

Toward a quantization of null dust collapse

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Spherically symmetric, null dust clouds, like their timelike counterparts, may collapse classically into black holes or naked singularities depending on their initial conditions. We consider the Hamiltonian dynamics of the collapse of an arbitrary distribution of null dust, expressed in terms of the physical radius R , the null coordinates, V for a collapsing cloud or U for an expanding cloud, the mass function m of the null matter, and their conjugate momenta. This description is obtained from the Arnowitt-Deser-Misner description by a Kuchař-type canonical transformation. The constraints are linear in the canonical momenta and Dirac's constraint quantization program is implemented. Explicit solutions to the constraints are obtained for both expanding and contracting null dust clouds with arbitrary mass functions.

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I. INTRODUCTION

Spherically symmetric dust clouds, depending on their initial matter and velocity distributions, will collapse in classical general relativity to form either black holes or naked singularities. Black holes are better understood than naked singularities. They are generally expected to evaporate via their associated Hawking radiation [1], although no agreement has yet been achieved regarding the end state of collapse, i.e., whether a remnant survives or whether all the matter contained in the original cloud is thermally radiated away. If a portion of the collapsing matter does manage to form a stable black hole, it is expected that the total mass of the remnant will be quantized. On the other hand, if all the matter is radiated away before a stable end state can form then one must explain what happens to the information that was contained in the initial matter distribution. The formation of black holes therefore presents a number of deep puzzles and various approaches to quantum gravity are being employed to address these at the present time [2–5]. On the contrary, naked singularities have received comparatively little attention. Yet, the formation of naked singularities (singularities that are visible either locally or asymptotically) is much more difficult to understand and for an entirely different reason: their existence implies the absence of a well defined Cauchy problem to the future of some lightlike surface (the Cauchy horizon); therefore, any attempt to describe the system to the future of this surface fails for lack of initial conditions. It seems that space-time must be terminated at the Cauchy horizon. In order to avoid the associated problems, Penrose proposed a cosmic censor [6], whose function

is essentially to ensure that naked singularities never form. The mechanism by which the cosmic censor operates, however, is still shrouded in mystery. The censor is most likely not classical because most models of classical collapse lead to the formation of both black holes and naked singularities in different domains of the initial phase space [7]. In fact very little is currently understood about the final stages of a collapse that leads to the formation of a classical naked singularity.

There are indications from the semiclassical treatment of naked singularities, in which the gravitational degrees of freedom are considered to be classical, that Penrose's cosmic censor may, in fact, be the quantum theory itself [8]. However, at the very final stages of collapse it is not possible to treat the gravitational degrees of freedom classically and a full blown quantum theory of the gravitational field becomes necessary to establish this possibility [9] firmly. Singularities in general relativity signal a breakdown of the classical theory, a regime in which the classical equations are meaningless. Cosmic censorship probably points to the need for quantum gravity in the same way as, more than eighty years ago, the electrodynamic instability of atoms pointed to the need for quantum mechanics. A good question is just how complete a theory of quantum gravity is required to begin addressing such issues as the cosmic censor. We take the attitude that, from past experience, it is not unreasonable to expect many of the key effects of quantum gravity to be understood from a more naive quantization of the gravitational field which, while it may be incomplete, incorporates the essential features of quantum mechanics.

This is what we propose to do in this paper. Our objective is to consider the midi-superspace quantization of a spherically symmetric cloud of null matter specified by an arbitrary mass distribution and collapsing in its own gravitational field. The model we are concerned with therefore is a solution of Einstein's equations with pressureless, null dust [10]

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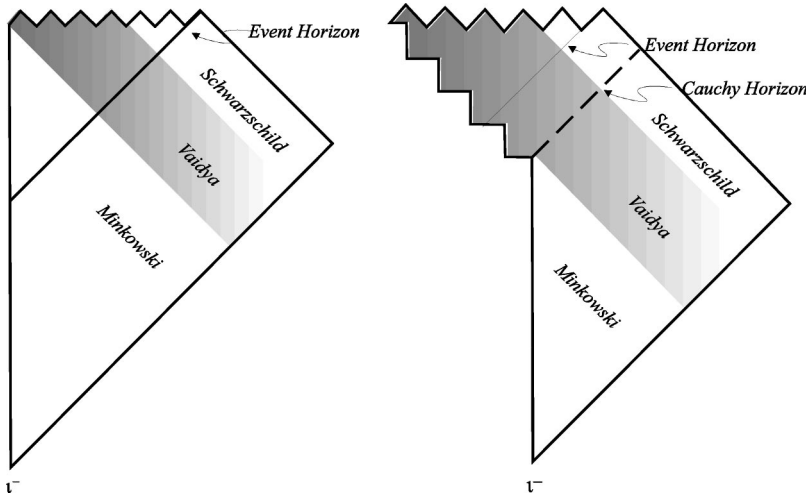


FIG. 1. Black hole (left) or naked singularity (right) formation in null dust collapse.

described by the stress energy tensor $T_{\mu\nu} = \epsilon(x)U_\mu U_\nu$, where $\epsilon(x)$ is the energy density of the cloud and $U^2 = 0$. When the cloud is contracting, the solution is characterized by an arbitrary function, $m(V)$, of the advanced null coordinate, $V \in (-\infty, +\infty)$. The mass function is generally taken to be vanishing for $V < V_o$ and constant, M , when $V > V_1$. The space-time is described by the metric

$$ds^2 = \left(1 - \frac{2m(V)}{R}\right) dV^2 - 2dRdV - R^2 d\Omega^2, \quad (1)$$

where $R \in [0, +\infty)$ is the area radius. The region $V > V_1$ is a part of the Schwarzschild space-time. In this region the metric may be written in terms of the Eddington-Finkelstein coordinates, \tilde{U} and \tilde{V} as

$$ds^2 = \left(1 - \frac{2M}{R}\right) d\tilde{U}d\tilde{V} - R^2 d\Omega^2, \quad (2)$$

where $\tilde{U} = \tilde{V} - 2R^*$ and R^* is the tortoise coordinate. The region $V < V_o$ is a part of Minkowski space-time, with metric

$$ds^2 = dudv - R^2 d\Omega^2, \quad (3)$$

where u and v are the ordinary retarded and advanced times, respectively $T \mp R$.

