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# Static axisymmetric solution of the Einstein equations

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We state a new axisymmetric, static, asymptotically flat solution of the Einstein equations in a vacuum. The solution depends on three parameters, $m$, $n$, and $\gamma$. When $n\to\infty$ and $\gamma\to 1$, the solution has the Schwarzschild solution as a limit. There are no horizons and the tetrad components of the conformal tensors are everywhere finite. The solution lacks elementary flatness on the whole axis of symmetry.

## I. THE $n\gamma$ SOLUTION

We describe a family of static axisymmetric solutions of the Einstein equations which depends on three parameters, $m$, $\gamma$, and $n$, which we shall call the $n\gamma$ solution. When $n\to\infty$ and $\gamma\to 1$, the solution is the Schwarzschild solution.

Static, axisymmetric solutions of the Einstein equations are given by the Weyl metric

$$ds^2=-e^{2\lambda}dt^2+e^{-2\lambda}[e^{2\mu}(d\rho^2+dz^2)+\rho^2 d\phi^2] \quad (1)$$

with

$$\lambda_{,\rho\rho}+\frac{\lambda_{,\rho}}{\rho}+\lambda_{,zz}=0 \quad (2)$$

and

$$\mu_{,\rho}=\rho(\lambda_{,\rho}{}^2-\lambda_{,z}{}^2),\quad \mu_{,z}=2\rho\lambda_{,\rho}\lambda_{,z}\ . \quad (3)$$

$\lambda$ is given by the Newtonian potential of an axisymmetric source. It is well known that the Schwarzschild solution is generated by a mass of density 1/2 distributed symmetrically along the axis for a length $2m$. The $\gamma$ solution is generated by a mass of density $\gamma/2$ distributed symmetrically along the axis for a length $2m$. Thus when $\gamma=1$, the solution is equal to the Schwarzschild solution. As described in Ref. 1 and in other references cited therein, the $\gamma$ solution has directional singularities at the points $\rho=0$, $|z|=m$. These singularities disappear when $\gamma=1$ (the Schwarzschild case); in this case the "line" $\rho=0$, $|z|\le m$ maps into the horizon surface $r=2m$ in the more usual coordinate system. It is worthwhile to try to understand the origin of the directional singularities. It may be that the discontinuous nature of the source function at $|z|=m$ produces directional singularities. We shall introduce a source which does not have discontinuities. The directional singularities are then indeed no longer present, however other problems arise along the axis. We take the source distribution along the axis to be

$$f=\frac{\gamma}{2\pi}\{\arctan[n(z+m)]-\arctan[n(z-m)]\}\ . \quad (4)$$

In the limit $n\to\infty$, this is the distribution for the $\gamma$ solution. The solution to Eq. (2) with this distribution is

$$\lambda=-\frac{\gamma}{4\pi}\int_{-\infty}^{\infty}\frac{dz'}{[(z'-z)^2+\rho^2]^{1/2}}\times\{\arctan[n(z'+m)]-\arctan[n(z'-m)]\}\ . \quad (5)$$

After an integration by parts

$$\lambda=\frac{\gamma}{4\pi}\int_{-\infty}^{\infty}dz'\ln|\{[(z'-z)^2+\rho^2]^{1/2}+(z'-z)\}|\left[\frac{1}{1+n^2(z'+m)^2}-\frac{1}{1+n^2(z'-m)^2}\right]\ . \quad (6)$$

The integration is now readily performed in the complex $z'$ plane. We state the result:

 

$$e^{2\lambda}=\left[\frac{(R_1+R_2-2m)(\bar{R}_1+\bar{R}_2-2m)}{(R_1+R_2+2m)(\bar{R}_1+\bar{R}_2+2m)}\right]^{\gamma/4}, \tag{7}$$

$$e^{2\mu}=\left[\frac{(R_1+R_2-2m+2i/n)(\bar{R}_1+\bar{R}_2-2m-2i/n)(R_1+R_2+2m-2i/n)(\bar{R}_1+\bar{R}_1+2m+2i/n)}{16R_1\bar{R}_1R_2\bar{R}_2}\right.$$
$$\left.\times\frac{(R_1+R_2-2m)(\bar{R}_1+\bar{R}_2-2m)(R_1+R_2+2m)(\bar{R}_1+\bar{R}_2+2m)}{(R_1+\bar{R}_1)^2+4/n^2)((R_2+\bar{R}_2)^2+4/n^2))}\right]^{\gamma^2/4}, \tag{8}$$

