

Toward a midisuperspace quantization of LeMaître-Tolman-Bondi collapse models

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LeMaître-Tolman-Bondi models of spherical dust collapse have been used and continue to be used extensively to study various stellar collapse scenarios. It is by now well known that these models lead to the formation of black holes and naked singularities from regular initial data. The final outcome of the collapse, particularly in the event of naked singularity formation, depends very heavily on quantum effects during the final stages. These quantum effects cannot generally be treated semiclassically as quantum fluctuations of the gravitational field are expected to dominate before the final state is reached. We present a canonical reduction of LeMaître-Tolman-Bondi space-times describing the marginally bound collapse of inhomogeneous dust, in which the physical radius R , the proper time of the collapsing dust τ , and the mass function F are the canonical coordinates $R(r)$, $\tau(r)$ and $F(r)$ on the phase space. Dirac's constraint quantization leads to a simple functional (Wheeler-DeWitt) equation. The equation is solved and the solution can be employed to study some of the effects of quantum gravity during gravitational collapse with different initial conditions.

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I. INTRODUCTION

In the standard treatment of Hawking radiation [1] from black holes, one begins with a black hole that is formed in some classical model of gravitational collapse, surrounds the hole by a quantum field and examines the propagation of this field on the classical background provided by the hole. The quantum field behaves as a thermometer. The quantum modes within the horizon are averaged over and one finds that the black hole radiates thermally leading to its "evaporation." The temperature that characterizes the evaporation of a black hole is inversely proportional to its mass and the radiation flux is inversely proportional to its mass square. For an astrophysical black hole, the radiation flux and temperature are therefore so small that the semi-classical approximation is expected to be an adequate description of the evaporation until the hole is roughly of Planck dimensions at which point higher order quantum gravity effects will undoubtedly become important. This means that the final state of the black hole will depend on quantum gravity. The black hole may evaporate completely or it may leave a remnant. If it evaporates completely, it is important to understand what happens to the information that was initially trapped within its horizon.

Classical models of collapse also lead to the formation of naked singularities for regular initial data [2]. One may then ask if the quantum radiation from naked singularities is similar to the radiation from black holes. By surrounding a clas-

sical naked singularity with a quantum field and examining its quantum modes at null infinity one finds that the evaporation of a naked singularity is qualitatively distinct from that of a black hole [3–6]. The radiation flux diverges as the Cauchy horizon [7–11] is approached and the spectrum of the radiation is non-thermal [5,12], falling off as the inverse frequency. Contrary to the case of an astrophysical black hole, the (divergent) flux should be observable and the unique spectrum should serve to distinguish objects undergoing this type of collapse from other celestial emitters. However, a closer look at the semi-classical approximation just described reveals that the flux of radiation is essentially negligible (on the order of one Planck mass) until about one Planck time before the putative Cauchy horizon is reached [13]. The semi-classical approximation therefore signals a quantum instability of naked singularities (and therefore a mechanism for the cosmic censor) but its quantitative predictions must be tested in a full quantization of *all* the degrees of freedom, including the gravitational field. When quantum gravitational effects are accounted for, does the flux continue to diverge and the radiation spectrum continue unique? Indeed, does the quantum theory serve as a cosmic censor and are there significant observational consequences of collapse into naked singularities as the semi-classical approximation suggests?

In a step toward answering these and other questions regarding the final stages of gravitational collapse Kuchař [14] examined a midisuperspace quantization of the Schwarzschild black hole, presenting in the process a remarkable series of canonical transformations that greatly simplified the dynamical equations. In the present paper, we describe a generalization of this work to spherically symmetric, marginally bound LeMaître-Tolman-Bondi space-times. We show

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below that there is an analogous description of the gravitational part of the action in terms of the “mass function,” the physical radius and their conjugate momenta. Furthermore, the coupling to dust introduces the proper time of the collapsing matter and its conjugate momentum. Thus time evolution appears naturally into the dynamical constraints.

The hypersurface action yields two constraints, viz. the Hamiltonian constraint and the momentum constraint, which are given in terms of this canonical chart consisting of the mass, $F[\rho(r)]$, contained within spherical shells of fixed shell-label coordinate, $\rho(r)$, the physical radius, $R(r)$, the dust proper time, $\tau(r)$, and their conjugate momenta, $P_F(r)$, $P_R(r)$ and $P_\tau(r)$ respectively. Here r is the radial label coordinate of a foliation of the space-time by spacelike hypersurfaces. The momentum conjugate to the mass function, $P_F(r)$, may be eliminated in the Hamiltonian constraint using the momentum constraint. This leads to a new and simpler constraint that is able to take the place of the original Hamiltonian constraint. Dirac’s constraint quantization then yields a two dimensional Klein-Gordon-like functional equation with a potential term that depends on the mass function and its derivative with respect to the label coordinate r . The simplest possible scenario, in which the mass function is constant throughout the space-time, the same for all shells, describes the Schwarzschild black hole. The Schwarzschild black hole will thus emerge as a special case of the general class of models we quantize below.

In Sec. II we review the classical collapse models being considered in this article and present the canonical formulation for spherically symmetric space-times in Sec. III, where we also discuss the fall-off conditions appropriate to the models and the resulting boundary terms. We reconstruct the mass and time from the canonical data in Sec. IV. This leads naturally to new variables, viz. the mass, the dust proper time, the physical radius and their conjugate momenta which are introduced along with the generator of the canonical transformation from the old to the new variables. We apply Dirac’s quantization program to the new constraints in Sec. V, showing how the new variables lead to a simplified Wheeler-DeWitt equation. We then present a solution for an arbitrary, but non-constant mass function.

