

## Quantum stress tensor in self-similar spherical dust collapse

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We calculate the quantum stress tensor for a massless scalar field in the 2D self-similar spherical dust collapse model which admits a naked singularity. We find that the outgoing radiation flux diverges on the Cauchy horizon. This may have two consequences. The resultant back reaction may prevent the naked singularity from forming, thus preserving cosmic censorship through quantum effects. The divergent flux may lead to an observable signature differentiating naked singularities from black holes in astrophysical observations. [S0556-2821(98)08020-5]

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### I. INTRODUCTION

A proof or disproof of the cosmic censorship hypothesis remains an important unsolved problem in classical general relativity. In physical terms, the hypothesis states that the singularities arising in the gravitational collapse of physically reasonable matter, starting from generic initial conditions, are not visible to an outside observer. Various model examples studied in recent years, especially in spherical collapse, show that either black holes or naked singularities may form during gravitational collapse (for recent reviews see [1]). While these models are not of a general enough nature to establish that the hypothesis does not hold, it still appears reasonable to start enquiring what the physical and possibly observational consequences of these naked singularities could be.

Although the censorship hypothesis is framed within classical general relativity, it is natural to expect that if the classical theory does predict a naked singularity, quantum effects will play a fundamental role in the extremely high curvature regions that are exposed near a naked singularity. When the curvatures approach Planck scales, the final fate of the collapse will be determined by quantum gravitational effects. However, even before the Planck epoch is reached, semiclassical processes (e.g. Hawking radiation and vacuum polarization) will affect the formation of the classical naked singularity. Unlike the quantum gravitational features, at least some of the semiclassical phenomena (corresponding to quantum theory on a classical curved background) are calculable, and these are the subject of the present paper.

Since not much is known about cosmic censorship, quantization of matter fields in the background of a classical dy-

namical spacetime containing a naked singularity has received only sparse attention. Two early works which have inspired some of the recent developments in this regard are those by Ford and Parker [2], and by Hiscock *et al.* [3]. Ford and Parker considered the quantization of a massless scalar field in a four dimensional background and gave an expression for the power radiated to infinity in the geometric optics approximation. One of the spacetimes they considered is the collapse of a dust cloud leading to the formation of a shell-crossing naked singularity. They found in this spacetime that the energy flux of the created scalar particles remains finite up to the time of formation of the naked singularity. They also considered the collapse of charged shells leading to naked singularities, and for this spacetime the flux of created particles is infinite.

Hiscock *et al.* studied quantum effects in the classical background of a collapsing self-similar spherical null dust cloud. This spacetime is described by the Vaidya metric, and admits a naked singularity if the rate of infall of the null dust is below a critical value. For the case in which the Cauchy horizon coincides with the event horizon, they could compute the spectrum of created particles, and the spectrum was found to be non-thermal. For the case in which the two horizons do not coincide, they performed an exact calculation of the vacuum stress tensor in two dimensions and found it to diverge along the Cauchy horizon in a positive fashion. Such a divergence is a likely indicator that the back reaction of the metric will prevent the formation of the naked singularity.

Calculations such as the above can be performed only for those spacetimes where the naked singularity formation has been demonstrated using analytical, rather than numerical, methods. Such analytical examples are rare. In addition to the above mentioned examples, another one which has been studied analytically in recent years is the shell-focusing naked singularity forming in the spherical collapse of dust [4], described by the Tolman-Bondi metric. Shell-focusing naked singularities are usually regarded as more serious violations of cosmic censorship as compared to the shell-crossing solu-

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tion studied, for instance, by Ford and Parker.

In a recent work we have used the Ford and Parker geometric optics formula to compute the radiated power in the model of self-similar dust collapse leading to a shell-focusing naked singularity [5]. The assumption of self-similarity (i.e. the existence of a homothetic Killing vector field in the spacetime) facilitates the integration of null rays, or equivalently, the construction of double null coordinates for the spacetime. Without such a construction such a computation would not be possible. We find that the radiated power diverges, in contrast to the finite flux found by Ford and Parker in the shell-crossing case—this again appears to suggest that shell focusing is a more serious example of a naked singularity, as compared to shell crossing. We have also recently calculated the spectrum of the radiation, and find it to differ significantly from the black body spectrum of Hawking radiation [6].

In the present work, we calculate the 2D quantum stress tensor for a minimally coupled massless scalar field in self-similar spherical dust collapse, in a manner analogous to the calculation of Hiscock *et al.* In two dimensions, the trace anomaly permits an exact calculation of the expectation value without having to resort to geometric optics. Once again, the assumption that the spacetime is self-similar is made. Our results are identical to those of Hiscock *et al.*—we also find a divergence of the outgoing flux on the Cauchy horizon, and the form of the divergence is same as that for the null dust model.

The plan of the paper is as follows. In Sec. II we briefly describe the self-similar dust collapse model. In Sec. III the standard expressions for the quantum stress tensor in the 2D case are recalled. In Sec. IV we construct the double null coordinates for our model and give a calculation showing the divergence of the outgoing flux on the Cauchy horizon. In Sec. V the detailed form of the quantum stress tensor is calculated, so as to find exact expressions that are valid everywhere in the spacetime, including on the Cauchy horizon.