In the time reversed situation the null dust cloud is expanding instead of contracting and the solution is written in an analogous fashion,

$$ds^2 = \left(1 - \frac{2m(U)}{R}\right) dU^2 + 2dRdU - R^2 d\Omega^2 \quad (4)$$

in terms of a *retarded* null coordinate, U . Again, the mass function is generally taken to vanish when $U > U_1$, having some constant value, M before some earlier retarded time, $U < U_o$. The region $U > U_1$ is then a part of Minkowski space-time, while the region $U < U_o$ is a part of the Schwarzschild space-time.

Depending on the distribution $m(V)$ of matter in the null cloud, either black holes or naked singularities may develop as the classical final state of the collapse. For example, in the

self similar model in which the mass is a linear function of the advanced null coordinate, $m(V) = \lambda V$, one finds that both outcomes described by the Penrose diagrams in Fig. 1 are possible, depending on whether $\lambda > 1/16$ (black hole) or $\lambda \leq 1/16$ (naked singularity) [11]. When the collapse evolves toward a naked singularity, spatial hypersurfaces in the future of the initial singularity cross the Cauchy horizon and collide with the central singularity but, because no sensible boundary conditions can be specified on a singularity, the evolution in the future of the initial singularity is arbitrary. The cosmic censor [6] should come into play *before* the Cauchy horizon has a chance to form. It is of interest, therefore, to understand how the system behaves close to the putative Cauchy horizon, where spatial hypersurfaces are well defined and the quantum evolution of the system may be studied.

Null shells classically collapse to form covered singularities, therefore, in order to examine such issues as the cosmic censor, mass distributions other than those representing shells should be considered. These require a quantization of a genuine field theory. The present paper is intended as a first step in this direction. The solution metric is written in Eddington-Finkelstein coordinates and the Kuchař transformation from Arnowitt-Deser-Misner (ADM) variables to Kuchař variables is established. The mass function is explicitly related to $m(V)$ and $m(U)$ of the metrics in Eqs. (1) and (4), respectively.

The quantization program employed in this paper involves a gauge fixing and it is known that quantum theories resulting from different gauge fixings are not necessarily equivalent. It is, nevertheless, the best approach to the quantization of realistic and pressing problems such as gravitational collapse at this time. Our choice of configuration space coordinates presents several advantages: their physical meaning is transparent, we are able to give explicit transformations between the original ADM phase space and the new, and the constraints expressed in terms of the new phase space variables are linear. The last enables us to obtain solutions for null dust clouds of arbitrary mass distributions. The second implies that operators representing observables, if known in one system, are easily constructed in the other, and the first makes our solutions easily interpretable in physical terms.

In Sec. II we summarize the canonical formulation of the action in ADM variables. The null dust action appropriate to the models being considered is also analyzed in this section. In Sec. III we explicitly perform a transformation of the phase-space to Kuchař variables [12]. In Sec. IV we apply Dirac's quantization program to these models. Solutions to the constraints for collapsing and expanding clouds are presented and the matter distribution representing a single shell is examined as a special case. However, the phase-space admits non-trivial boundaries and this generally leads to complications in the quantization procedure. Our solutions are valid subject to the condition that a suitable measure can be found so that the replacement $P_X \rightarrow -i\delta/\delta X$ leads to self-adjoint operators. We conclude in Sec. V with a discussion of the strengths and weaknesses of our approach, the issues that remain to be resolved (such as the question of observables and of the measure on the Hilbert space) as well as some suggestions for future directions.

II. CANONICAL FORMULATION IN ADM VARIABLES

Consider the line element, $d\sigma$ on a spherically symmetric three dimensional Riemann surface, Σ . It is completely characterized by two functions, $L(r)$ and $R(r)$ of the radial label coordinate

$$ds_{(3)}^2 = L^2(r)dr^2 + R^2(r)d\Omega^2, \quad (5)$$

where Ω is the solid angle. The angular coordinates play no role and will be integrated over. We take both $L(r)$ and $R(r)$ to be positive definite except, possibly, at the center. $R(r)$ represents the physical radius of a shell labeled by r on the surface. It behaves as a scalar under transformations of r , whereas $L(r)$ behaves as a scalar density. The corresponding four dimensional line element may be written in terms of two additional functions, the lapse, $N(t, r)$, and the shift, $N^r(t, r)$, as

$$ds^2 = N^2 dt^2 - L^2(dr + N^r dt)^2 - R^2 d\Omega^2. \quad (6)$$

In this spherically symmetric space-time, we will consider the Einstein-Dust system described by the action

$$S = -\frac{1}{16\pi} \int d^4x \sqrt{-g} \mathcal{R} - \frac{1}{8\pi} \int d^4x \sqrt{-g} \epsilon(x) g^{\alpha\beta} U_\alpha U_\beta \quad (7)$$

where \mathcal{R} is the scalar curvature. As is well known, the gravitational part of this action can be cast into the form

$$S^g = \int dt \int_0^\infty dr [P_L \dot{L} + P_R \dot{R} - NH^g - N^r H_r^g] + S_{\partial\Sigma}^g \quad (8)$$