II. CLASSICAL MODELS

The LeMaître-Tolman-Bondi models [15] constitute a complete solution of the Einstein equations for a matter continuum of inhomogeneous dust; i.e., they are solutions of the spherically symmetric Einstein’s field equations, $G_{\mu\nu} = -8\pi GT_{\mu\nu}$, with vanishing cosmological constant and with stress-energy describing inhomogeneous, pressureless dust given by $T_{\mu\nu} = \epsilon u_\mu u_\nu$. The solution is the LTB metric, given in co-moving coordinates as [16]

$$ds^2 = d\tau^2 - \frac{\bar{R}^2}{1+f} d\rho^2 - R^2 d\Omega^2, \quad \epsilon = \frac{\bar{F}}{R^2 \bar{R}}, \quad R^* = \pm \sqrt{f + \frac{F}{R}} \quad (2.1)$$

(we have set $8\pi G = 1 = c$) where R is the physical radius. A tilde represents a derivative with respect to ρ and an asterisk represents a derivative with respect to the dust proper time τ . The functions $f(\rho)$ and $F(\rho)$ are arbitrary functions only of ρ , interpreted respectively as the energy and mass functions. The energy density of the collapsing matter is $\epsilon(\tau, \rho)$, and the negative sign in the third equation above is required to describe a collapsing cloud. Its general solution is given up to an arbitrary function $\psi(\rho)$ of the shell label coordinate. This arbitrariness reflects only a freedom in our choice of units; i.e., at any given time, say τ_0 , the function $R(\tau_0, \rho)$ can be chosen to be an arbitrary function of ρ .

The mass function, $F(\rho)$, represents the weighted mass (weighted by the factor $\sqrt{1+f}$) contained within the matter shell labeled by ρ . If a scaling is chosen so that the physical radius coincides with the shell label coordinate, ρ , at $\tau=0$, then it can be expressed in terms of the energy density at $\tau=0$ according to

$$F(\rho) = \int \epsilon(0, \rho) \rho^2 d\rho, \quad (2.2)$$

while the energy function, $f(\rho)$, can be expressed in terms of the initial velocity profile, $v(\rho) = R^*(0, \rho)$, according to

$$f(\rho) = v^2(\rho) - \frac{1}{\rho} \int \epsilon(0, \rho) \rho^2 d\rho. \quad (2.3)$$

The marginally bound models, which we will consider in this paper, are defined by $f(\rho) = 0$. For the scaling referred to above, we must choose $\psi(\rho) = \rho^{3/2}$, whence the solution of Eq. (2.1) can be written as

$$R^{3/2}(\tau, \rho) = \rho^{3/2} - \frac{3}{2} \sqrt{F(\rho)} \tau. \quad (2.4)$$

The epoch $R=0$ describes a physical singularity, whose singularity curve

$$\tau(\rho) = \frac{2\rho^{3/2}}{3\sqrt{F(\rho)}}, \quad (2.5)$$

gives the proper time when successive shells meet the central physical singularity. Various models are obtained from choices of the mass function, $F(\rho)$. For example, the Schwarzschild black hole is the marginally bound solution with $F(\rho) = 2M$, a constant.

A collapsing star does not have sharp boundaries. In the simplest possible approximation to reality, it will consist of a dense core surrounded by a crust of lower density which, itself, is encased in a cloud whose density will smoothly go to zero with distance from the star’s center [17]. Nevertheless, as an approximation, one could consider a sharp boundary at some constant shell label, ρ_b , with an exterior Schwarzschild metric. One could also consider several regions described by different mass functions describing successively lower matter densities as one moves out from the

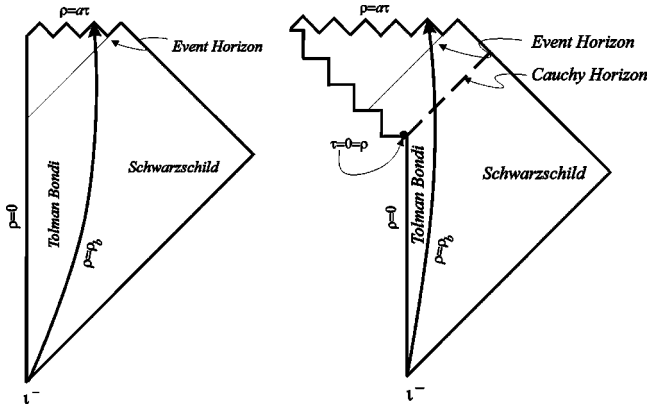


FIG. 1. Black hole (left) or naked singularity (right).

center. The mass function as a whole would not be differentiable at the boundary or boundaries, but we require it to be continuous.

One can then examine outgoing families of non-spacelike geodesics and check if there exist congruences that terminate in the past at the central singularity [18]. If such congruences exist, then the collapse leads to a naked singularity, and if they do not exist, then the collapse leads to a black hole. A naked singularity may further be characterized as globally naked or locally naked depending on whether the outgoing geodesics succeed in reaching null infinity or not.

In general one finds that both black holes and naked singularities may develop as the end states of collapse, depending on initial data, i.e. on the initial density and velocity profiles of the collapsing dust. For example, in the marginally bound self-similar collapse model, $F(\rho) = \lambda\rho$, where $\lambda = \text{const}$, both outcomes described by the Penrose diagrams in Fig. 1 are possible, depending on whether $\lambda > 0.1809$ (black hole) or $\lambda \leq 0.1809$ (naked singularity). It is in fact believed by many that sets of initial data may evolve in general relativity toward either naked singularities or black holes independently of the equation of state or the type of matter used.

Geometrodynamics views space-time as the dynamical evolution of spatial hypersurfaces. When the collapse evolves toward a black hole such spatial hypersurfaces exist, starting at infinity, crossing the horizon and continuing to $\rho = 0$ without encountering the central singularity. On the contrary, when the collapse evolves toward a naked singularity, the spatial hypersurfaces in the future of the initial singularity cross the Cauchy horizon and collide with the central singularity. No sensible boundary conditions can be specified on a singularity and evolution in the future of the initial singularity is arbitrary. To avoid the consequent breakdown of predictability, Penrose proposed the cosmic censor [19] which, as mentioned in the Introduction, is likely to be the quantum theory itself and will come into play *before* the Cauchy horizon has a chance to form. It is of particular interest, therefore, to understand precisely how the system behaves close to, but in the past of, the putative Cauchy horizon. Spatial hypersurfaces in the past of the Cauchy horizon are well defined and the quantum evolution of the system may be studied until the time of its formation.