Quantum effects have also been studied in dynamic naked singular spacetimes in string theory inspired gravity models, and here too, features similar to those in general relativistic collapse models (e.g. divergence of stress-tensor on the Cauchy horizon) have been found [7].

## II. THE CLASSICAL SOLUTION

The Tolman-Bondi solution represents a spherically symmetric cloud of dust collapsing under the action of its own gravity. The energy-momentum tensor is diagonal with the pressure terms zero and  $\varepsilon$  as the energy density. The metric is assumed to be diagonal and spherically symmetric. The line element is given by

$$ds^2 = dt^2 - e^{\lambda(r,t)} dr^2 - R(r,t)^2 (d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where  $t$  is coordinate time and  $r$ ,  $\theta$  and  $\phi$  are the comoving space like polar coordinates.  $R(r,t)$  is called the ‘‘area radius’’ as the area of a 2D spherical surface is  $4\pi R^2$ .

The field equations simplify to

$$4\pi\varepsilon(r,t) = \frac{M'(r)}{R^2(r,t)R'(r,t)} \quad (2)$$

and

$$\dot{R}^2 = f(r) + \frac{2M(r)}{R}, \quad (3)$$

where the dot indicates partial derivative with respect to time  $t$ , and the prime indicates partial derivative with respect to radial coordinate  $r$ .  $f(r)$  is defined through the relation

$$e^{\lambda(r,t)} = \frac{R'^2}{1+f(r)}. \quad (4)$$

$M(r)$  and  $f(r)$  are smooth functions with the restriction that  $M(r) \geq 0$  for every  $r$  and  $f(r) \geq -1$  for every  $r$ .  $M(r)$  is the mass within a sphere and  $f(r)$  resembles total energy at  $r$ . Hence they are called mass and energy functions, respectively.

It turns out that  $r = \text{const}$  are geodesics in this spacetime. Thus we are using a comoving frame of reference, as mentioned above.

The evolution of the cloud according to the above equations results in the formation of a curvature singularity and the Kretschmann scalar diverges in the approach to the singularity. The singularity at  $r=0$  (usually known as the central singularity) is known to be a naked singularity for certain initial data, and a covered singularity for other initial data [4].

We will be interested in those initial conditions for which the collapse of the cloud is self-similar, i.e. it admits a homothetic Killing vector field [8]. It can be shown that this corresponds to choosing  $2M(r) = \lambda r$ , where  $\lambda$  is a constant, and  $f(r) = 0$ . We assume that the central singularity forms at  $t=0$ , and that at  $t=0$  the scaling  $R(r,0) = (3/2)^{2/3} \lambda^{1/3} r$  holds. (Actually the scaling  $R=r$  is simpler and conventional; in our case the present choice is necessitated by our having to extend the Tolman-Bondi coordinates in the exterior of the cloud, as discussed in Sec. IV.) The self-similar collapse solution is then

$$R = (2M(r))^{1/3} \left( \frac{3}{2}(r-t) \right)^{2/3}. \quad (5)$$

Collapse begins at some coordinate time  $t = -t_i$ , and the initial density profile  $\varepsilon(r, -t_i) \equiv \rho(r)$  can be shown to be of the form

$$\rho(r) = \rho_0 + \frac{1}{3!} \rho_3 r^3 + \frac{1}{6!} \rho_6 r^6 + \dots \quad (6)$$

$t(\partial/\partial t) + r(\partial/\partial r)$  is the homothetic Killing vector field. The central singularity forming in self-similar collapse is naked for the range

$$\frac{\lambda^{3/2}}{12} \leq \frac{26}{3} - 5\sqrt{3} \quad (7)$$

and covered when  $\lambda$  exceeds this value. The naked singularity is locally as well as globally naked [5]. The derivative  $R'$  in the case of self-similar collapse will be needed subsequently, and is given by

$$R' = \left(\frac{9\lambda}{4}\right)^{1/3} \frac{1-z/3}{(1-z)^{1/3}}, \quad (8)$$

where  $z = t/r$ .

The geometry to the exterior of the boundary  $r_b$  of the cloud is Schwarzschild, with the Schwarzschild mass  $M$  given by  $M = M(r_b) = \lambda r_b/2$ . The first and the second fundamental forms for the two spacetime metrics can be easily shown to match. The comoving time measured on the boundary, which is the geodesic  $r = r_b$ , is the same independently of whether it is thought of as comoving time of the self-similar metric or that of the Schwarzschild exterior.

We shall work with the two-dimensional Tolman-Bondi spacetime obtained by setting the angular part  $d\Omega^2 = 0$  in the spherical metric given in Eq. (1).