with the momenta conjugate to L and R respectively given by

$$P_L = \frac{R}{N} [-\dot{R} + N^r R'] \quad (9)$$

$$P_R = \frac{1}{N} [-L\dot{R} - \dot{L}R + (N^r LR)']$$

and where the overdot and the prime refer respectively to partial derivatives with respect to the label time, t , and coordinate, r . The lapse, shift and phase-space variables are required to be continuous functions of the label coordinates. The boundary action, $S_{\partial\Sigma}^g$, is required to cancel unwanted boundary terms in the hypersurface action, ensuring that the hypersurface evolution is not frozen on the frontiers. It is determined after fall-off conditions appropriate to the models under consideration are specified. The super-Hamiltonian and super-momentum constraints are given by

$$\mathcal{H}^g = -\left[\frac{P_L P_R}{R} - \frac{L P_L^2}{2R^2} \right] + \left[-\frac{L}{2} - \frac{R'^2}{2L} + \left(\frac{R R'}{L} \right)' \right] \quad (10)$$

$$\mathcal{H}_r^g = +R' P_R - L P_L'.$$

We will assume that the matter distribution is such that at infinity Kuchař's fall-off conditions [12] are suitable and we will adopt them here. These conditions would be applicable, for example, in models in which the collapsing metric asymptotically approaches or is smoothly matched to an exterior Schwarzschild background at some boundary. They read

$$L(t, r) = 1 + M_+(t) r^{-1} + \mathcal{O}^\infty(r^{-1-\epsilon})$$

$$R(t, r) = r + \mathcal{O}^\infty(r^{-\epsilon})$$

$$P_L(t, r) = \mathcal{O}^\infty(r^{-\epsilon})$$

$$P_R(t, r) = \mathcal{O}^\infty(r^{-1-\epsilon}) \quad (11)$$

$$N(t, r) = N_+(t) + \mathcal{O}^\infty(r^{-\epsilon})$$

$$N^r(t, r) = \mathcal{O}^\infty(r^{-\epsilon})$$

and imply that the asymptotic regions are flat with the spatial hypersurfaces asymptotic to surfaces of constant Minkowski time. Again, as $r \rightarrow 0$ we require that [14]

$$L(t, r) = L_0(t) + \mathcal{O}(r^2)$$

$$R(t, r) = R_1(t)r + \mathcal{O}(r^3)$$

$$P_L(t, r) = P_{L_2}(t)r^2 + \mathcal{O}(r^4)$$

$$P_R(t, r) = P_{R_1}(t)r + \mathcal{O}(r^3) \quad (12)$$

$$N(t, r) = N_0(t) + \mathcal{O}(r^2)$$

$$N^r(t, r) = N_1^r(t)r + \mathcal{O}(r^3).$$

With these conditions, it is easy to see that the appropriate choice of surface action involves only the contribution,

$$S_{\partial\Sigma}^g = - \int_{\partial\Sigma_\infty} dt N_+(t) M_+(t) \quad (13)$$

at the boundary at infinity.

Let us now consider the null dust action in Eq. (7). We note first that the energy density, $\epsilon(x)$, plays the role of a

Lagrange multiplier enforcing null dust, i.e., $U^2=0$, and variation with respect to $g_{\alpha\beta}$ yields the standard dust stress tensor, $T_{\alpha\beta}=\epsilon(x)U^\alpha U^\beta$. The canonical form of the null dust action in various forms has been studied by Kuchař and Bičák [15]. In particular, for the action in the form given in Eq. (7) one may expand U_α as a Pfaff form of six scalar fields, the three co-moving coordinates of the null dust particles, Z^k , and three scalars (velocities), w_k ,

$$U_\alpha = w_k Z^k_{,\alpha}. \quad (14)$$

This representation is redundant because, by Pfaff's theorem, only four scalars are required to describe an arbitrary covector in a four dimensional space. Suppose we require one of the scalars, say w_3 to be unity and drop the index from the associated co-moving coordinate, $Z^3:=Z$, then

$$U_\alpha = Z_{,\alpha} + w_k Z^k_{,\alpha}, \quad k \in \{1,2\}. \quad (15)$$

Consider the independent variations

$$\begin{aligned} 0 &= \frac{\delta S}{\delta \epsilon} = -\sqrt{-g} g^{\alpha\beta} U_\alpha U_\beta \\ 0 &= \frac{\delta S}{\delta Z} = [\sqrt{-g} \epsilon(x) U^\alpha]_{,\alpha} = \nabla_\alpha [\epsilon(x) U^\alpha] \\ 0 &= \frac{\delta S}{\delta w_k} = -\sqrt{-g} \epsilon(x) g^{\alpha\beta} Z^k_{,\alpha} U_\beta \\ 0 &= \frac{\delta S}{\delta Z^k} = [\sqrt{-g} \epsilon(x) w_k U^\alpha]_{,\alpha} \\ &= \nabla_\alpha [\epsilon(x) w_k U^\alpha] \\ T_{\alpha\beta} &= \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\alpha\beta}} = \epsilon(x) U_\alpha U_\beta. \end{aligned} \quad (16)$$

The conservation of the stress energy tensor in the last equation implies that

$$\epsilon(x) U^\alpha \nabla_\alpha U^\beta + U^\beta \nabla_\alpha [\epsilon(x) U^\alpha] = 0, \quad (17)$$

which says that the particles follow geodesic curves. Using the second equation above we find $U^\alpha \nabla_\alpha U^\beta = 0$, implying affine parametrization. The third equation says that $\mathcal{L}_U Z^k = Z^k_{,\alpha} U^\alpha = 0$, i.e., all of the Z^k are constant along flow lines and none of them are timelike. And finally, multiplying the third equation by w_k we find

$$Z_{,\alpha} U^\alpha = 0, \quad (18)$$

saying that $Z_{,\alpha}$ may be space-like or null. If the twist, $U_{[\alpha;\beta]}$, also vanishes, then $Z_{,\alpha}$ is null, which would imply that $w_k Z^k_{,\alpha} = 0$, or $w_k = 0 \forall k \in \{1,2\}$, because the $Z^k_{,\alpha}$ are taken to form a (linearly independent) cobasis.

