{aligned} A_r\cdot B_s &= \langle A_rB_s\rangle_{|r-s|} & \qquad A_r\wedge B_s &= \langle A_rB_s\rangle_{r+s} \\ A*B &= \langle AB\rangle & \qquad A\times B &= \tfrac{1}{2}(AB-BA), \end{aligned} \tag{1}$$

and reversion as:

$$\begin{aligned} (AB)\tilde{} &= \tilde{B}\tilde{A} \\ \tilde{a} &= a \qquad \text{for any vector } a. \end{aligned} \tag{2}$$

Upon introducing an orthonormal frame $\{\gamma_\mu\}$ ($\mu = 0\ldots 3$), satisfying

$$\gamma_\mu\cdot\gamma_\nu = \eta_{\mu\nu} = \mathrm{diag}(+\,-\,-\,-), \tag{3}$$

the full STA is spanned by the quantities

$$1, \qquad \{\gamma_\mu\}, \qquad \{\sigma_k,\, i\sigma_k\}, \qquad \{i\gamma_\mu\}, \qquad i, \tag{4}$$

where

$$i \equiv \gamma_0\gamma_1\gamma_2\gamma_3, \qquad \sigma_k \equiv \gamma_k\gamma_0. \tag{5}$$

Linear functions are written as $\underline{h}(a)$ and are extended via outermorphism [3, 10] to act on the entire algebra. The adjoint to a linear function is defined by

$$\overline{h}(a) = \partial_b\langle a\underline{h}(b)\rangle \tag{6}$$

where

$$\partial_b = \gamma^\mu \frac{\partial}{\partial b^\mu}, \tag{7}$$

and $b^\mu = \gamma^\mu \cdot b$ is the (scalar) component of b along the γ_μ axis. An important class of linear functions is obtained from differentiating a (possibly non-linear) transformation $f(x)$. For these we write

$$\underline{f}(a) = \underline{f}_x(a) = a\cdot\nabla_x f(x) \tag{8}$$

where ∇_x is the vector derivative with respect to x. The dependence on spacetime position of $\underline{f}$ is indicated by a subscript, to distinguish it from the argument of the linear function, but we will drop this subscript whenever no confusion can arise. The adjoint to $\underline{f}$ (defined by (6)) satisfies the integrability condition that

$$\nabla\wedge\overline{f}(a) = 0, \quad \text{for all constant } a. \tag{9}$$

2 The Matter-free Field Equations

We are concerned in this paper with the gravitational field equations in the absence of matter. In the full treatment [6] the field equations are derived by demanding invariance of the Dirac equation under local, active Poincaré transformations. For our present purpose it suffices to consider the simpler equation

$$\nabla\psi = \nabla_x\psi(x) = 0, \tag{10}$$

where ψ is some arbitrary multivector function. This equation encompasses the neutrino equation (ψ = even multivector) and the free-field Maxwell equations (ψ = bivector). Given a solution $\psi(x)$ to (10), we can obtain a new solution by a translation; that is, if we define

$$\psi(x) \mapsto \psi(x') \equiv \psi'(x), \quad \text{where } x' = x + a \tag{11}$$

for some constant vector a, then $\psi'(x)$ also satisfies (10). To make this symmetry local we let a become an arbitrary function of position, which is done by replacing the translation with

$$x \mapsto f^{-1}(x) \equiv x' \tag{12}$$

so that the field transforms to

$$\psi'(x) \equiv \psi(x'). \tag{13}$$

Here f is an arbitrary non-linear mapping of spacetime vectors. Now $\psi'(x)$ no longer satisfies (10), but satisfies the new equation

$$\overline{f}_{x'}(\nabla)\psi' = \nabla_{x'}\psi(x') = 0. \tag{14}$$

This immediately tells us how to generalise (10). We introduce an arbitrary position-dependent linear function $\overline{h}$, and replace (10) by

$$\overline{h}(\nabla)\psi = 0. \tag{15}$$

If $\overline{h}$ is now transformed according to

$$\overline{h}_x \mapsto \overline{h}_{x'}\overline{f}_{x'} \equiv \overline{h}'_x, \tag{16}$$

then the transformed functions $\overline{h}'$ and ψ' together satisfy (15), provided that $\overline{h}$ and ψ do so.