III. CANONICAL DYNAMICS

The line element $d\sigma$ on a spherically symmetric three-dimensional Riemann surface Σ is completely characterized by two functions, $L(r)$ and $R(r)$, of the radial label coordinate, r , according to

$$d\sigma^2 = L^2(r)dr^2 + R^2(r)d\Omega^2 \quad (3.1)$$

where Ω is the solid angle. Neither $L(r)$ nor $R(r)$ can be negative and we take them to be positive definite except possibly at the center. $R(r)$ represents the physical radius of the point labeled by r on the surface. It behaves as a scalar under transformations of r , whereas $L(r)$ behaves as a scalar density. The corresponding four dimensional line element may be written in terms of two additional functions, the lapse, $N(t, r)$, and the shift, $N^r(t, r)$, as

$$ds^2 = N^2 dt^2 - L^2 (dr - N^r dt)^2 - R^2 d\Omega^2. \quad (3.2)$$

In this spherically symmetric space-time, we will consider the Einstein-Dust system described by the action

$$S = -\frac{1}{16\pi} \int d^4x \sqrt{-g} \mathcal{R} - \frac{1}{8\pi} \int d^4x \sqrt{-g} \epsilon(x) [g_{\alpha\beta} U^\alpha U^\beta + 1] \quad (3.3)$$

where \mathcal{R} is the scalar curvature. As is well known, the gravitational part of this action can be cast into the form

$$S^g = \int dt \int_0^\infty dr [P_L \dot{L} + P_R \dot{R} - NH^g - N^r H_r^g] + S_{\partial\Sigma}^g \quad (3.4)$$

with the momenta conjugate to L and R respectively given by

$$P_L = \frac{R}{N} [-\dot{R} + N^r R']$$

$$P_R = \frac{1}{N} [-L\dot{R} - \dot{L}R + (N^r LR)'] \quad (3.5)$$

and where the overdot and the prime refer respectively to partial derivatives with respect to the label time, t , and coordinate, r . The lapse, shift and phase-space variables are required to be continuous functions of the label coordinates. The boundary action, $S_{\partial\Sigma}^g$, is required to cancel unwanted boundary terms in the hypersurface action, thereby ensuring that the hypersurface evolution is not frozen. It is determined after fall-off conditions appropriate to the models under consideration are specified. The super-Hamiltonian and super-momentum constraints are given by

$$H^g = -\left[\frac{P_L P_R}{R} - \frac{L P_L^2}{2R^2} \right] + \left[-\frac{L}{2} - \frac{R'^2}{2L} + \left(\frac{RR'}{L} \right)' \right]$$

$$H_r^g = R' P_R - L P_L'. \quad (3.6)$$

We will assume that the chosen mass function, $F(\rho)$, is such that at infinity Kuchař's fall-off conditions [14] are suitable and we will adopt them here. These conditions would be applicable, for example, in models in which the collapsing metric asymptotically approaches or is smoothly matched to an exterior Schwarzschild background at some boundary, ρ_b . They read

$$\begin{aligned} L(t,r) &= 1 + M_+(t)r^{-1} + \mathcal{O}^\infty(r^{-1-\epsilon}) \\ R(t,r) &= r + \mathcal{O}^\infty(r^{-\epsilon}) \\ P_L(t,r) &= \mathcal{O}^\infty(r^{-\epsilon}) \\ P_R(t,r) &= \mathcal{O}^\infty(r^{-1-\epsilon}) \\ N(t,r) &= N_+(t) + \mathcal{O}^\infty(r^{-\epsilon}) \\ N^r(t,r) &= \mathcal{O}^\infty(r^{-\epsilon}). \end{aligned} \quad (3.7)$$

Again, because the label radial coordinate $r \in [0, \infty)$, we must also consider the boundary conditions at $r=0$. Let the mass function near the center ($\rho=0$) have a series expansion of the form

$$F(\rho) = F_0 + F_1\rho + F_2\rho^2 + \dots = \sum_n F_n\rho^n. \quad (3.8)$$

If $F_n=0 \forall n>0$ but $F_0>0$, the solution describes a Schwarzschild black hole of mass $F_0/2$. The marginally bound, self-similar model mentioned in the previous section corresponds to $F_1=\lambda>0$ and $F_n=0 \forall n \neq 1$ or a density profile that behaves as $\epsilon(0,\rho) \sim \rho^{-2}$. Thus $\tau=0$ is the singular epoch for the self-similar model and this singular density profile arises from a regular initial profile at some $\tau<0$.

Consider models with $F(\rho) = F_n\rho^n$. As far as the fall-off conditions at the center are concerned, two classes of models arise, viz. $n \leq 3$ and $n > 3$. The considerations below are applicable to models with $n \leq 3$, although conditions appropriate to models with $n > 3$ may likewise be given. Referring to Eq. (2.4) we find that as $\rho \rightarrow 0$, $R(\tau, \rho) \approx R_0(\tau)\rho^{n/3} + \mathcal{O}(\rho^{n/3+k})$ and $L(\tau, \rho) \approx L_0(\tau)\rho^{n/3-1} + \mathcal{O}(\rho^{n/3-1+k})$, where $k > 0$. We will exclude the black hole ($n=0$) in the following because in that case the space-time may be analytically continued to $r = -\infty$ where the boundary conditions given in Eqs. (3.7) may be applied and no conditions at $r=0$ need be given. For a genuine collapse, the cases of interest are those with $n > 0$ around the central region.