Substituting the decomposition (15) into the dust action in Eq. (7), using Eq. (6) and integrating over the angular coordinates the action may be put in the form

$$S^d = \int dt \int_0^\infty dr [P_Z \dot{Z} + P_k \dot{Z}^k - N \mathcal{H}^d - N^r \mathcal{H}_r^d], \quad (19)$$

where the momenta conjugate to $\{Z, Z^k\}$ are, respectively,

$$P_Z = \frac{LR^2}{N} \epsilon(r) [(\dot{Z} + w_k \dot{Z}^k) - N^r (Z' + w_k Z^{k'})] \quad (20)$$

$$P_k = w_k P_Z,$$

and the constraints, \mathcal{H}^d and \mathcal{H}_r^d are

$$\begin{aligned} \mathcal{H}^d &= \left[\frac{P_Z^2}{2LR^2\epsilon} + \frac{\epsilon R^2 (Z' + w_k Z^{k'})^2}{2L} \right] \\ \mathcal{H}_r^d &= P_Z (Z' + w_k Z^{k'}). \end{aligned} \quad (21)$$

Setting $\delta \mathcal{L} / \delta \epsilon = 0$ gives the final form of the dust Hamiltonian and momentum constraints

$$\begin{aligned} \mathcal{H}^d &= \pm \frac{P_Z (Z' + w_k Z^{k'})}{L} \\ \mathcal{H}_r^d &= P_Z (Z' + w_k Z^{k'}), \end{aligned} \quad (22)$$

where the positive (negative) signs in the dust Hamiltonian density represent incoming (outgoing) dust. In the spherically symmetric collapse we are considering, we take $w_k = 0 = P_k$. Thus we have arrived at the canonical form of our theory, which we will write as

$$\begin{aligned} S &= + \int dt \int_0^r dr [\dot{Z} P_Z + \dot{L} P_L + \dot{R} P_R - N \mathcal{H} - N^r \mathcal{H}_r] \\ \mathcal{H} &= - \left[\frac{P_L P_R}{R} - \frac{L P_L^2}{2R^2} \right] + \left[-\frac{L}{2} - \frac{R'^2}{2L} + \left(\frac{R R'}{L} \right)' \right] \\ &\quad - \eta \frac{P_Z Z'}{L} \end{aligned} \quad (23)$$

$$\mathcal{H}_r^g = Z' P_Z + R' P_R - L P_L'$$

$$S_{\partial\Sigma} = - \int_{\partial\Sigma_\infty} N_+(t) M_+(t),$$

where $\eta = \text{sgn}(Z')$. In the following section we will show that the co-moving coordinate Z may be identified with the null coordinates according to $Z = -U$ for an expanding solution and $Z = -V$ for a collapsing one. $P_Z \leq 0$ and the dust Hamiltonian density is chosen to be always non-negative. When P_Z is non-vanishing the phase-space is made up of two disconnected sectors, labeled by η . An initial data set with $P_Z = 0$ cannot evolve into a set with $P_Z \neq 0$ and we will assume from now on that $P_Z \neq 0$.

III. CANONICAL TRANSFORMATION

The description of contracting and expanding clouds is seen to be related by time reversal. The two descriptions may be formally unified in the following way. Introduce a null coordinate W_η , which can be the ‘‘advanced’’ time or the ‘‘retarded’’ time, satisfying only the requirement that W_η increases toward the future. If $W'_\eta < 0$ (primes denote differentiation with respect to the ADM label coordinate r) then it represents the retarded coordinate, U , and if, on the contrary, $W'_\eta > 0$, it represents the advanced coordinate, V . Let us write both solutions in terms of a parameter η that represents the behavior of the matter (whether it is expanding or collapsing)

$$ds^2 = \left(1 - \frac{2m}{R}\right) dW_\eta^2 + 2\eta dW_\eta dR - R^2 d\Omega^2. \quad (24)$$

The metric (24) is appropriate for either expansion or contraction of the dust cloud depending on whether $\eta = -\text{sgn}(W'_\eta)$ is $+1$ or -1 . η is the same as appears in Eqs. (23), as we argue below. The null coordinate, \bar{W}_η , whose spatial direction is opposite to W_η is obtained by integrating

$$d\bar{W}_\eta = \sigma(W_\eta, R) \left[dW_\eta + 2\eta \frac{dR}{\mathcal{F}} \right], \quad (25)$$

where $\sigma(W_\eta, R)$ is an integrating factor. This coordinate must also be always increasing toward the future.

The hypersurfaces (5) from which Eq. (6) is constructed must be embedded in the space-time described by the metric (24). Substituting the foliation $W_\eta(t, r)$ and $R(t, r)$ in Eq. (24) gives the density $L(t, r)$ and the lapse and shift functions, $N(t, r)$ and $N^r(t, r)$, as

$$\begin{aligned} \mathcal{F} \dot{W}_\eta^2 + 2\eta \dot{W}_\eta \dot{R} &= N^2 - L^2 N'^2 \\ -\mathcal{F} W_\eta'^2 - 2\eta W_\eta' R' &= L^2 \\ -\eta(\dot{W}_\eta R' + W_\eta' \dot{R}) - \mathcal{F} \dot{W}_\eta W_\eta' &= N^r L^2, \end{aligned} \quad (26)$$

where we have set $\mathcal{F} = 1 - 2m/R$. These relations can be used to determine,

$$\begin{aligned} N^r &= \frac{\mathcal{F} \dot{W}_\eta W_\eta' + \eta(\dot{W}_\eta R' + W_\eta' \dot{R})}{\mathcal{F} W_\eta'^2 + 2\eta W_\eta' R'} \\ N &= \frac{\dot{W}_\eta R' - W_\eta' \dot{R}}{L}, \end{aligned} \quad (27)$$

where we have chosen the sign of the square root so that N is positive. Inserting these expressions for N and N^r in the expression (9) for P_L , we find

$$\frac{LP_L}{R} = -\eta R' - \mathcal{F} W_\eta', \quad (28)$$

which, when substituted into the expression for L^2 in Eq. (26), gives

$$\mathcal{F} = \frac{R'^2}{L^2} - \frac{P_L^2}{R^2}, \quad (29)$$

or, equivalently, the mass function in terms of the canonical data

$$m = \frac{R}{2} \left[1 - \frac{R'^2}{L^2} + \frac{P_L^2}{R^2} \right]. \quad (30)$$