### III. THE 2D QUANTUM STRESS TENSOR

We now calculate the quantum stress tensor  $\langle 0_{\text{out}} | T_{\mu\nu} | 0_{\text{in}} \rangle$  for a minimally coupled massless scalar field  $\phi$  in the background of a 2D collapsing Tolman-Bondi dust cloud, in a manner analogous to the calculation of Hiscock *et al.* [3]. As is well known, this stress tensor exhibits a trace anomaly [9], and the trace of  $\langle T_{\mu\nu} \rangle$  is equal to  $\mathcal{R}/24\pi$ , where  $\mathcal{R}$  is the Ricci scalar for the background spacetime. Furthermore, according to the Wald axioms [10], the quantum stress tensor is conserved,

$$\langle T^{\mu\nu} \rangle_{; \nu} = 0. \quad (9)$$

Any two-dimensional spacetime is conformally flat, and its metric can hence be expressed using double null coordinates  $\hat{u}$  and  $\hat{v}$  as

$$ds^2 = C^2(\hat{u}, \hat{v}) d\hat{u} d\hat{v}. \quad (10)$$

It can be shown that the Wald axioms and the trace anomaly lead to the following form of the quantum stress tensor in two dimensions [9]:

$$\langle T_{\hat{u}\hat{u}} \rangle = -\frac{1}{12\pi} C \left( \frac{1}{C} \right)_{,\hat{u},\hat{u}} + A(\hat{u}), \quad (11)$$

$$\langle T_{\hat{v}\hat{v}} \rangle = -\frac{1}{12\pi} C \left( \frac{1}{C} \right)_{,\hat{v},\hat{v}} + B(\hat{v}), \quad (12)$$

$$\langle T_{\hat{u}\hat{v}} \rangle = \frac{\mathcal{R}C^2}{96\pi}. \quad (13)$$

Expressing all quantities in terms of null coordinates makes the interpretation of  $\langle T_{\mu\nu} \rangle$  easier. The scalar field being massless, its modes propagate along null rays. The functions  $A(\hat{v})$  and  $B(\hat{v})$  are ‘‘constants’’ of integration which can be shown to be zero from the following argument.

We choose these null coordinates  $\hat{u}$  and  $\hat{v}$  according to the following prescription. The initial quantum state of the scalar field is taken to be the standard Minkowski vacuum on  $I^-$ . This state is also the vacuum with respect to the normal modes of the scalar wave equation in  $\hat{u}, \hat{v}$  coordinates. Since spacetime is asymptotically flat and there is no incoming radiation from  $I^-$  it means that  $\langle T_{\hat{v}\hat{v}} \rangle$  is zero on  $I^-$  and also that the first term on the right hand side of Eq. (12) vanishes there; showing that  $B(\hat{v})$  is zero. Further, at the center ( $r = 0$ ) of the cloud we require  $\hat{u} = \hat{v}$  (see e.g. Section 8.1 of [12]) and  $\langle T_{\hat{u}\hat{u}} \rangle = \langle T_{\hat{v}\hat{v}} \rangle$ , which gives  $A(\hat{u}) = 0$ .

### IV. OUTGOING FLUX ON THE CAUCHY HORIZON

In terms of the similarity parameter  $z = t/r$ , a set of double null coordinates for the self-similar interior metric is

$$\eta = r e^{\int dz/(z-R')}, \quad \zeta = r e^{\int dz/(z+R')}. \quad (14)$$

In terms of these null coordinates the 2D part of the metric (1) can be written as

$$ds^2 = \frac{e^{2\phi}}{\zeta \eta} \left( 1 - \frac{R'^2}{z^2} \right) d\zeta d\eta, \quad (15)$$

where  $t = e^\phi$ . It is understood that  $z$  is an implicit function of  $\zeta$  and  $\eta$ . We now want to relate these null coordinates to the global null coordinates  $\hat{u}$  and  $\hat{v}$ . We will assume that there exist relations  $\hat{u} = \hat{u}(\eta)$  and  $\hat{v} = \hat{v}(\zeta)$ . In the exterior of the cloud the Eddington-Finkelstein double null coordinates  $u$  and  $v$  for the Schwarzschild metric

$$ds^2 = \left( 1 - \frac{2M}{R} \right) dudv + R^2 d\Omega^2 \quad (16)$$

are given by

$$u = T - R^*, \quad v = T + R^*, \quad R^* = R + 2M \ln \left( \frac{R}{2M} - 1 \right). \quad (17)$$

Since the ingoing scalar field modes in Schwarzschild coordinates reduce to the standard ingoing Minkowski modes at past null infinity, we have  $\hat{v} = v$ . The Tolman-Bondi coordinate system can be extended to the exterior of the cloud, and the Schwarzschild metric  $(T, R)$  can be expressed in Tolman-Bondi coordinates  $(t, r)$  as follows [11]:

$$T = t - 2\sqrt{2MR} - 2M \ln \frac{\sqrt{R} - \sqrt{2M}}{\sqrt{R} + \sqrt{2M}}, \quad (18)$$

$$R = (2M)^{1/3} \left( \frac{3}{2}(r-t) \right)^{2/3}. \quad (19)$$

At the boundary  $r_b$  of the cloud we get

$$R = (2M)^{1/3} \left( \frac{3}{2} r_b \right)^{2/3} (1-z)^{2/3} \quad (20)$$

and this substitution in Eq. (18), along with  $t=ZR_b$ , gives  $T=T(z)$  on the boundary. Hence  $v=T(z)+R^*(z)=v(z)$  on the boundary. Also,

$$\frac{d\zeta}{dz} = \frac{\zeta}{z+R'}, \quad (21)$$

giving on the boundary the relation

$$\frac{d\hat{v}}{d\zeta} = \frac{dv}{d\zeta} = \frac{dv/dz}{d\zeta/dz} \equiv E(z, r_b). \quad (22)$$