Let us assume that, at the center, (ρ, τ) approach (r, t) as $\rho = r + \mathcal{O}(r^\alpha)$ ($\alpha > 1$) and $\tau = f(t) + \mathcal{O}(r^\alpha)$. Now, in the following section we will show that the mass function is recovered locally from the canonical data according to

$$F = R \left[1 + \frac{P_L^2}{R^2} - \frac{R'^2}{L^2} \right]. \quad (3.9)$$

Again, with $\rho = r + \mathcal{O}(r^\alpha)$ and $\tau = f(t) + \mathcal{O}(r^\alpha)$, it follows that

$$\frac{R'}{L} \approx 1 + \mathcal{O}(r^{\alpha-1}) \quad (3.10)$$

and therefore

$$F = F_n\rho^n = F_n r^n + \mathcal{O}(r^{n+\alpha-1}) = \frac{P_L^2}{R} + \mathcal{O}(r^{n/3+\alpha-1}) \quad (3.11)$$

or

$$F_n = \frac{P_L^2}{R_0} r^{-4n/3} + \mathcal{O}(r^{-2n/3+\alpha-1}). \quad (3.12)$$

The last equation can be satisfied if P_L falls off faster than $r^{2n/3}$ and $\alpha = 1 + 2n/3$. Furthermore, requiring both terms in the Liouville form to have the same behavior at the origin, we choose the following conditions near $r=0$ when $n = \{1, 2, 3\}$:

$$\begin{aligned} R(r,t) &= R_0(t)r^{n/3} + \mathcal{O}(r^{n/3+\epsilon}) \\ L(r,t) &= L_0(t)r^{n/3-1} + \mathcal{O}(r^{n/3-1+\epsilon}) \\ P_L(r,t) &= \mathcal{O}(r^{2n/3+1+\epsilon}) \\ P_R(r,t) &= \mathcal{O}(r^{2n/3+\epsilon}) \\ N(r,t) &= N_0(t) + \mathcal{O}(r^\epsilon) \\ N^r(r,t) &= \mathcal{O}(r^\epsilon). \end{aligned} \quad (3.13)$$

The conditions (3.7) and (3.13) ensure that the Liouville form is well behaved both at the origin [$\mathcal{O}(r^{(n+\epsilon)})$] as well as at infinity [$\mathcal{O}^\infty(r^{-\epsilon})$]. The Hamiltonian and momentum densities behave as

$$H^g = \mathcal{O}(r^{n+1+\epsilon}), \quad H_r^g = \mathcal{O}(r^{n-1+\epsilon}) \quad (3.14)$$

as the origin is approached and

$$H^g = \mathcal{O}^\infty(r^{-2(1+\epsilon)}), \quad H_r^g = \mathcal{O}^\infty(r^{-(1+\epsilon)}) \quad (3.15)$$

asymptotically. Thus the total Hamiltonian and momentum are well defined and the surface action is meaningful. The potential contributions to the surface action can be read off the constraint equations (3.6) by considering variations of the phase space variables. Applying the fall-off conditions at infinity, one finds that only one of these is non-vanishing and behaves as

$$\int_{\partial\Sigma_\infty} dt N_+(t) \delta M_+(t). \quad (3.16)$$

On the other hand, with the fall-off conditions in Eqs. (3.13) as $r \rightarrow 0$, all the variations vanish at the origin. Therefore the only contribution to the boundary action comes from the surface term at infinity and we find

$$S_{\partial\Sigma} = - \int_{\partial\Sigma_\infty} dt N_+(t) M_+(t). \quad (3.17)$$

We will return to this surface action shortly.

Let us now turn to the dust portion of the action in Eqs. (3.3):

$$S^d = \int d^4x \sqrt{-g} \epsilon(x) [g_{\alpha\beta} U^\alpha U^\beta + 1]. \quad (3.18)$$

It has been exhaustively analyzed by Kuchař and Torre [20] and by Brown and Kuchař [21]. It may be understood in two ways: either as a consequence of imposing coordinate conditions (the Gaussian conditions) or as a realistic material medium. For the collapse problem it is a realistic material medium and for the LTB models being considered it is non-rotating. Dust is described by eight space-time scalars, ϵ , τ , Z^k and W_k ($k \in \{1,2,3\}$). The physical interpretation of these variables which follows from an analysis of the equations of motion was given in Ref. [21] and will be summarized here for completeness. τ is the proper time measured along particle flow lines, Z^k are the comoving coordinates of the dust, W^k are the spatial components of the four-velocity in the dust frame, and ϵ is the dust proper energy density. All these scalars are assumed to be functions of the space-time coordinates. In particular, the four variables, $Z^K = (\tau, Z^k)$, are independent functions, $\det|Z^K_{,\mu}| \neq 0$, and the four-velocity of the dust particles may be defined by its decomposition in the co-basis $Z^K_{,\mu}$ by

$$U_\mu = -\tau_{,\mu} + W_k Z^k_{,\mu}. \quad (3.19)$$

In the spherically symmetric geometry described by Eq. (3.2), the dust action may be cast into the form

$$S^d = \int dt \int dr [P_\tau \dot{\tau} + P_k \dot{Z}^k - NH^d - N^r H_r^d], \quad (3.20)$$

where

$$P_\tau = \frac{LR^2}{N} \epsilon(r,t) [-U_t + N^r U_r] \\ P_k = -W_k P_\tau \quad (3.21)$$

are the momenta conjugate to the dust proper time and the frame variables respectively, and

$$H^d = P_\tau \sqrt{1 + \frac{U_r^2}{L^2}} \\ H_r^d = -U_r P_\tau. \quad (3.22)$$

This expression for H^d is obtained by exploiting the fact that $\epsilon(t,r)$ is a Lagrange multiplier, and therefore $\delta\mathcal{L}/\delta\epsilon = 0$. When the dust is non-rotating Eqs. (3.22) are further simplified by requiring that the dust motion be described with respect to the frame orthogonal foliation. Then we may impose the additional constraints $P_k = 0$ (when imposed on the state functional these constraints simply mean that the state func-

tional does not depend on the frame variables Z^k). Using this in Eqs. (3.21) we see that $W_k = 0$ and so $U_r = -\tau'$.