By directly taking Poisson brackets, the momentum conjugate to m can now be shown to be simply

$$P_m = \frac{LP_L}{R\mathcal{F}}. \quad (31)$$

Kuchař [12] proposed that (R, m, \bar{P}_R, P_m) should form a canonical chart whose coordinates are spatial scalars, whose momenta are scalar densities and which is such that $H_r(r)$ generates Diff \mathbf{R} . This means that

$$\mathcal{H}_r^g = R' P_R - L P_L' = R' \bar{P}_R + m' P_m \approx 0. \quad (32)$$

Substituting the expressions derived for m and P_m into the above constraint one arrives at

$$\bar{P}_R = P_R - \frac{LP_L}{2R} - \frac{LP_L}{2R\mathcal{F}} - \frac{\Delta}{RL^2\mathcal{F}}, \quad (33)$$

where $\Delta = (RR')(LP_L)' - (RR')'(LP_L)$. One can then show that the transformation,

$$(R, L, P_R, P_L) \rightarrow (R, m, \bar{P}_R, P_m), \quad (34)$$

is canonical, and generated by

$$\mathcal{G} = \int_0^\infty dr \left[LP_L - \frac{1}{2} RR' \ln \left| \frac{RR' + LP_L}{RR' - LP_L} \right| \right]. \quad (35)$$

By computing the difference between the old and the new Liouville forms and using the fall off conditions in Eqs. (11) and (12), one can show that the transformation has introduced no fresh boundary terms.

There are (infinite) boundary terms at the horizon, when $\mathcal{F} = 0$. It can be shown, however, that the contributions from the interior and the exterior cancel each other. There will also be contributions at the boundary between the interior of the star and its exterior or more generally at any frontier between two regions described by mass functions with different derivatives. Again, if the mass function is continuous across the boundary and regions are consistently matched by equating both the first and second fundamental forms, then the contribution from one side will cancel the contribution from the other.

Before rearranging the action, we will consider the coordinates $\{Z, Z^k\}$ for the collapsing Vaidya null congruence. Returning to the metric in Eq. (24) with $\eta = -1$ we find that the incoming null congruence is given by $V = \text{const}$, θ

$=\text{const}$ and $\phi=\text{const}$. The coordinates $Z=-V$, $Z^1=\theta$ and $Z^2=\phi$ are co-moving. Let us form the basis

$$\begin{aligned} Z_{,\mu} &= (-1,0,0,0) \\ Z_{,\mu}^1 &= (0,0,1,0) \\ Z_{,\mu}^2 &= (0,0,0,1). \end{aligned} \quad (36)$$

It is easily shown that $\xi=-R$ is an affine parameter and the covariant components of the velocity $dx^\mu/d\xi$ are $U_\mu=(-1,0,0,0)$, whose decomposition in the co-basis $Z_{,\alpha}^k$ yields $W_1=0=W_2$. Similarly treating the outgoing null congruence shows that the affine parameter is $\xi=+R$ and that Z is to be identified with $-U$. Both cases may be treated simultaneously by letting $Z=-W_\eta$, $P_Z=-P_{W_\eta}$ and $\xi=\eta R$. This identification shows that the η used in the section is identical to that used in the previous section.

Note also that taking the spatial derivative of m in Eq. (30), yields

$$m' = -\frac{R'}{L}\mathcal{H}^s - \frac{P_L}{RL}\mathcal{H}_r^s. \quad (37)$$

This may be used to write the action in Eq. (7) as

$$\begin{aligned} S = \int dt \int_0^\infty dr [P_{W_\eta} \dot{W}_\eta + \bar{P}_R \dot{R} + P_m \dot{m} \\ - N\mathcal{H} - N^r \mathcal{H}_r] + S_{\partial\Sigma}, \end{aligned} \quad (38)$$

with

$$\begin{aligned} \mathcal{H} &= -\left[\frac{m' \mathcal{F}^{-1} R' + \mathcal{F} P_m \bar{P}_R}{L} \right] - \eta \frac{P_{W_\eta} W_\eta'}{L} \\ \mathcal{H}_r &= R' \bar{P}_R + m' P_m + W_\eta' P_{W_\eta} \\ S_{\partial\Sigma} &= -\int_{\partial\Sigma_\infty} N_+(t) M_+(t). \end{aligned} \quad (39)$$

The surface term contains the mass at spatial infinity and may be re-expressed in a more convenient form. Use the fall-off conditions (11) at infinity and the expression for N in Eq. (27) to write $N_+=\dot{W}_{\eta+}$ and

$$S_{\partial\Sigma} = -\int_{\partial\Sigma_\infty} \dot{W}_{\eta+}(t) M_+(t). \quad (40)$$