This gives the relation between  $\hat{v}$  and  $\zeta$  on the boundary, and the same functional relation holds inside the cloud. Hence we get that  $d\hat{v}/d\zeta=B(z, r; r_b)$  inside the cloud. It is understood in the above equation that  $z$  is a function of  $\zeta$ , and the function  $B(z, r; r_b)$  is obtained from  $E(z, r_b)$  by expressing  $\zeta$  in terms of  $r$  and  $z$  in the cloud. It may be noted that relation (22) holds on the boundary and is subsequently extended inside the cloud. The latter is possible because we could write Eq. (22) in terms of null coordinates.

In order to relate  $\hat{u}$  to  $\eta$  we begin by noting that

$$\frac{\zeta}{\eta} = \frac{e^{\int dz/(z+R')}}{e^{\int dz/(z-R')}}. \quad (23)$$

Consider the center  $r=0$ . Here, for  $t<0$ ,  $z$  diverges ( $z \rightarrow -\infty$ ), and the ratio on the right hand side of the above equation goes to a constant, say  $\epsilon$ . ( $\epsilon$  may be equal to unity, but that does not affect the argument.) Thus, at  $r=0$ ,  $\zeta=\epsilon\eta$ . We also assume that at the center,  $\hat{u}=\hat{v}$ , which gives, at  $r=0$ ,

$$\frac{d\hat{u}}{d\zeta} = \epsilon \frac{d\hat{v}}{d\zeta} = \epsilon B(z, r=0). \quad (24)$$

Hence, at a point  $(t, r)$  in the cloud the relation between  $\hat{u}$  and  $\eta$  is given by

$$\frac{d\hat{u}}{d\eta} = \epsilon B(z, r). \quad (25)$$

Inside the cloud we can now write the metric as

$$ds^2 = \frac{e^{2\phi}}{\zeta\eta} \left(1 - \frac{R'^2}{z^2}\right) \frac{d\zeta}{d\hat{v}} \frac{d\eta}{d\hat{u}} d\hat{u}d\hat{v}. \quad (26)$$

We will need  $\partial\phi/\partial\eta$  and  $\partial z/\partial\eta$ , and these derivatives can be had by inverting the relations (14). We get

$$\frac{\partial\phi}{\partial\eta} = \frac{z-R'}{2z\eta}, \quad \frac{\partial z}{\partial\eta} = \frac{z^2-R'^2}{2\eta R'}. \quad (27)$$

We thus have in the cloud

$$C^2(\hat{u}, \hat{v}) = \frac{e^{2\phi}}{\zeta\eta} \left(1 - \frac{R'^2}{z^2}\right) \frac{d\zeta}{d\hat{v}} \frac{d\eta}{d\hat{u}} \equiv \frac{e^{2\phi}}{\zeta\eta} \psi(z, \phi), \quad (28)$$

where

$$\psi(z, \phi) = \frac{1-R'^2/z^2}{\epsilon B^2(z, \phi)}. \quad (29)$$

We can compute  $\partial C/\partial\eta$  to get

$$\frac{\partial C}{\partial\eta} = \frac{C}{2\eta} \left( \frac{z-R'}{z} - 1 + \frac{z^2-R'^2}{2R'} \frac{\partial\psi/\partial z}{\psi} + \frac{z-R'}{2z} \frac{\partial\psi/\partial\phi}{\psi} \right). \quad (30)$$

We are interested in computing the quantum stress tensor in the approach to the Cauchy horizon, which is given by the smaller root  $z_-$  of  $(z-R')=0$  [5]. In this limit,  $\eta$  diverges, and the function  $B(z, \phi)$  and its derivatives are well behaved and finite. As a result, we get that as  $z \rightarrow z_-$ ,

$$\frac{\partial C}{\partial\eta} = -\frac{C}{2\eta} \frac{dR'}{dz}, \quad \frac{\partial^2 C}{\partial\eta^2} = \frac{C}{2\eta^2} \frac{dR'}{dz} \left(1 + \frac{1}{2} \frac{dR'}{dz}\right). \quad (31)$$

The outgoing radiation flux  $T_{\hat{u}\hat{u}}$ , which is given by Eq. (11), can be written as

$$T_{\hat{u}\hat{u}} = -\frac{1}{C^2} (2C_{\hat{u}}^2 - CC_{\hat{u}\hat{u}}), \quad (32)$$

and by noting that

$$C_{,\hat{u}} = \frac{d\eta}{d\hat{u}} C_{,\eta} = \frac{C_{,\eta}}{\epsilon B}, \quad (33)$$

and by similarly converting  $C_{,\hat{u}\hat{u}}$  we find that in the approach to the Cauchy horizon

$$T_{\hat{u}\hat{u}} = \frac{dR'/dz(2-dR'/dz)}{4\epsilon^2 B^2 \eta^2}, \quad T_{\eta\eta} = \frac{dR'/dz(2-dR'/dz)}{4\eta^2}. \quad (34)$$