Equation (15) is also invariant under rotations, although not in the conventional manner in which the spacetime dependence of ψ is also rotated (with the opposite orientation to the rotation of the fields). The gauging of translations has already allowed for the most general type of transformation of position dependence, so rotational invariance is instead achieved by transforming both ψ and $\overline{h}$:

$$\begin{aligned} \psi &\mapsto R\psi\tilde{R} \\ \overline{h}(a) &\mapsto R\overline{h}(a)\tilde{R}, \end{aligned} \tag{17}$$

where R is a constant rotor ($R\tilde{R} = 1$) and we have assumed a double-sided transformation law for ψ. (If ψ is a spinor function, which has a single-sided transformation law, the analysis is similar and the resultant field equations are the same [6].) To make this symmetry local, we write $\overline{h}(\nabla)$ as $\overline{h}(\partial_a)a{\cdot}\nabla$ and replace the directional derivative $a{\cdot}\nabla$ by a directional coderivative defined by

$$\mathcal{D}_a \equiv a{\cdot}\nabla + \Omega(a)\times. \tag{18}$$

Here $\Omega(a)$ is a position-dependent bivector-valued linear function of the vector a, which behaves under local rotations as

$$\Omega(a) \mapsto R\Omega(a)\tilde{R} - 2a{\cdot}\nabla R\tilde{R} \tag{19}$$

and under translations as

$$\Omega_x(a) \mapsto \Omega_{x'}\underline{f}^{-1}_{x'}(a). \tag{20}$$

The full generalisation of equation (10) now reads

$$\overline{h}(\partial_a)\mathcal{D}_a\psi \equiv \mathcal{D}\psi = 0, \tag{21}$$

and is invariant under both local translations and rotations. Local Poincaré invariance has been achieved at the expense of introducing two gauge fields, $\overline{h}$ and Ω, with a total of $(4\times 4) + (4\times 6) = 40$ degrees of freedom. (This is precisely the number expected from gauging the 10-dimensional Poincaré group.) The advantage of this approach over previous formulations of gravity as a gauge theory [11, 12] is that the freedom from coordinates has shown us exactly how the gauge fields enter into the field equations.

We must now construct an action integral for the gauge fields $\overline{h}$ and Ω. We first define the field-strength tensor $R(a\wedge b)$ by

$$\begin{aligned} &[\mathcal{D}_a, \mathcal{D}_b]\psi = R(a\wedge b)\times\psi \\ \Rightarrow\quad &R(a\wedge b) = a{\cdot}\nabla\Omega(b) - b{\cdot}\nabla\Omega(a) + \Omega(a)\times\Omega(b), \end{aligned} \tag{22}$$

so that $R(B)$ is a bivector-valued function of the bivector B. This transforms under local translations as

$$R_x(a\wedge b) \mapsto R_{x'}\underline{f}_{x'}^{-1}(a\wedge b), \tag{23}$$

and under rotations as

$$R(a\wedge b) \mapsto RR(a\wedge b)\tilde{R}. \tag{24}$$

From $R(a\wedge b)$ we define the contractions

$$\begin{aligned} R(b) &= \overline{h}(\partial_a)\cdot R(a\wedge b) && (25)\\ \mathcal{R} &= \overline{h}(\partial_b\wedge\partial_a)\cdot R(a\wedge b), && (26) \end{aligned}$$

and of these the ('Ricci') scalar $\mathcal{R}$ has the simple transformation property $\mathcal{R}(x) \mapsto \mathcal{R}(x')$ under local translations. It follows that the action integral

$$S = \int |d^4x|(\det h)^{-1}\mathcal{R} \tag{27}$$

is invariant under local Poincaré transformations, as will be the field equations derived from S. The derivation of the field equations from S is carried out in full in [6]; this derivation is similar to the Palatini formulation of general relativity, but there is no need for the concept of a metric associated with a curved space. Here we simply quote the required equations [6]:

$$\mathcal{D}\wedge\overline{h}(a) \equiv \overline{h}(\partial_b)\wedge(\mathcal{D}_b\overline{h}(a)) = \overline{h}(\nabla\wedge a) \quad \text{for all } a, \tag{28}$$

and

$$R(a) = 0 \quad \text{for all } a. \tag{29}$$

These constitute our flat-space matter-free gravitational field equations.