Then define $\Gamma_\eta(r)$ by

$$m(r) = M_+ - \int_r^\infty dr' \Gamma_\eta(r') \quad (41)$$

i.e., $m'(r) = \Gamma_\eta(r)$ and M_+ is clearly the mass at infinity. The one form

$$\Omega = \int_0^\infty dr P_m(r) \delta m(r) - M_+ \delta W_{\eta+} \quad (42)$$

can now be written as

$$\begin{aligned} \Omega &= \left(W_{\eta+} + \int_0^\infty dr P_m(r) \right) \delta M_+ \\ &\quad - \int_0^\infty dr P_m(r) \int_r^\infty dr' \delta \Gamma_\eta(r') - \delta\omega, \end{aligned} \quad (43)$$

where $\delta\omega = \delta(M_+ W_{\eta+})$ is an exact form. The second term in the expression for Ω continues to be inconvenient, but may be cast into a more appropriate form using the identity [12]

$$\begin{aligned} &\left(\int_0^r dr' P_m(r') \times \int_r^\infty dr' \delta \Gamma_\eta(r') \right)' \\ &= P_m(r) \int_r^\infty dr' \delta \Gamma_\eta(r') - \left(\int_0^r dr' P_m(r') \right) \delta \Gamma_\eta(r). \end{aligned} \quad (44)$$

Integrating from $r=0$ to $r=\infty$, the left-hand side of the above equation vanishes identically and one finds

$$\int_0^\infty dr P_m(r) \int_r^\infty dr' \delta \Gamma_\eta(r') = \int_0^\infty dr \delta \Gamma_\eta(r) \int_0^r dr' P_m(r'), \quad (45)$$

so that Ω can be cast into the form

$$\Omega = p_+ \delta M_+ + \int_0^\infty dr P_{\Gamma_\eta} \delta \Gamma_\eta - \delta\omega, \quad (46)$$

where we have defined

$$\begin{aligned} p_+ &= W_{\eta+} + \int_0^\infty dr P_m(r) \\ P_{\Gamma_\eta} &= -\int_0^r dr' P_m(r'). \end{aligned} \quad (47)$$

Eliminating a total time derivative turns the action in Eq. (38) into

$$\begin{aligned} S = \int dt \left(p_+ \dot{M}_+ + \int_0^\infty dr [P_{W_\eta} \dot{W}_\eta + \bar{P}_R \dot{R} \right. \\ \left. + P_{\Gamma_\eta} \dot{\Gamma}_\eta - N\mathcal{H} - N^r \mathcal{H}_r] \right), \end{aligned} \quad (48)$$

where

$$\begin{aligned} \mathcal{H} &= -\left[\frac{\Gamma_\eta \mathcal{F}^{-1} R' - \mathcal{F} P_{\Gamma_\eta} \bar{P}_R}{L} \right] - \eta \frac{P_{W_\eta} W_\eta'}{L} \\ \mathcal{H}_r &= R' \bar{P}_R - \Gamma_\eta P_{\Gamma_\eta}' + W_\eta' P_{W_\eta}. \end{aligned} \quad (49)$$

Furthermore, it follows from Eq. (28) that

$$P'_{\Gamma_\eta} = W'_\eta + \eta \frac{R'}{\mathcal{F}}. \quad (50)$$

The constraints, $\mathcal{H} \approx 0 \approx \mathcal{H}_r$, can be further simplified by using $\mathcal{H}_r \approx 0$ to eliminate $W'_\eta P_{W_\eta}$ from the Hamiltonian constraint. This gives

$$(\mathcal{F}P'_{\Gamma_\eta} + \eta R') \left(\bar{P}_R - \eta \frac{\Gamma_\eta}{\mathcal{F}} \right) \approx 0. \quad (51)$$

Consider the first of the two factors above. Using Eq. (50) to substitute for P'_{Γ_η} , we find

$$\mathcal{F}P'_{\Gamma_\eta} + \eta R' = \mathcal{F} \left(W'_\eta + 2\eta \frac{R'}{\mathcal{F}} \right) = \mathcal{F} \frac{\bar{W}'_\eta}{\sigma}, \quad (52)$$

where $\sigma(R, W_\eta)$ is the integrating factor introduced in the

previous section. But $\bar{W}'_\eta \neq 0$ because it is a null coordinate and is required to increase in time, therefore,

$$\bar{P}_R - \eta \frac{\Gamma_\eta}{\mathcal{F}} \approx 0 \quad (53)$$

is equivalent to the Hamiltonian constraint, $\mathcal{H} \approx 0$. Inserting this into either of the two constraints then gives $P_{W_\eta} \approx \eta \mathcal{F} \bar{P}_R \approx \Gamma_\eta$.

The configuration space is made of the set of variables $\{W_\eta, R, \Gamma_\eta, M_+\}$, whose physical significance is transparent. This is an advantage of Kuchař variables. W_η is a null coordinate, R is the area radius of a point labeled (r, t) , Γ_η is the energy density of the collapsing cloud and M_+ is the mass measured at spatial infinity. M_+ is a constant of the motion and may be viewed as part of the initial data. Our gravity-matter system may be re-written as

$$S = \int dt \left(p_+ \dot{M}_+ + \int_0^\infty dr [P_{W_\eta} \dot{W}_\eta + \bar{P}_R \dot{R} + P_{\Gamma_\eta} \dot{\Gamma}_\eta - N^W (P_{W_\eta} - \Gamma_\eta) - N^r (R' \bar{P}_R - \Gamma_\eta P'_{\Gamma_\eta} + W'_\eta P_{W_\eta})] \right). \quad (54)$$

The canonical action (54) can be a starting point for Dirac quantization. The physical meaning of all the configuration space coordinates is clear: W_η and R locate the hypersurface and Γ_η (along with M_+) determines the matter distribution.