On the Cauchy horizon  $\eta$  diverges and hence the  $(\zeta, \eta)$  coordinate system is singular, and the metric goes to zero as  $(z-R')/\eta$ . Hence we transform from  $\eta$  to a new coordinate  $\bar{\eta}$  that is well behaved on the Cauchy horizon so that in the approach to the horizon

$$\frac{\partial\eta}{\partial\bar{\eta}} \equiv \frac{\eta}{h(\eta)} \approx \frac{\eta}{z-R'}. \quad (35)$$

That such a transformation is possible is demonstrated in Sec. V. In these new coordinates  $(\bar{\eta}, \zeta)$  the outgoing radiation flux, in the approach to the Cauchy horizon, is

$$\begin{aligned} T_{\bar{\eta}\bar{\eta}} &= \left(\frac{\partial\eta}{\partial\bar{\eta}}\right)^2 T_{\eta\eta} \approx \frac{dR'/dz(2-dR'/dz)}{4(z-R')^2} \\ &= \frac{dR'/dz(2-dR'/dz)}{4(z-z_-)^2}. \end{aligned} \quad (36)$$

The outgoing flux  $T_{\bar{\eta}\bar{\eta}}$  thus diverges on the Cauchy horizon. Furthermore, it diverges in a positive fashion—this follows from noting that

$$\frac{dR'}{dz} = \left( \frac{4\sqrt{\lambda}}{9} \right)^{2/3} \frac{z}{2(1-z)^{4/3}} > 0. \quad (37)$$

This quantity can be shown by a numerical check to be less than unity on the Cauchy horizon.

We now demonstrate the divergence of the outgoing flux on the Cauchy horizon in the Schwarzschild region outside the cloud, for which the metric is Eq. (16), and  $R$  is an implicit function of  $u$  and  $v$ . We have to relate  $u$  to the global coordinate  $\hat{u}$  (we already know that  $\hat{v}=v$ ). Since we know  $d\hat{u}/d\eta$  and  $du/d\eta$ , we can find  $du/d\hat{u}$  as

$$\frac{du}{d\hat{u}} = \frac{du/d\eta}{d\hat{u}/d\eta}. \quad (38)$$

To get  $du/d\eta$  explicitly we recall that  $u = T - R^* = u(z, r_b)$  on the boundary, from which  $du/dz$  on the boundary can be calculated. Also,

$$\frac{d\eta}{dz} = \frac{\eta}{z - R'}, \quad (39)$$

so that

$$\frac{du}{d\eta} = \frac{du/dz}{d\eta/dz} = \frac{z - R'}{\eta} \frac{du}{dz} \quad (40)$$

on the boundary. Hence

$$\frac{du}{d\hat{u}} \equiv \chi(z)(z - R') \quad (41)$$

on the boundary. We assume this relation to extend in the Schwarzschild region outside, where  $z = t/r$  is related to the Schwarzschild coordinates through the relations (18) and (19), and is to be thought of as a function of  $u$ , by putting  $r = r_b$  and imagining  $t$  to be a function of  $u$ . The function  $\chi(z)$  is well behaved on the Cauchy horizon, and hence  $du/d\hat{u}$  goes to zero as  $(z - R')$  on the Cauchy horizon. We may now write the 2D part of the Schwarzschild metric in global coordinates as

$$ds^2 = C(\hat{u}, \hat{v}) d\hat{u} d\hat{v} = \left( 1 - \frac{2M}{R} \right) \frac{du}{d\hat{u}} d\hat{u} d\hat{v}. \quad (42)$$

It can be checked that  $\partial C/\partial \hat{u}$  and  $\partial^2 C/\partial \hat{u}^2$  both go to zero as  $du/d\hat{u}$  as the Cauchy horizon is approached. From Eq. (32) it follows that  $T_{\hat{u}\hat{u}}$  goes to a finite value on the horizon. Transforming to  $(u, v)$  coordinates we get

$$T_{uu} = \left( \frac{\partial \hat{u}}{\partial u} \right)^2 T_{\hat{u}\hat{u}} \sim \frac{1}{(z - R')^2} \sim \frac{1}{(z - z_-)^2} \quad (43)$$

showing the divergence of  $T_{uu}$  on the Cauchy horizon in the Schwarzschild region.