3 Radially-symmetric Static Solutions

In order to find radially-symmetric static solutions to the field equations (28) and (29) we introduce a set of polar coordinates (t, r, θ, ϕ), and define the coordinate frame

$$\begin{aligned} e_t &= \partial_t x = \gamma_0 \\ e_r &= \partial_r x = \sin\theta\cos\phi\gamma_1 + \sin\theta\sin\phi\gamma_2 + \cos\theta\gamma_3 \\ e_\theta &= \partial_\theta x = r(\cos\theta\cos\phi\gamma_1 + \cos\theta\sin\phi\gamma_2 - \sin\theta\gamma_3) \\ e_\phi &= \partial_\phi x = r\sin\theta(-\sin\phi\gamma_1 + \cos\phi\gamma_2). \end{aligned} \tag{30}$$

For our initial ansätz we choose $\overline{h}$ to be of the form

$$\begin{aligned} \overline{h}(e_t) &= f_1 e_t + f_2 e_r & \qquad \overline{h}(e_\theta) &= e_\theta \\ \overline{h}(e_r) &= g_1 e_r + g_2 e_t & \qquad \overline{h}(e_\phi) &= e_\phi, \end{aligned} \tag{31}$$

where f_i and g_i are functions of r only. We can write $\overline{h}$ in a more compact form as

$$\overline{h}(n) = n + n\cdot e_t\left((f_1-1)e_t + f_2 e_r\right) - n\cdot e_r\left((g_1-1)e_r + g_2 e_t\right). \tag{32}$$

We also take a trial form for the bivector field Ω; abbreviating $\Omega(e_\mu)$ to Ω_μ, this is

$$\begin{array}{rclrcl} \Omega_t & = & a e_r e_t & \Omega_\theta & = & (b_1 e_r + b_2 e_t) e_\theta / r \\ \Omega_r & = & 0 & \Omega_\phi & = & (b_1 e_r + b_2 e_t) e_\phi / r, \end{array} \tag{33}$$

where a and b_i are also scalar functions of r only. This is not the most general form of radially-symmetric Ω, but the general form can be generated from (33) by local gauge transformations.

The first of the field equations (28) can be written as

$$\overline{h}(e^\mu)\wedge(\mathcal{D}_\mu \overline{h}(e^\nu)) = 0, \tag{34}$$

where $\{e^\mu\}$ is the frame reciprocal to the $\{e_\mu\}$. Inserting (31) and (33) into (34) generates the four equations

$$g_2 f_2' - g_1 f_1' - a{f_1}^2 + a{f_2}^2 = 0 \tag{35}$$
$$g_1 g_2' - g_1' g_2 + a f_1 g_2 - a f_2 g_1 = 0 \tag{36}$$
$$g_1 = b_1 + 1 \tag{37}$$
$$g_2 = b_2, \tag{38}$$

where the primes denote differentiation with respect to r. We use (37) and (38) to eliminate b_1 and b_2. Next, calculating the 6 quantities $R_{\mu\nu} \equiv R(e_\mu \wedge e_\nu)$, yields

$$R_{tr} = -a' e_r e_t \tag{39}$$
$$R_{t\theta} = a(g_1 e_t + g_2 e_r) e_\theta / r \tag{40}$$
$$R_{r\theta} = (g_1' e_r + g_2' e_t) e_\theta / r \tag{41}$$
$$R_{\theta\phi} = ({g_1}^2 - {g_2}^2 - 1) e_\theta e_\phi / r^2, \tag{42}$$

with $R_{t\phi}$, $R_{r\phi}$ having the same form as $R_{t\theta}$, $R_{r\theta}$ respectively. By forming the contraction $\overline{h}(e^\mu)\cdot R_{\mu\nu}$ and setting the result equal to zero, we find that

$$2a + a'r = 0 \tag{43}$$
$$2g_1' + f_1 a' r = 0 \tag{44}$$
$$2g_2' + f_2 a' r = 0 \tag{45}$$
$$ar(f_1 g_1 - f_2 g_2) + r(g_1 g_1' - g_2 g_2') + {g_1}^2 - {g_2}^2 - 1 = 0. \tag{46}$$