IV. QUANTIZATION

The configuration space consists of two disconnected components: for expanding null matter it is spanned by the set $\{W_+, R, \Gamma_+, M_+\}$ and for collapsing matter by the set $\{W_-, R, \Gamma_-, M_+\}$. In each case the phase-space has non-trivial boundaries,

$$\begin{aligned} M_+ \geq 0, \quad P_Z \leq 0, \quad (Z = -W_\pm) \\ R > 0. \end{aligned} \quad (55)$$

The last is due to the fact that $R(t, r) = 0$ describes the central singularity, i.e., the final singularity for the collapsing solution and the initial singularity for the expanding one. In Dirac's approach, when the phase-space admits trivial boundaries, the canonical momenta P_X , are raised to operator status,

$$P_X \rightarrow -i \frac{\delta}{\delta X} \quad (56)$$

and the constraints are considered as operator restrictions on the state functional. When the boundaries are non-trivial, as is the case in (55), this naive exchange of momenta for functional derivatives in (56) may lead to operators that are not self-adjoint [13], but we will assume here that a suitable measure on the Hilbert space can be found, with respect to

which the above replacement leads indeed to self-adjoint operators. Subject to this caveat, the state-functional obeys

$$-i \frac{\delta \Psi}{\delta W_\eta} = \Gamma_\eta \Psi$$

$$R' \frac{\delta \Psi}{\delta R} + W'_\eta \frac{\delta \Psi}{\delta W_\eta} - \Gamma_\eta \left(\frac{\delta \Psi}{\delta \Gamma_\eta} \right)' = 0. \quad (57)$$

The last constraint is solved by any functional that is a spatial scalar. Consider a solution of this constraint that is of the form

$$\Psi = C_\eta(M_+) \exp \left[i \int_0^\infty dr \Gamma_\eta(r) \cdot \mathcal{K}(\eta, W_\eta, R, M_+) \right], \quad (58)$$

where C_η is a constant depending only on η and M_+ , and \mathcal{K} is an arbitrary complex valued function of its arguments (and not their derivatives) that is to be evaluated so that Ψ satisfies the other constraints. The wave-functional Ψ in Eq. (58) is evidently a spatial scalar because $\Gamma_\eta(r)$ is a spatial density and \mathcal{K} is a spatial scalar. It is therefore a solution of the constraint providing that \mathcal{K} has no explicit dependence on r .

The solution, which agrees with all the constraints considered as operator restrictions on the state functional, is given by

$$\Psi = C_\eta \exp \left[i \int_0^\infty dr \Gamma_\eta(r) [W_\eta(r) + \eta R_*(r)] \right], \quad (59)$$

where $R_*(R, M_+, \Gamma_\eta)$ is a ‘‘tortoise’’-like coordinate defined by

$$R_* = R + 2m \ln \left| \frac{R}{2m} - 1 \right|. \quad (60)$$

It is not, of course, the tortoise coordinate R^* except in a Schwarzschild region when $m(M_+, \Gamma_\eta) = M_+$ is constant.

The parameter η represents the direction of the flow, being $+1$ for outgoing null matter and -1 for incoming matter. The combination $\eta R(r)$ represents the affine parameter, $\xi(r)$. Re-expressing the wave-functional in Eq. (58) to make the dependence on the affine parameter explicit, we find

$$\Psi = C_\eta e^{i \int_0^\infty dr \Gamma_\eta(r) \xi_*(r)} \exp \left[i \int_0^\infty dr \Gamma_\eta(r) \cdot W_\eta(r) \right], \quad (61)$$

where $\xi_* = \eta R_*$.

For $\eta = -1$, $\xi \in (-\infty, 0)$ and $W_\eta = V$, the functionals are of the form

$$\Psi_{-1} = A(M_+) e^{-i \int_0^\infty dr \Gamma_-(r) R_*(r)} e^{i \int_0^\infty dr \Gamma_-(r) V(r)}, \quad (62)$$

and describe collapsing null matter. Likewise, for $\eta = +1$, $\xi \in (0, \infty)$ and $W_\eta = U$, the functionals

$$\Psi_{+1} = B(M_+) e^{i \int_0^\infty dr \Gamma_+(r) R_*(r)} e^{i \int_0^\infty dr \Gamma_+(r) U(r)}, \quad (63)$$

describe expanding null matter.

A given classical collapse problem is specified by a choice of mass function, $m(W_\eta)$, which determines an initial energy distribution, thus a collapse ‘‘model.’’ As an example, we shall consider the function,

$$m(W_\eta) = M_+ \theta(W_\eta - w), \quad (64)$$

where θ is the Heaviside unit step-function and w is constant. The matter energy vanishes when $W_\eta < w$ and is M_+ when $W_\eta \geq w$. The mass function evidently makes sense only as a thin shell that is collapsing toward the center and we must have $W_\eta = V$ ($\eta = -1$). Calling the corresponding mass at spatial infinity M_+^{in} , we find the energy density by differentiating with respect to the ADM label coordinate, r ,

$$\begin{aligned} \Gamma_{\eta \text{in}}(r) &= M_+^{\text{in}} V'(r) \delta(V - v) \\ &= M_+^{\text{in}} \frac{V'(r)}{V'(r)} \delta(r - \tau), \end{aligned} \quad (65)$$

where $\tau(t)$ is the solution of $V(r, t) = v$. Likewise, a thin shell that expands out of the center is represented by the mass function

$$m(W_\eta) = M_+^{\text{out}} \theta(w - W_\eta), \quad (66)$$

for $W_\eta = U$ ($\eta = +1$). The energy density is

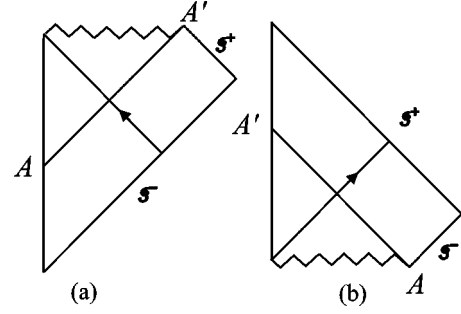


FIG. 2. A thin shell (a) collapsing and (b) expanding.