## V. EXACT CALCULATION OF THE QUANTUM STRESS TENSOR

In this section we compute exact expressions for the quantum stress tensor which are valid everywhere in the spacetime, including the approach to the Cauchy horizon. We make the following choice of the null coordinates in the interior, for convenience:

$$U = \ln \eta = \ln r + \int \frac{1}{z - R'(z)} dz, \quad (44)$$

$$V = \ln \zeta = \ln r + \int \frac{1}{z + R'(z)} dz.$$

We assume that the internal and external null coordinates are related by

$$U = \alpha(u), \quad v = \beta(V). \quad (45)$$

We impose the requirement that the field modes vanish at the center of the cloud  $V = U + 2R_0$  where  $R_0$  is a constant. Comparing the solutions to the equations of motion at this one-dimensional locus, we obtain  $\hat{u} = \hat{v}$  at the center. Furthermore, on the same locus,

$$\hat{u} = v = \beta(V). \quad (46)$$

As the relationship above is valid on a 1D region, it should be valid all over spacetime. Hence

$$\hat{u} = \beta(U + 2R_0) \quad (47)$$

all over spacetime. Also, it is trivial to see that

$$\hat{u} = \beta[\alpha(u) + 2R_0].$$

Having obtained all the coordinate relations one can transform the quantum stress tensor given by Eqs. (11)–(13), expressing the components in whichever coordinate system one finds convenient. We express the components in the  $(U, V)$  coordinates in the interior and  $(u, v)$  coordinates in the exterior. We get

$$\langle T_{UU} \rangle = F_U(\beta') - F_U(A^2), \quad (48)$$

$$\langle T_{VV} \rangle = F_V(\beta') - F_V(A^2), \quad (49)$$

and

$$\langle T_{UV} \rangle = -\frac{1}{24\pi} (\ln A^2)_{,U,V}, \quad (50)$$

where

$$F_x(y) = \frac{1}{12\pi} \sqrt{y} \left( \frac{1}{\sqrt{y}} \right)_{,x,x}, \quad (51)$$

and  $ds^2 = A^2(U, V) dU dV$  is the interior Tolman-Bondi metric.

In the Schwarzschild region we get

$$\langle T_{uu} \rangle = -F_u(D^2) + \alpha'^2 F_U(\beta') + F_u(\alpha'), \quad (52)$$

$$\langle T_{vv} \rangle = -F_v(D^2), \quad (53)$$

$$\langle T_{uv} \rangle = -\frac{1}{24\pi} (\ln(D^2))_{,u,v}, \quad (54)$$

where  $ds^2 = D^2(u,v)dudv$  is the line element outside the cloud. The  $\beta'$  in Eqs. (48) and (52) is to be understood as  $\beta'(U+2R_0)$ .

The advantage of writing the quantum stress tensor in this form is that this allows for an easy comparison with the quantum stress tensor for black-hole evaporation. Thus one may note that the 2D tensor for Hawking radiation can be written in precisely the above form (see Sec. 8.2 in [12]). The various terms in the stress tensor have the following interpretation. Field modes starting from  $I^-$  pass through the cloud and reach  $I^+$ . While going in, they make a contribution  $F_V(\beta')$  to  $\langle T_{VV} \rangle$ . While they are outgoing, they contribute an amount  $F_U(\beta')$  to  $\langle T_{UU} \rangle$ . Being affected by the cloud, they contribute now even in the Schwarzschild region as  $\alpha'^2 F_U(\beta')$ . These modes do not carry any new information about the singular boundary. Apart from this effect, the quantum stress tensor contains vacuum polarization terms. They are  $-F_v(D^2)$ ,  $-F_u(D^2)$ ,  $-F_U(A^2)$  and  $-F_V(A^2)$  in the respective components of the stress tensor. Finally,  $F_u(\alpha')$  represents the Hawking radiation contribution.

We now obtain the explicit expressions for the quantum stress tensor in terms of the null coordinates, all over the spacetime. Consider first the interior metric. We define  $z = 1 - q^3$  and calculate

$$A^2 = t^2 \left[ 1 - \left( \frac{9\lambda}{4} \right)^{2/3} \left( \frac{2+q^3}{3q(1-q^3)} \right)^2 \right], \quad (55)$$

$$(A^2)_{,U} = A^2 \left[ \frac{2}{9} \left( \frac{9\lambda}{4} \right)^{1/3} \left( \frac{q^3-1}{q^4} \right) + 1 \right], \quad (56)$$

$$(A^2)_{,V} = A^2 \left[ \frac{2}{9} \left( \frac{9\lambda}{4} \right)^{1/3} \left( \frac{1-q^3}{q^4} \right) + 1 \right], \quad (57)$$

$$F_U(A^2) = \frac{1}{12\pi} \left\{ \frac{1}{2} \left[ \frac{2}{9} \left( \frac{9\lambda}{4} \right)^{1/3} \left( \frac{q^3-1}{q^4} \right) + 1 \right]^2 - \frac{2}{9} \left( \frac{9\lambda}{4} \right)^{2/3} \frac{(q^3+2)(q^3-4)}{18q^8} \frac{A^2}{t^2 - A^2} \right\}, \quad (58)$$

$$F_V(A^2) = \frac{1}{12\pi} \left\{ \frac{1}{2} \left[ \frac{2}{9} \left( \frac{9\lambda}{4} \right)^{1/3} \left( \frac{1-q^3}{q^4} \right) + 1 \right]^2 - \frac{2}{9} \left( \frac{9\lambda}{4} \right)^{2/3} \frac{(q^3+2)(q^3-4)}{18q^8} \frac{A^2}{t^2 - A^2} \right\}. \quad (59)$$