$$R_1{}^2=\rho^2+\left[\frac{i}{n}-z+m\right]^2, \tag{9}$$

$$R_2{}^2=\rho^2+\left[\frac{i}{n}-z-m\right]^2. \tag{10}$$

## II. LIMITING BEHAVIOR OF THE $n\gamma$ SOLUTION

The $n\gamma$ solution is asymptotically flat; as $\rho\to\infty$, or $|z|\to\infty$, $e^{2\lambda}\to e^{2\mu}\to 1$. The solution does not have the property of elementary flatness on the axis. The ratio of the circumference of a circle surrounding the axis to its radius in the limit of vanishing radius is easily calculated to be $\exp(-\mu)$. Elementary flatness requires $\mu=0$ on the axis. We examine the behavior of the metric components on the axis ($\rho=0$):

$$R_1=\frac{i}{n}-(z-m),\quad R_2=\frac{i}{n}-(z+m)\,. \tag{11}$$

In fact either or both $R_1$ and $R_2$ can have the opposite sign from that indicated above when $\rho=0$. We choose the signs indicated and identify the range of $z$ for which these signs hold by comparison with the known properties of the $\gamma$ metric ($n\to\infty$). Of course the metric is symmetric about the $z=0$ plane:

$$e^{2\lambda}(\rho=0)=\left[\frac{(z+m)^2+1/n^2}{(z-m)^2+1/n^2}\right]^{\gamma/2}. \tag{12}$$

This shows that the choice of signs in Eq. (11) is valid for $z<-m$ since when $n\to\infty$, $\exp(2\lambda)\to 0$ for $z=m$ or $z=-m$ in the $\gamma$ metric (as in the Schwarzschild metric $\gamma=1$, where $|z|=m$ are the poles of the horizon). We see that for the $n\gamma$ solution there is no singularity at $z=-m$, and we can take Eq. (11) to be valid for $z<0$:

$$e^{2\mu}(\rho=0)=\left[\frac{[(z+m)^2+4/n^2[(z-m)^2]}{[(z-m)^2+1/n^2][(z+m)^2+1/n^2]}\right]^{\gamma^2/4}. \tag{13}$$

Elementary flatness is recovered as $n\to\infty$. The lack of elementary flatness along the axis is probably related to the fact that the source function is finite along the entire axis [Eq. (4)], but vanishes off axis (hence is singular along the axis).

The axis for the Schwarzschild and $\gamma$ solutions could be best studied by making a coordinate transformation

$$\rho^2=(r^2-2mr)\sin^2\theta,\quad z=(r-m)\cos\theta\,. \tag{14}$$

This transforms the line $\rho=0$, $|z|\le m$ into the sphere $r=2m$. The properties of the $n\gamma$ solution in the neighborhood of the axis will quite probably also require a coordinate change for greater understanding.

## III. TETRAD COMPONENTS OF THE CONFORMAL TENSOR

We now examine the tetrad components of the conformal tensor. They are everywhere finite. Define a complex null tetrad $(m,\bar{m},l,k)$ in the usual way: $k$ and $l$ are real, $m$ and $\bar{m}$ are complex and complex conjugates of each other:

$$k^\mu l_\mu=-1,\quad m^\mu \bar{m}_\mu=1\ ,$$
$$k^\mu k_\mu=l^\mu l_\mu=m^\mu m_\mu=k^\mu m_\mu=l^\mu m_\mu=0\ , \tag{15}$$
$$g_{\mu\nu}=2m_{(\mu}\bar{m}_{\nu)}-2k_{(\mu}l_{\nu)}\ .$$

The complex Weyl tetrad components are defined as

$$\begin{aligned}
\psi_0&\equiv C_{\mu\nu\rho\sigma}k^\mu m^\nu k^\rho m^\sigma\ ,\\
\psi_1&\equiv C_{\mu\nu\rho\sigma}k^\mu l^\nu k^\sigma m^\sigma\ ,\\
\psi_2&\equiv \tfrac{1}{2}C_{\mu\nu\rho\sigma}k^\mu l^\nu (k^\rho l^\sigma-m^\rho\bar{m}^\sigma)\ ,\\
\psi_3&\equiv C_{\mu\nu\rho\sigma}l^\mu k^\nu l^\rho \bar{m}^\sigma\ ,\\
\psi_4&\equiv C_{\mu\nu\rho\sigma}l^\mu \bar{m}^\nu l^\rho \bar{m}^\sigma\ .
\end{aligned} \tag{16}$$