Combining the results thus far obtained, we now give the full form of the canonical description of the action in Eqs. (3.3) as

$$S = \int dt \int_0^\infty dr [P_\tau \dot{\tau} + P_L \dot{L} + P_R \dot{R} - NH - N^r H_r] + S_{\partial\Sigma}^g \quad (3.23)$$

with

$$H = - \left[-\frac{P_L P_R}{R} - \frac{L P_L^2}{2R^2} \right] + \left[-\frac{L}{2} - \frac{R'^2}{2L} + \left(\frac{R R'}{L} \right)' \right] \\ + P_\tau \sqrt{1 + \frac{\tau'^2}{L^2}} \approx 0 \\ H_r = \tau' P_\tau + R' P_R - L P_L' \approx 0 \quad (3.24)$$

and

$$S_{\partial\Sigma} = - \int_{\partial\Sigma_\infty} dt N_+(t) M_+(t) \quad (3.25)$$

for the boundary contribution.

IV. NEW VARIABLES

The hypersurfaces we consider, from which Eq. (3.2) is constructed, must eventually be embedded in a space-time described by the metric given in Eq. (2.1) with $f=0$. We imagine that they are leaves of the foliation $\tau(t,r)$ and $\rho(t,r)$. Then the functions $L(t,r)$ and $R(t,r)$ appearing in Eq. (3.2) are easily determined by substituting the foliation in Eq. (2.1). We find

$$L^2 = \tilde{R}^2 \rho'^2 - \tau'^2 \\ N^r = \frac{\tilde{R}^2 \dot{\rho} \rho' - \dot{\tau} \tau'}{L^2} \quad (4.1) \\ N = \frac{\tilde{R}}{L} (\dot{\tau} \rho' - \dot{\rho} \tau').$$

The last of the equations above involves taking a square root. We must check that the positive square root taken leads to a positive lapse function in all the regions of the space-time. Call $\mathcal{F} = 1 - F/R$ where F is the mass function and note that $R^* = -\sqrt{1 - \mathcal{F}}$ according to Eq. (2.1) with $f=0$.

Substituting the expression for $N(t,r)$ and $N^r(t,r)$ obtained in Eqs. (4.1) into the expressions for the canonical momenta in Eqs. (3.5), one obtains the relation

$$\frac{L P_L}{R} = \frac{1}{\tilde{R} (\dot{\tau} \rho' - \dot{\rho} \tau')} [-\dot{R} (\tilde{R}^2 \rho'^2 - \tau'^2) \\ + R' (\tilde{R}^2 \dot{\rho} \rho' - \dot{\tau} \tau')], \quad (4.2)$$

which, after some algebra, can be used to obtain τ' in terms of the canonical variables:

$$\tau' = -\frac{LP_L}{R\mathcal{F}} + \frac{R'\sqrt{1-\mathcal{F}}}{\mathcal{F}}. \quad (4.3)$$

Inserting this into the expression for L^2 in Eqs. (4.1), we find that

$$L^2 = \frac{R'^2}{\mathcal{F}} - \frac{L^2 P_L^2}{R^2 \mathcal{F}}, \quad (4.4)$$

which determines \mathcal{F} ,

$$\mathcal{F} = \frac{R'^2}{L^2} - \frac{P_L^2}{R^2}, \quad (4.5)$$

and, through \mathcal{F} , recovers the mass function

$$F = R \left[1 + \frac{P_L^2}{R^2} - \frac{R'^2}{L^2} \right]. \quad (4.6)$$

This is the relation used in the previous section when discussing the fall-off conditions at the center. It enables us to determine the mass function *locally* from the canonical data. Furthermore, once the dust proper time is fixed at some point on the hypersurface, say at spatial infinity, Eq. (4.3), which determines the difference in dust proper times between any two points r_1 and r_2 on a spatial hypersurface, will determine it at any point on the hypersurface. Note that at the horizon, when $\mathcal{F}=0$, i.e.,

$$\frac{R'}{L} = \frac{P_L}{R}; \quad (4.7)$$

τ' continues to be well behaved, as expected.

It turns out that the functions P_F , defined by¹

$$P_F = \frac{LP_L}{2R\mathcal{F}}, \quad (4.8)$$

and the mass function, F , form a conjugate pair of variables. Moreover, because neither F nor P_F depends on P_R , they have vanishing Poisson brackets with R . They also have vanishing Poisson brackets with τ and P_τ . Their Poisson brackets with P_R , however, do not vanish and one cannot directly replace the pair (L, P_L) with the more transparent variables (F, P_F) to form a new chart. Instead one asks if it is possible to determine a new momentum, \bar{P}_R , conjugate to R and such that the set $(\tau, R, F, P_\tau, \bar{P}_R, P_F)$ forms a canonical chart. We

¹ $P_F(r)$ is the equivalent of Kuchař's $P_M(r)$, the momentum conjugate to the black hole mass function, $M(r)$. The parallel between our construction for LTB metrics and Kuchař's construction for the Schwarzschild black hole, which inspired this work, is remarkable because the Schwarzschild metric [with a varying mass, $M(r)$] is not diffeomorphic to the LTB metric in Eq. (2.1) *except* in the black hole case, i.e. when $F=2M=\text{const}$ and $f=0$.

proceed by constructing \bar{P}_R in exactly the same way as Kuchař did for the Schwarzschild black hole. Then we show that the transition to the new chart is indeed a canonical transformation by displaying its generator.