Next, define  $x$  by the relation

$$\frac{r_b - v}{2M} = \frac{2}{3x^3} + \frac{2}{x} - \frac{1}{x^2} - 2 \ln \left| \frac{1+x}{x} \right|. \quad (60)$$

$\beta'$  can be shown to be

$$\beta'(V) = \frac{2M}{1+x} \left( \frac{1}{\lambda} - \frac{2}{3x^3} + \frac{x}{\lambda} + \frac{1}{3x^2} \right). \quad (61)$$

Calculating  $F_V(\beta')$  yields

$$F_V(\beta') = \frac{1}{12\pi} \left[ \beta'^2 \left( \frac{8x^7 + 7x^8}{16(2M)^2} \right) - \beta' \left( \frac{4x^3}{\lambda} - \frac{2}{3} \right) \times \left( \frac{x^4(1+x)}{8(2M)} \right) + \frac{1}{16} \left( 2 + \frac{x^4}{\lambda} - \frac{2x}{3} \right)^2 \right]. \quad (62)$$

$F_U(\beta')$  is the same expression but with  $x$  related to the retarded time as

$$\frac{r_b - \beta(\alpha(u) + 2R_0)}{2M} = \frac{2}{3x^3} + \frac{2}{x} - \frac{1}{x^2} - 2 \ln \left| \frac{1+x}{x} \right|. \quad (63)$$

It can be easily shown that  $F_U(\beta')$  turns out to be finite on the Cauchy horizon.

The vacuum polarization depends only on the ratio  $z$ , which reflects the fact that the cloud is self-similar. The contribution of the modes from past infinity is finite inside the cloud.

Now, we turn to the Cauchy horizon. This is the outer homothetic Killing horizon and turns out to be the locus  $A = 0$  in the interior. This presents a technical difficulty with the coordinate system as the metric becomes non-invertible on the Cauchy horizon. This problem is solved by transforming to another null system ( $W$  and  $Z$ ) described below, and a better physical picture is obtained.

Let  $W = e^{-\tau U}$  and  $Z = e^{\tau V}$  where

$$\tau = -1 + \frac{dR'(z)}{dz} \Big|_{z=z_-}. \quad (64)$$

The metric component in the  $W$  and  $Z$  system becomes  $\tau^{-2} e^{\tau(U-V)} A^2$ . From expressions for  $U$ ,  $V$ , and  $A$ , this becomes

$$-\frac{1}{\tau^2} \exp \left[ \int \frac{2\tau R'(z)}{z^2 - R'^2(z)} dz \right] t^2 \left( 1 - \frac{R'^2(z)}{z^2} \right). \quad (65)$$

Expanding about the Cauchy horizon  $z = z_-$ , it becomes

$$-\frac{1}{\tau^2} \left[ \frac{1}{z - z_-} + O(z - z_-)^0 \right] \times \left[ \frac{2(z - z_-)}{z_-} \left( 1 - \frac{dR'(z)}{dz} \Big|_{z=z_-} \right) \right] \times t^2 + O(z - z_-)^2. \quad (66)$$

This simplifies to

$$-\frac{2}{\tau^2}(1-dR'(z)/dz|_{z=z_-})t^2+O(z-z_-). \quad (67)$$

This shows that the new component is non-zero and finite at the Cauchy horizon. Also,

$$\frac{dU}{dW} = -\frac{r^\tau}{\tau} \exp\left[\int \frac{\tau}{z-R'(z)} dz\right]. \quad (68)$$

This can be shown to be

$$-\frac{1}{\tau}(z-z_-)^{\tau(1-dR'(z)/dz|_{z=z_-})^{-1}}+O(z-z_-). \quad (69)$$

Using the definition of  $\tau$ , one now obtains

$$\langle T_{WW} \rangle = \frac{-F_U(A^2)+F_U(\beta')}{\tau^2(z-z_-)^2}+O(z-z_-). \quad (70)$$

The numerator of the first term may be shown to be finite. The  $\langle T_{ZZ} \rangle$  contribution remains finite. In the  $\langle T_{WW} \rangle$  component, both the vacuum polarization as well as the contribution from the modes from past infinity diverge. The divergence is inversely proportional to the square of separation in  $z$  from the horizon, a feature also observed in collapse of a null dust shell. Thus we reproduce the results of the previous section, while also obtaining an exact form for  $\langle T_{\mu\nu} \rangle$ .