Equation (43) yields a immediately, and equations (44) and (45) define f_1 and f_2 in terms of g_1 and g_2. Upon combining these with (35) we find that

$$\det h \equiv \overline{h}(i) i^{-1} = f_1 g_1 - f_2 g_2 = \text{constant}, \tag{47}$$

and we set this constant equal to 1, since $\overline{h}$ is required to reduce to the identity at large distances. All that remains is a simple equation for ${g_1}^2 - {g_2}^2$, and the full solution to our field equations is

$$\begin{array}{rcl} a & = & GM/r^2 \\ {g_1}^2 - {g_2}^2 & = & 1 - 2GM/r \\ GMf_1 & = & r^2 g_1' \\ GMf_2 & = & r^2 g_2', \end{array} \tag{48}$$

subject to the boundary conditions that

$$\left.\begin{array}{l} f_1, g_1 \rightarrow 1 \\ f_2, g_2 \rightarrow 0 \end{array}\right\} \quad \text{as} \quad r \rightarrow \infty. \tag{49}$$

These boundary conditions guarantee that at large distances the effects of the fields fall away to zero. The solution (48) contains a single arbitrary function, g_2 say, subject to the condition that

$$g_2{}^2(r) \geq 2GM/r - 1, \tag{50}$$

together with the boundary conditions. From our initial restricted choice of $\overline{h}$ and Ω we have found a one-parameter family of solutions. This is extended to a four-parameter family by considering radially-symmetric gauge transformations. The four classes of transformation which preserve radial symmetry are

$$\begin{array}{rl} \text{Radial boost :} & R = \exp(\alpha(r) e_r e_t/2) \\ \text{Rotation :} & R = \exp(\alpha(r) i e_r e_t/2) \\ \text{Time translation :} & f(x) = x + \alpha(r) e_t \\ \text{Radial translation :} & f(x) = x + \alpha(r) e_r. \end{array} \tag{51}$$

These transformations induce more general forms of $\overline{h}$ and Ω, and combinations of the first and third can be used to move within the one-parameter family described by (48). All transformations in (51) leave $g_1{}^2 - g_2{}^2$ and Ω_t unchanged.

We are now in a position to compare our solutions with the Schwarzschild metric of general relativity. The 'line element' in our flat-space theory is given by:

$$\begin{aligned} ds^2 &= \underline{h}^{-1}(e_\mu)\underline{h}^{-1}(e_\nu) dx^\mu dx^\nu \\ &= (1 - 2GM/r)dt^2 - (f_1 g_2 - f_2 g_1) 2dr\, dt - (f_1{}^2 - f_2{}^2) dr^2 \\ &\quad - r^2(d\theta^2 + sin^2\theta\, d\phi^2). \end{aligned} \tag{52}$$

We see that the exterior Schwarzschild metric is recovered by setting $g_2 = 0$, which can only be done outside the horizon ($r > 2GM$). If we wish to extend the same line element inside the horizon, we must set $g_1 = 0$ for $r < 2GM$. However this solution is then strongly discontinuous at the horizon: something has gone wrong! To see exactly what, we focus on the cross-term in the line element, since standard treatments of the Schwarzschild solution always assume that this term can be transformed away. The coefficient of this term is $f_1 g_2 - f_2 g_1$, and since our solutions have $g_1 = \pm g_2$ at the horizon and $f_1 g_1 - f_2 g_2 = 1$ everywhere, we find that

$$f_1 g_2 - f_2 g_1 = \pm 1 \quad \text{at } r = 2GM. \tag{53}$$

The assumption that $f_1 g_2 - f_2 g_1$ can be transformed away fails at the horizon, so that the standard form of the Schwarzschild solution is not admissible in our theory. We shall shortly see that this removes much of the pathological behaviour of test particles at the horizon of a Schwarzschild black hole. The reason for our more restrictive class of solutions is that we retain a notion of position in a flat spacetime, and demand that the $\overline{h}$ and Ω functions be well-defined throughout this spacetime (except possibly where a point source is present). General relativity, by contrast, does not place such restrictions on the components of the metric. These components are scalar functions which can be transformed by a coordinate transformation. General relativity admits coordinate transformations which result in patches of spacetime not being covered; such transformations have no counterpart in our theory.

The shift from the Schwarzschild solution, with $f_1g_2 - f_2g_1 = 0$, to two distinct families of solutions, with $f_1g_2 - f_2g_1 = \pm 1$ at the horizon, is characteristic of the transition from second-order to first-order theories. A similar phenomenon is seen in the theory of propagation of electromagnetic waves, for example, where the first-order formulation correctly fixes the obliquity factors which have to be put in by hand in the second-order theory [7]. Furthermore, the appearance of two disconnected families of solutions is precisely as expected from a gauge theory of the Poincaré group, since the disconnected families are related by the discrete symmetry of time-reversal, which switches the sign of $f_1g_2 - f_2g_1$.