Kuchař [14] proposed that (R, F, \bar{P}_R, P_F) should form a canonical chart whose coordinates are spatial scalars, whose momenta are scalar densities and which is such that $H_r(r)$ generates Diff \mathbf{R} . This means that

$$H_r = \tau' P_\tau + R' P_R - LP'_L = \tau' P_\tau + R' \bar{P}_R + F' P_F \approx 0. \quad (4.9)$$

Substituting the expressions derived earlier for F and P_F into the above one finds

$$\bar{P}_R = P_R - \frac{LP_L}{2R} - \frac{LP_L}{2R\mathcal{F}} - \frac{\Delta}{RL^2\mathcal{F}} \quad (4.10)$$

where $\Delta = (RR')(LP_L)' - (RR')'(LP_L)$. We must now show that the transformation

$$(\tau, R, L, P_\tau, P_R, P_L) \rightarrow (\tau, R, F, P_\tau, \bar{P}_R, P_F) \quad (4.11)$$

is a canonical transformation.

To do this we solve the set

$$\begin{aligned} p_i(r) &= \sum_j \int_0^\infty dr' P_j(r') \frac{\delta Q_j(r')}{\delta q_i(r)} + \frac{\delta \mathcal{G}}{\delta q_i(r)} \\ 0 &= \sum_j \int_0^\infty dr' P_j(r') \frac{\delta Q_j(r')}{\delta p_i(r)} + \frac{\delta \mathcal{G}}{\delta p_i(r)} \end{aligned} \quad (4.12)$$

where (p_i, q_i) and (P_i, Q_i) are respectively the old and the new phase-space variables and where $\mathcal{G}[q_i, p_i]$ generates the transformation. Because we know F and P_F in terms of (R, L, P_R, P_L) , the four non-trivial equations, viz.

$$\begin{aligned} P_L(r) &= \int_0^\infty dr' P_F(r') \frac{\delta F(r')}{\delta L(r)} + \frac{\delta \mathcal{G}}{\delta L(r)} \\ P_R(r) &= \bar{P}_R(r) + \int_0^\infty dr' P_F(r') \frac{\delta F(r')}{\delta R(r)} + \frac{\delta \mathcal{G}}{\delta R(r)} \\ 0 &= \int_0^\infty dr' P_F(r') \frac{\delta F(r')}{\delta P_L(r)} + \frac{\delta \mathcal{G}}{\delta P_L(r)} \\ 0 &= \int_0^\infty dr' P_F(r') \frac{\delta F(r')}{\delta P_R(r)} + \frac{\delta \mathcal{G}}{\delta P_R(r)}, \end{aligned} \quad (4.13)$$

can be solved for \mathcal{G} , and \bar{P}_R can be recovered using the second of the above equations. The last equation of Eqs. (4.13) implies that \mathcal{G} is independent of P_R . The third equation in Eqs. (4.13) reads

$$\frac{\delta \mathcal{G}}{\delta P_L} = -\frac{2P_F P_L}{R} = -\frac{LP_L^2}{R^2} \left[\frac{R'^2}{L^2} - \frac{P_L^2}{R^2} \right]^{-1} \quad (4.14)$$

and can be integrated to give

$$\mathcal{G} = \int_0^\infty dr \left[LP_L - \frac{1}{2} RR' \ln \left| \frac{RR' + LP_L}{RR' - LP_L} \right| \right] + \mathcal{G}_2[R, L] \quad (4.15)$$

and the first equation ensures that \mathcal{G}_2 is independent of L . We will take \mathcal{G}_2 to be independent also of R and use \mathcal{G} to determine \bar{P}_R from the second equation in Eqs. (4.13). This gives precisely Eq. (4.10). Next, we verify that the transformation has not introduced fresh boundary terms by computing the difference between the old and the new Liouville forms:

$$\begin{aligned} & \int_0^\infty dr [P_R \delta R + P_L \delta L - \bar{P}_R \delta R - P_F \delta F] \\ &= \int_0^\infty dr \left\{ \delta \left[LP_L - \frac{1}{2} RR' \ln \left| \frac{RR' + LP_L}{RR' - LP_L} \right| \right] \right. \\ & \quad \left. + \left[\frac{1}{2} R \delta R \ln \left| \frac{RR' + LP_L}{RR' - LP_L} \right| \right]' \right\}. \end{aligned} \quad (4.16)$$

Kuchař [14] has shown that the fall-off conditions (3.7) at infinity imply that

$$\left[\frac{1}{2} R \delta R \ln \left| \frac{RR' + LP_L}{RR' - LP_L} \right| \right]_{r \approx \infty} \approx \frac{LP_L \delta R}{R'} \approx \mathcal{O}(r^{-\epsilon}) \quad (4.17)$$

and hence vanish at infinity. Again, as $r \rightarrow 0$, the fall-off conditions in Eqs. (3.13) imply that $R \rightarrow R_0(t)r^{n/3}$, $\delta R \rightarrow \delta R_0(t)r^{n/3}$, $R' \rightarrow R_0(t)r^{n/3-1}$, $L \rightarrow L_0(t)r^{n/3-1}$ and $P_L \rightarrow \mathcal{O}(r^{2n/3+1+\epsilon})$. We find that

$$\left[\frac{1}{2} R \delta R \ln \left| \frac{RR' + LP_L}{RR' - LP_L} \right| \right]_{r \approx 0} \approx \frac{LP_L \delta R}{R'} \approx \mathcal{O}(r^{n+1+\epsilon}) \quad (4.18)$$

which therefore also vanishes at the origin. The functional \mathcal{G} defined by Eq. (4.15) is well defined. The integrand is of order $r^{-(1+\epsilon)}$ at infinity and of order $r^{4n/3+1+\epsilon}$ at the origin, avoiding divergences at both places.

There are (infinite) boundary terms at the horizon, when $\mathcal{F}=0$. It can be shown, however, that the contributions from the interior and the exterior cancel each other. In the same way, there will be contributions at the boundary between the interior of the star and its exterior or more generally at any frontier between two LTB regions described by different mass functions. Again, if the mass function is continuous across the boundary and the regions are consistently matched by equating both the first and second fundamental forms, then the contribution from one side will cancel the contribution from the other.