Choose the $(t,\rho,\phi,z)$ components of the null tetrad to be

$$\begin{aligned}
k^\mu&=[1,(-g_{tt})^{1/2}g_{\rho\rho}{}^{1/2},0,0]\ ,\\
l^\mu&=\tfrac{1}{2}[(g_{tt})^{-1}\ ,\\
&\qquad (-g_{tt})^{-1/2}g_{\rho\rho}{}^{-1/2},0,0]\ ,\\
m^\mu&=\tfrac{1}{2}[0,0,(g_{\phi\phi})^{-1/2},-i(g_{zz})^{-1/2}]\ .
\end{aligned} \tag{17}$$

With this tetrad, $\psi_1=\psi_3=0$, $\psi_4=(\psi_0/4)\exp(-4\lambda)$,

$$\begin{aligned}
\psi_0=e^{4\lambda-2\mu}\Bigg\{&\frac{\gamma}{4\rho^2}\left[\frac{(z+m-i/n)^3}{R_2{}^3}+\frac{(z+m+i/n)^3}{\bar{R}_2{}^3}-\frac{(z-m+i/n)^3}{\bar{R}_1{}^3}-\frac{(z-m-i/n)^3}{R_1{}^3}\right]\\
&-\frac{3\gamma^2}{16}\left[\frac{1}{\bar{R}_1{}^2}+\frac{1}{R_1{}^2}+\frac{2}{R_1\bar{R}_1}+\frac{1}{R_2{}^2}+\frac{1}{\bar{R}_2{}^2}+\frac{2}{R_2\bar{R}_2}-\frac{2}{\bar{R}_1R_2}-\frac{2}{R_1R_2}-\frac{2}{\bar{R}_1\bar{R}_2}-\frac{2}{R_1\bar{R}_2}\right]\\
&-\frac{\gamma^3}{64\rho^2}\left[\frac{z+m-i/n}{R_2}+\frac{z+m+i/n}{\bar{R}_2}-\frac{z-m+i/n}{\bar{R}_1}-\frac{z-m-i/n}{R_1}\right]\\
&\times\Bigg[\frac{4\rho^2-\bar{R}_1{}^2}{\bar{R}_1{}^2}+\frac{4\rho^2-R_1{}^2}{R_1{}^2}+\frac{4\rho^2-R_2{}^2}{R_2{}^2}+\frac{4\rho^2-\bar{R}_2{}^2}{\bar{R}_2{}^2}+\frac{6\rho^2-2(z-m+i/n)(z-m-i/n)}{R_1\bar{R}_1}\\
&\quad+\frac{6\rho^2-2(z+m-i/n)(z+m+i/n)}{R_2\bar{R}_2}-\frac{6\rho^2-2(z+m-i/n)(z-m+i/n)}{R_2\bar{R}_1}\\
&\quad-\frac{6\rho^2-2(z+m+i/n)(z-m+i/n)}{\bar{R}_2\bar{R}_1}-\frac{6\rho^2-2(z+m-i/n)(z-m-i/n)}{R_2R_1}\\
&\quad-\frac{6\rho^2-2(z+m+i/n)(z-m-i/n)}{\bar{R}_2R_1}\Bigg]\Bigg\}\ ,
\end{aligned} \tag{18}$$