Consider next the exterior Schwarzschild region. We first compute the Hawking radiation contribution  $F_U(\alpha')$ . Let  $w$  be related to the advanced time coordinate as follows:

$$\frac{r_b-u}{2M} = \frac{2}{3w^3} + \frac{2}{w} + \frac{1}{w^2} + 2 \ln \left| \frac{1-w}{w} \right|. \quad (71)$$

The retarded time relation between interior and exterior regions becomes

$$\frac{1}{\alpha'(u)} = \frac{2M}{1-w} \left( \frac{1}{\lambda} - \frac{2}{3w^3} - \frac{w}{\lambda} - \frac{1}{3w^2} \right). \quad (72)$$

The expression for  $F_u(\alpha')$  becomes

$$\begin{aligned} F_u(\alpha') = & \frac{1}{12\pi} \left[ -\frac{\alpha'^2}{4} \left( 1 - \frac{w^2}{2\lambda} + \frac{w}{3} \right)^2 \right. \\ & + \frac{\alpha'}{4(2M)} \left( \frac{w^4}{3} + \frac{2w^8}{\lambda} - \frac{2w^7}{\lambda} - \frac{w^5}{3} \right) \\ & \left. + \frac{1}{16(2M)^2} (8w^7 - 7w^8) \right]. \quad (73) \end{aligned}$$

Let us recall how the Hawking radiation result is recovered from this expression. One can see that as  $w$  tends to  $1^-$ ,  $u$  tends to  $\infty$ , trivially by inspecting the definition of  $w$  in terms of  $u$ . This is the event horizon. Naturally, one would be interested in the limit of  $F_u(\alpha')$  in such a case. It is easy to see that the first and second terms vanish in this limit. The third term becomes  $1/16 \times (2M)^{-2}$ . Thus in the approach to the event horizon,

$$F_u(\alpha') = \frac{1}{48\pi} \times (4M)^{-2}. \quad (74)$$

This is the well known inverse square dependence on Hawking temperature which leads to the interpretation of  $F_u(\alpha')$  as the Hawking radiation contribution.

We show next that this expression behaves very differently when the classical collapse ends in a naked singularity rather than a black hole. Now there is a Cauchy horizon which lies outside the event horizon, and  $u$  takes a finite value on the Cauchy horizon. As we know, the Cauchy horizon occurs when  $R'(z)-z=0$  in the interior. This means that the time coordinate of the intersection of the Cauchy horizon on the boundary  $r=r_b$  is given by

$$\frac{t}{r_b} - R' \left( \frac{t}{r_b} \right) = 0. \quad (75)$$

The comoving time  $t$  is the same along the boundary whether taken from the interior or Schwarzschild exterior because of matching of the first fundamental form. Therefore, when  $r=r_b$ ,

$$t = \left( 1 - \frac{2\lambda}{3w^3} \right) r_b. \quad (76)$$

Using this in the locus of the Cauchy horizon, one finds that

$$\frac{1}{\lambda} - \frac{2}{3w^3} - \frac{w}{\lambda} - \frac{1}{3w^2} = 0. \quad (77)$$

This holds all over the Cauchy null ray in the exterior, as  $w$  is constant once it is known at the event of intersection with the boundary.

Going back to  $F_u$ , one finds that the third term is finite on the Cauchy horizon. Apart from  $\alpha'$ , the rest of the factors in the first and the second term are finite in the limit.  $\alpha'$  diverges in this limit. The first term goes as  $\alpha'^2$  and hence gives the dominant divergence. To check its behavior near the Cauchy horizon,  $\alpha'$  is written as follows:

$$\alpha'(u) = \frac{w^4 - w^3}{3(2M)} \left[ 1 - \frac{3w^3}{\lambda} + \frac{3w^4}{\lambda} + w \right]^{-1}. \quad (78)$$

We define

$$z_{\text{out}}(w) = 1 - \frac{2\lambda}{3w^3}. \quad (79)$$

It is physically analogous to the ratio  $z=t/r$  of the interior. Then  $\alpha'^2$  behaves like

$$[z_{\text{out}}(w) - z_{\text{out}}(w \text{ at the Cauchy horizon})]^{-2}. \quad (80)$$

Thus  $F_u(\alpha')$  has an inverse square divergence in the above sense. This occurs because of the first term in Eq. (73). The second term diverges only linearly as  $\alpha'$ .

Finally, if we go back to expression (52) for the outgoing flux, we can conclude that it diverges on the Cauchy horizon. The vacuum polarization term goes to zero at  $I^+$ . The components (53) and (54) for the stress tensor are the same as in the black hole case.

It is important to note here that the Cauchy horizon is an ordinary null ray ( $u = \text{const}$ ). The feature of inverse square divergence, in this case of a function resembling the ratio  $z$  of the interior [called  $z_{\text{out}}(w)$ ], is observed for the contribution of the modes from past infinity. The Hawking radiation contribution goes as inverse square of the function  $z_{\text{out}}$ .

As far as the Cauchy horizon is concerned, if one ignores the effect of modes from past infinity, the vacuum polarization diverges in the interior but remains finite outside, the Hawking radiation diverging there. This suggests that field modes originating from the singular boundary, in the naked case, produce energy momentum on encountering a high curvature region and transfer it to the exterior of the cloud as Hawking radiation.

Our results support our earlier 4D flux calculation, which was carried out in the geometric optics approximation [5]. The inverse square divergence obtained is identical to what is found in the collapse of a shell of self-similar null dust and could be generic to self-similar collapse models. The diver-

gence also serves to distinguish a naked singularity from a black hole, and may be of significance if naked singularities do occur in nature.

The divergence on the Cauchy horizon suggests that when the back reaction of the flux on the metric is taken into account, the formation of the naked singularity will be avoided. This is an example of the instability of the Cauchy horizon and a possible way of preserving cosmic censorship.

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