In terms of the new variables, the action in Eqs. (3.3) can be expressed as

$$S = \int dt \int_0^\infty dr [P_\tau \dot{\tau} + \bar{P}_R \dot{R} + P_F \dot{F} - NH^g - N'H_r^g] + S_{\partial\Sigma} \quad (4.19)$$

with

$$H = - \left[\frac{F' \mathcal{F}^{-1} R' + \mathcal{F} P_F \bar{P}_R}{2L} \right] + P_\tau \sqrt{1 + \frac{\tau'^2}{L^2}} \quad (4.20)$$

$$H_r = \tau' P_\tau + R' \bar{P}_R + F' P_F$$

$$S_{\partial\Sigma} = - \int_{\partial\Sigma_\infty} N_+(t) M_+(t).$$

Let us now re-express the boundary action in a more convenient form. As Kuchař [14] has emphasized, $N_+(t)$ must be treated as a *prescribed* function of t . This is to avoid the conclusion that the total mass M_+ as measured at infinity is zero, which would follow from varying the lapse function at infinity. $N_+(t)$ must be chosen and, once chosen, held fixed. The freedom in choosing this function can be combined with the freedom we have of setting the dust proper time at infinity to correspond to the parametrization clocks there. The lapse function is the rate of change of the proper time with the coordinate time at infinity, so it is natural to set $N_+(t) = \dot{\tau}_+(t)$ and write the surface action as

$$S_{\partial\Sigma} = - \int dt [M_+ \dot{\tau}_+]. \quad (4.21)$$

It is linear in the time derivative, $\dot{\tau}_+$, and defines a one-form which can be re-written in terms of the mass function, F , as

$$\begin{aligned} -M_+ \delta\tau_+ &= - \frac{1}{2} \int_0^\infty dr (F \delta\tau)' \\ &= - \frac{1}{2} \int_0^\infty dr [F' \delta\tau - \tau' \delta F + \delta(F\tau')]. \end{aligned} \quad (4.22)$$

The first two terms on the right-hand side may be absorbed into the Liouville form of the hypersurface action (they modify the canonical momenta) and the last term is an exact form which can be dropped. The action is thus expressed entirely as a hypersurface action. Defining $\bar{P}_\tau = P_\tau - F'/2$ and $\bar{P}_F = P_F + \tau'/2$, we have

$$S = \int dt \int_0^\infty dr [\bar{P}_\tau \dot{\tau} + \bar{P}_R \dot{R} + \bar{P}_F \dot{F} - NH^g - N'H_r^g] \quad (4.23)$$

where the constraints in the new chart read

$$\begin{aligned} H &= - \left[\frac{F' \mathcal{F}^{-1} R' + \mathcal{F} (\bar{P}_F - \tau'/2) \bar{P}_R}{2L} \right] \\ & \quad + (\bar{P}_\tau + F'/2) \sqrt{1 + \frac{\tau'^2}{L^2}} \\ H_r &= \tau' \bar{P}_\tau + R' \bar{P}_R + F' \bar{P}_F. \end{aligned} \quad (4.24)$$

Finally, eliminating the momentum \bar{P}_F from the Hamiltonian constraint by using the momentum constraint, we obtain a new and dynamical constraint which takes the place of the Hamiltonian constraint:

$$(\bar{P}_\tau + F'/2)^2 + \mathcal{F}\bar{P}_R^2 - \frac{F'^2}{4\mathcal{F}} \approx 0. \quad (4.25)$$

This is the simplified constraint referred to in the Introduction. It takes the place of the Hamiltonian constraint in Eqs. (4.24). In the following section we quantize the constraints and solve the Wheeler-DeWitt equation for arbitrary non-constant mass functions.

V. QUANTIZATION

From the dynamical constraint in the previous section, one reads off the DeWitt supermetric, γ_{ab} , on the configuration space $X^a = (\tau, R)$:

$$\gamma_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1/\mathcal{F} \end{pmatrix}. \quad (5.1)$$

To quantize the system, the momenta must be turned into operators which act on the state functional. There is a standard procedure to do this, which follows a proposal due to Vilkovisky [22,23] and DeWitt [24]: exchange the classical momenta for covariant functional derivatives:

$$\hat{P}_a = -i\nabla_a = -i\left(\frac{\delta}{\delta X^a(r)} + \Gamma_a\right) \quad (5.2)$$

where Γ is the connection belonging to the configuration space metric, γ_{ab} . Then defining

$$\Psi[\tau, R, F] = \exp\left(-\frac{i}{2} \int_0^\infty F'(r) \tau(r) dr\right) \tilde{\Psi}[\tau, R, F], \quad (5.3)$$

we see that $\tilde{\Psi}$ obeys the ‘‘Klein-Gordon’’ equation with a potential,

$$\left[\gamma^{ab} \nabla_a \nabla_b + \frac{F'^2}{4\mathcal{F}}\right] \tilde{\Psi}[X, F] = 0. \quad (5.4)$$

The metric in Eq. (5.1) is positive definite outside the horizon ($\mathcal{F} > 0$) and indefinite inside ($\mathcal{F} < 0$). The functional equation is therefore elliptic outside the horizon and hyperbolic inside. Furthermore, the configuration space is flat and this metric is brought to a manifestly flat form by the coordinate transformation

$$R_* = \int \frac{dR}{\sqrt{|\mathcal{F}|}}. \quad (5.5)$$

In terms of R_* , the Wheeler-DeWitt equation reads

$$\left[\frac{\delta^2}{\delta \tau^2} \pm \frac{\delta^2}{\delta R_*^2} + \frac{F'^2}{4\mathcal{F}}\right] \tilde{\Psi}[\tau, R, F] = 0 \quad (5.6)$$