4 Test-particle Motion under Radial Gravitational Forces

To find the equations of motion for a test particle we need the version of the geodesic equation appropriate to our flat-space theory. The required equation is

$$\mathcal{D}_v v = 0, \tag{54}$$

and, on defining

$$\dot{x} = \underline{h}(v) \tag{55}$$

$$\dot{v} = \partial_\tau v = \dot{x}\cdot\nabla v, \tag{56}$$

our geodesic equation (54) becomes

$$\dot{v} = -\Omega(\dot{x})\cdot v. \tag{57}$$

In relativistic physics, accelerations should be thought of as bivectors [13] and in (57) we can identify the bivector $\Omega(\dot{x})$ as the acceleration of the test particle.

If we assume that all motion takes place in the azimuthal plane ($\theta = \pi/2$), we can write

$$\dot{x} = \dot{t}e_t + \dot{r}e_r + \dot{\phi}e_\phi \tag{58}$$

and the equations of motion (57) give

$$\begin{aligned} r^2\dot{\phi} &= L \quad \text{(constant)} && (59) \\ \dot{r}^2 &= A^2 - 1 - (1-2GM/r)L^2/r^2 + 2GM/r && (60) \\ \dot{t}(1-2GM/r) &= \dot{r}(f_1g_2 - g_2f_1) + A. && (61) \end{aligned}$$

Equations (59) and (60) agree with those found in general relativity using the Schwarzschild metric, and (60) can be differentiated to give

$$\ddot{r} - r\dot{\phi}^2 = -(1+3L^2/r^2)GM/r^2. \tag{62}$$

This will reproduce the standard results for the shape of the orbit and the precession rate per orbit, which have been checked to high accuracy by measurements on binary pulsar systems [14]. Equation (61), which describes the rate of change of coordinate time with respect to proper time, differs from that of the Schwarzschild metric through the inclusion of the

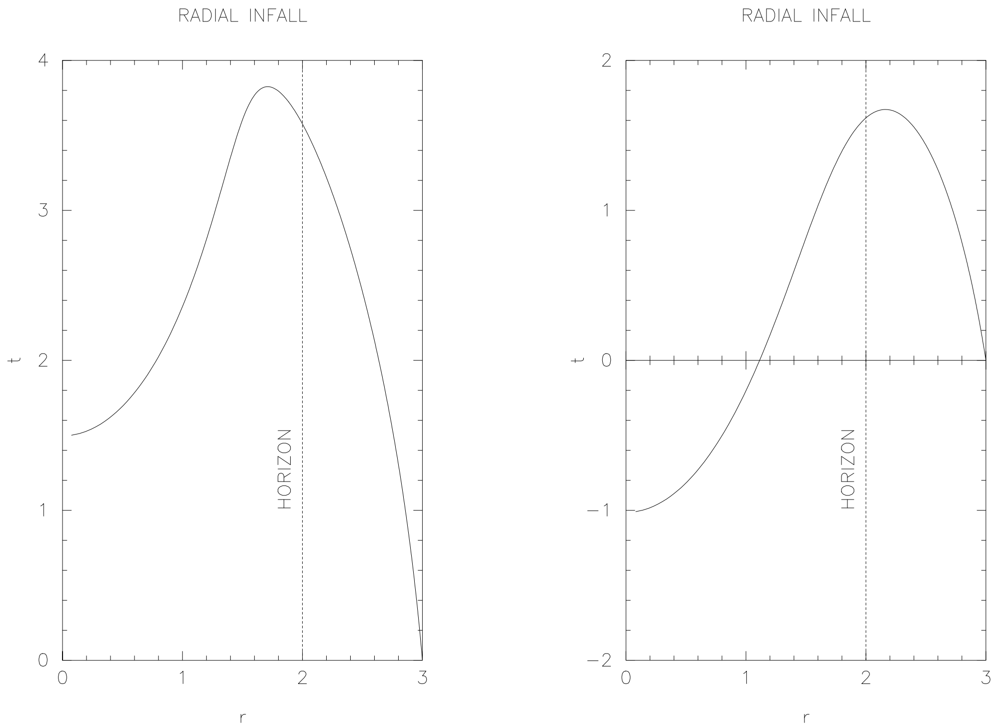

Figure 1: *Radial infall for $g_2 = 1.4/r^2$ (left) and $g_2 = 2.0/r^2$ (right).*