$$\Gamma_{\eta \text{out}}(r) = -M_+^{\text{out}} U'(r) \delta(u - U)$$

$$= M_+^{\text{out}} \frac{U'(r)}{U'(\tau)} \delta(r - \tau), \quad (67)$$

where $\tau(t)$ is the solution of $U(r, t) = u$. The energy density is always positive and in either case we find

$$\Gamma = M_+^{\text{in(out)}} \delta(r - \tau). \quad (68)$$

The shell trajectories, $\tau = \tau(t)$ are different in the two cases, being $v = \text{const}$ in the first and $u = \text{const}$ in the second. Using these expressions, let us relate the constraints in Eqs. (23) to the constraints that have been used by others to describe thin shells. The dust Hamiltonian and momentum density turn out to be ($P_Z = -\Gamma$)

$$\begin{aligned} \mathcal{H}^d &= \eta \frac{M_+ Z'}{L} \delta(r - \tau) = \eta \frac{p}{L} \delta(r - \tau) \\ \mathcal{H}_r^d &= -M_+ Z' \delta(r - \tau) = -p \delta(r - \tau), \end{aligned} \quad (69)$$

where we have defined $p = M_+ Z'(\tau)$ and therefore $\eta = \text{sgn}(Z') = \text{sgn}(p)$. These expressions were used as a starting point in [14, 16, 18]. The gravitational contributions to the constraints are, of course, the same.

The corresponding classical solutions are represented in the Penrose diagram of Fig. 2, where AA' represents the event horizon in (a) and the Cauchy horizon in (b). Inserting Eq. (68) into Eqs. (62) and (63) one finds that the quantum mechanics of a single collapsing shell is defined on the reduced configuration space (v, R, M_+^{in}) and described by the wave-function

$$\Psi_{-1} = A(M_+^{\text{in}}, v) e^{-i M_+^{\text{in}} R^*(\tau)}, \quad (70)$$

whereas, for an outgoing shell the reduced configuration space is (u, R, M_+^{out}) and the wave-function

$$\Psi_{+1} = B(M_+^{\text{out}}, u) e^{i M_+^{\text{out}} R^*(\tau)}, \quad (71)$$

where R^* as given by Eq. (60) is, in this case, the usual tortoise coordinate.

V. DISCUSSION

In this paper we have examined the collapse and expansion of a null dust cloud of arbitrary mass distribution and shown that there exists a canonical transformation that brings the corresponding Vaidya system to the Kuchař form, in which the dynamics is expressed in terms of embedding variables whose physical meaning is transparent. Written in terms of these variables, the constraints take on a simple form.

The physical content of the solution wave-functionals discussed in the previous section is given by the expectation values of observables. These are, for example, the geometric invariants and matter invariants (such as curvature scalars and trace of the stress energy tensor) and in general any C^∞ function of the phase-space variables, old or new, that weakly commutes with the constraints. They must be written as operators on the Hilbert space, but there are two associated difficulties. Firstly, because they are generally non-linear, there will be operator ordering ambiguities. Secondly, one must ensure that they are self-adjoint with respect to the chosen measure. However, as our transformation between the spaces is explicit, knowing these functions in one system is equivalent to knowing them in the other. It should be noted that a complete set of Dirac observables has been constructed in the case of a single shell [16] where a single degree of freedom is present. For the general case, this issue is quite complicated and will be discussed in the future.

Subject to the condition that suitable boundary conditions or a suitable inner product can be found so that (56) leads to self-adjoint operators, we have found solution wave-functionals for arbitrary mass distributions in each sector (collapse and expansion) independently (as mentioned in the Introduction, solutions with non-trivial mass distributions are essential for the description of issues such as naked singularities and the cosmic censor). These sectors are disjoint, separated by the central singularity (at $R=0$). However, if the solutions are viewed as describing the evolution, in the affine parameter ξ , of a collapsing matter cloud beginning on \mathcal{I}^- , they seem to suggest that it should be possible to define the wave-functional over the entire interval $\xi \in (-\infty, \infty)$ by extending the range of R to include the center. Yet, for any matter distribution, the solution space-time in Eq. (24) ad-

mits a strong curvature singularity at the origin, so the classical dynamics cannot be extended even to it. Any attempt to continue the quantum dynamics through the origin must therefore ensure that at least the expectation values of the observables in the consequent quantum theory are well behaved there. Thus the central singularity would be made harmless by the quantum theory. It can be so if, for example, the wave-functional were to vanish there. In this way the matter would collapse and re-expand through the (benign) center in one continuous history, the solutions being given by Eq. (62) for $\xi \in (-\infty, 0]$ and Eq. (63) for $\xi \in [0, \infty)$. This has been proposed for a single shell by Hájíček and Kiefer [16–18], who merged the two solutions into one bouncing solution. Their bounce was obtained by working in double-null coordinates and employing group quantization techniques (see [16] and [19]) to the problem. Group quantization is beautifully adapted to the quantization of systems with non-trivial boundary restrictions such as those in (55) on the phase space, but its application to problems with more than a few degrees of freedom and in particular to the collapse of general matter distributions, being dependent on the construction of a complete set of observables, remains a topic for future investigation.

The Eddington-Finkelstein coordinates we have employed in this paper present several advantages over the double null system with regard to the problem of collapse or re-expansion of arbitrary matter distributions. The new variables have a clear physical and geometrical meaning. This is useful when comparing the quantum behavior with the classical. Our transformations are explicit, which means that operators and, in particular, observables that are known in one coordinate system can be expressed in the other. The matter-gravity constraints in the new phase space are linear for all matter distributions. This simplification, achieved on the classical level, has allowed us to obtain exact solution wave-functionals describing the respective physical processes.

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