or

$$i \frac{\delta \tilde{\Psi}}{\delta \tau} = \hat{h} \tilde{\Psi} = \pm \sqrt{\mp \hat{P}_*^2 + \frac{F'^2}{4\mathcal{F}}} \tilde{\Psi}[\tau, R, F] = 0, \quad (5.7)$$

where the negative sign within the square root refers to the region outside the horizon, the positive sign to the interior and where \hat{P}_* is conjugate to R_* . Invariance under spatial diffeomorphisms is enforced by the momentum constraint,

$$\left[\tau' \frac{\delta}{\delta \tau} + R'_* \frac{\delta}{\delta R_*} + F' \frac{\delta}{\delta F}\right] \tilde{\Psi}[\tau, R, F] = 0, \quad (5.8)$$

and, together, Eqs. (5.6) and (5.8) define the quantum theory whose inner product is given by

$$\langle \Psi_1, \Psi_2 \rangle = \int_{R_*(0)}^\infty dR_* \Psi_1^\dagger \Psi_2 = \int_{R_*(0)}^\infty \prod_r dR_*(r) \Psi_1^\dagger \Psi_2 \quad (5.9)$$

where $R(0)$ represents the physical radius at the center, $r = 0$. The inner product ensures the Hermiticity of the momentum, \hat{P}_* , conjugate to R_* . The norm of a quantum state under this scalar product is formally τ independent provided that \hat{h} defined in Eq. (5.7) is self-adjoint. However, this occurs only in the linear sub-space in which the operator $\mp \hat{P}_*^2 + F'^2/4\mathcal{F}$ admits positive eigenvalues. When $F' = 0$, for example for the Schwarzschild black hole, this is true of all states with support only in the interior of the hole [25]. Such states alone may be ascribed a probabilistic interpretation.

While specific models of stellar collapse must be analyzed individually, a solution for general $F(r) \neq \text{const}$ can be given as follows. The momentum constraint requires the wave functional to be a spatial scalar; therefore consider a solution of the form

$$\tilde{\Psi}[\tau, R, F] = \exp\left[\frac{1}{2} \int_0^\infty dr F'(r) \mathcal{W}(\tau(r), R(r), F(r))\right] \quad (5.10)$$

where $\mathcal{W}(\tau, R, F)$ is an arbitrary (complex valued) function of the coordinates (and not their derivatives) which is to be determined. The integrand in the exponent is clearly a spatial density because, while $\tau(r)$, $R(r)$ and $F(r)$ are spatial scalars, $F'(r)$ is a density. It follows that the wave functional so defined will obey the momentum constraint. Indeed, Eq. (5.8) simply requires that \mathcal{W} admit no explicit dependence on r . It is well known that the Wheeler-DeWitt equation requires some regularization, involving as it does two functional derivatives taken at the same point. We will follow DeWitt [26] (see also [27]) and require that the delta function vanish in coincidence limit, as does the coincidence limit of its derivatives to all orders; i.e., we let $\delta(0) = 0 = \delta^{(n)}(0)$ in what follows [where the superscript (n) refers to the n th spatial derivative of the delta function]. Then the dynamical equation (5.6) reads

$$\frac{F'^2}{4} \left[\left(\frac{\partial \mathcal{W}}{\partial \tau} \right)^2 \pm \left(\frac{\partial \mathcal{W}}{\partial R_*} \right)^2 + \frac{1}{\mathcal{F}} \right] \tilde{\Psi} = 0. \quad (5.11)$$

Let us choose \mathcal{W} to be of the form $\mathcal{W} = -i\tau + \mathcal{U}(R, F)$. This ansatz is motivated by the fact that $F'/2$ actually represents the radial mass-energy density within spherical shells. Inserting it into Eq. (5.11) gives the following equations for $\mathcal{U}(R, F)$:

$$\begin{aligned} \frac{\partial \mathcal{U}}{\partial R_*} &= \pm i \sqrt{\frac{F}{R-F}} \\ \mathcal{W} &= -i\tau \pm i\sqrt{F} \int dR \sqrt{\frac{R}{R-F}} \end{aligned} \quad (5.12)$$

outside the horizon ($R > F$) and

$$\begin{aligned} \frac{\partial \mathcal{U}}{\partial R_*} &= \pm i \sqrt{\frac{F}{F-R}} \\ \mathcal{W} &= -i\tau \pm i\sqrt{F} \int dR \sqrt{\frac{R}{F-R}} \end{aligned} \quad (5.13)$$

inside ($R < F$). In both regions the equations are easily integrated. They give

$$\mathcal{W} = -i\tau \pm 2i\sqrt{F} \left[\sqrt{R} - \sqrt{F} \tanh^{-1} \sqrt{\frac{F}{R}} \right] \quad (5.14)$$

outside and

$$\mathcal{W} = -i\tau \mp 2i\sqrt{F} \left[\sqrt{R} - \sqrt{F} \tanh^{-1} \sqrt{\frac{R}{F}} \right] \quad (5.15)$$

inside.

Any collapse model that approximates reality must involve two or more regions of differing mass functions (the simplest example would be a constant mass function representing the Schwarzschild exterior of a star matched to another non-constant mass function representing the interior of the star). In general the mass function must be continuous but it may not necessarily be differentiable across the boundary between these regions. At such boundaries, the wave functional must be appropriately matched, i.e. required to be both continuous and differentiable. Specific models which exhibit some of the interesting classical features whose correct understanding cannot be had without appealing to quantum gravity, as mentioned in the Introduction, will be described in a forthcoming publication.

We have shown that there exists a canonical chart analogous to that used by Kuchař to describe the Schwarzschild black hole and which describes the marginally bound collapse of inhomogeneous, pressureless dust. This chart enjoys several advantages over the original. For one, the proper time of the collapsing dust enters naturally and serves as a time variable in the Wheeler-DeWitt equation. Again, the chart describes the collapse in terms of a more transparent and physically meaningful set of variables, yielding a dynamical constraint that is greatly simplified and whose solution we have displayed over the entire class of models being considered.

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