$\dot{r}(f_1g_2 - g_2f_1)$ term. We saw in Section 3 that $f_1g_2 - g_2f_1$ must equal ± 1 at the horizon, so, taking the positive sign and radial infall ($\dot{r} < 0$, $A > 0$), we find that

$$\dot{r}(f_1g_2 - g_2f_1) + A = 0 \quad \text{at } r = 2GM. \tag{63}$$

This result removes the pole present in the corresponding equation for the Schwarzschild metric, and allows particles to cross the horizon in *finite* external coordinate time. This is demonstrated in Figure 1, which plots radial infall of a particle for two distinct choices of the function g_2. Different field configurations have different $\dot{t}$ equations, and lead to different trajectories relative to the flat background. The trajectories are mapped onto each other by gauge transformations.

Solutions with $f_1g_2 - g_2f_1 = 1$ at the horizon act as 'one-way valves', in that particles can cross from the outside in finite coordinate time, but once inside can never get back out again. No part of an outgoing trajectory from the past is inside the horizon. The solutions for which $f_1g_2 - g_2f_1 = -1$ at the horizon have the reverse properties — matter can escape in finite coordinate time, but can never pass through the horizon from the outside (without going to infinite coordinate time). A geodesic for an escaping particle is plotted in Figure 2.

The pictures presented by these two (separate) types of solution have a counterpart in general relativity as the extension of the Schwarzschild metric written using advanced- and retarded-time Eddington-Finkelstein coordinates. General relativity understands these as representing the same solution, both obtained from the Schwarzschild solution by (passive) coordinate transformations. In our theory, however, they are *different* solutions, with different physical properties. They are related to each other via the active transformation of time-reversal. These differences from general relativity are seen most starkly when we consider

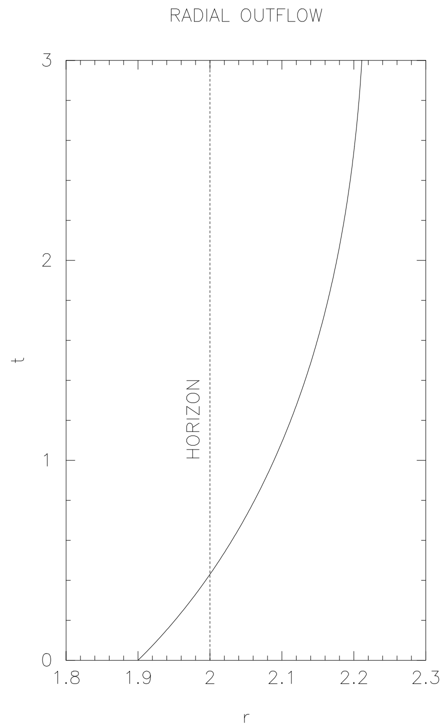

Figure 2: *Radial outflow for* $g_2 = -1.4/r^2$.

the maximal analytic extension of the Schwarzschild metric found by Kruskal (see Hawking and Ellis [15], for example). The Kruskal metric mixes advanced and retarded coordinates and therefore has no counterpart in our theory, since it would combine elements of distinct solutions. The formulation of gravity as a first-order theory in flat spacetime rules out Kruskal's extension. Further, related differences from general relativity are dealt with in greater detail in a forthcoming paper [6].

5 Conclusions

Gravity can be formulated in geometric algebra as a first-order gauge theory of multilinear functions in a flat spacetime. The dynamical variables are the functions $\overline{h}$ and Ω, and these are arrived at by gauging the Poincaré group of transformations of spacetime fields. All the dynamical quantities have coordinate-free definitions, and become much easier to manipulate than in tensor analysis. The insistence on a globally-defined underlying flat space has implications for cosmology which are discussed in the following paper [16].

The search for radially-symmetric static solutions has led us naturally to two distinct families of solutions, both different from the standard Schwarzschild solution of general relativity. This in turn has produced a new picture of geodesic motion around massive bodies, in which particles cross the horizon in finite external coordinate time. Implications for the formation of horizons will be discussed in future work, in which models of collapsing matter will be presented.

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