$$\begin{aligned}
\psi_2=\tfrac{1}{2}e^{2\lambda-2\mu}\Bigg\{&\frac{\gamma}{4\rho^2}\left[\frac{2\rho^2(z-m+i/n)+(z-m+i/n)^3}{\bar{R}_1{}^3}+\frac{2\rho^2(z-m-i/n)+(z-m-i/n)^3}{R_1{}^3}\right.\\
&\left.-\frac{2\rho^2(z+m-i/n)+(z+m-i/n)^3}{R_2{}^3}-\frac{2\rho^2(z+m+i/n)+(z+m+i/n)^3}{\bar{R}_2{}^3}\right]\\
&+\frac{\gamma^2}{16\rho^2}\Bigg[\frac{2R_2{}^2-3\rho^2}{R_2{}^2}+\frac{2\bar{R}_2{}^2-3\rho^2}{\bar{R}_2{}^2}+\frac{2\bar{R}_1{}^2-3\rho^2}{\bar{R}_1{}^2}+\frac{2R_1{}^2-3\rho^2}{R_1{}^2}\\
&\quad+\frac{4(z+m-i/n)(z+m+i/n)-2\rho^2}{R_2\bar{R}_2}+\frac{4(z-m+i/n)(z-m-i/n)-2\rho^2}{R_1\bar{R}_1}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{4(z+m-i/n)(z-m+i/n)-4\rho^2}{R_2\bar{R}_1}-\frac{4(z+m+i/n)(z-m+i/n)-4\rho^2}{\bar{R}_2\bar{R}_1}\\
&-\frac{4(z+m-i/n)(z-m-i/n)-4\rho^2}{R_2R_1}-\frac{4(z+m+i/n)(z-m-i/n)-4\rho^2}{\bar{R}_2R_1}\Bigg]\\
&+\frac{\gamma^3}{64\rho^2}\left[\frac{z+m-i/n}{R_2}+\frac{z+m+i/n}{\bar{R}_2}-\frac{z-m+i/n}{\bar{R}_1}-\frac{z-m-i/n}{R_1}\right]\\
&\times\Bigg[\frac{4\rho^2-\bar{R}_1{}^2}{\bar{R}_1{}^2}+\frac{4\rho^2-R_1{}^2}{R_1{}^2}+\frac{4\rho^2-R_2{}^2}{R_2{}^2}+\frac{4\rho^2-\bar{R}_2{}^2}{\bar{R}_2{}^2}\\
&\quad+\frac{6\rho^2-2(z-m+i/n)(z-m-i/n)}{R_1\bar{R}_1}+\frac{6\rho^2-2(z+m-i/n)(z+m+i/n)}{R_2\bar{R}_2}\\
&\quad-\frac{6\rho^2-2(z+m-i/n)(z-m+i/n)}{R_2\bar{R}_1}-\frac{6\rho^2-2(z+m+i/n)(z-m+i/n)}{\bar{R}_2\bar{R}_1}\\
&\quad-\frac{6\rho^2-2(z+m-i/n)(z-m-i/n)}{R_2R_1}-\frac{6\rho^2-2(z+m+i/n)(z-m-i/n)}{\bar{R}_2R_1}\Bigg]\Bigg\}\,.
\end{aligned}
\tag{19}
$$

As $n\to\infty$, these become equal to the tetrad components of the conformal tensor for the $\gamma$ metric.[1] They vanish asymptotically; i.e., as $|z|\to\infty$ or $\rho\to\infty$. They are all finite for finite $n$ as $\rho\to 0$.

## IV. CONCLUDING REMARKS

The $n\gamma$ solution describes an interesting vacuum solution of the Einstein equations. The most widely studied solution of the Einstein equations is the Schwarzschild solution ($\gamma=1$, $n\to\infty$). In Weyl coordinates, this solution has strange behavior on the axis for $|z|\le m$. It turns out that the solution is not geodesically complete on this portion of the axis and must be extended. A natural generalization of the Schwarzschild solution is the $\gamma$ solution ($n\to\infty,\gamma\neq 1$). This solution is generally singular on the axis at $|z|<m$ and has a directional singularity at $|z|=m$. Hence a coordinate change near the $|z|=m$ is required, followed usually by an analytic extension (Ref. 1). For the $n\gamma$ solution, the algebraic invariants of the Riemann tensor seem to be always well behaved. However the axis everywhere lacks elementary flatness. This suggests the possibility that the coordinate system is bad in the neighborhood of the entire axis.

The $n\gamma$ solution or the $n$ ($\gamma=1$) solution seems to have interesting suggestions to make regarding gravitational collapse to the Schwarzschild solution. As $n\to\infty$, the solution approaches the Schwarzschild solution, but in a discontinuous sort of way. The geometric properties of the $n$ solution are discontinuous in nature as $n\to\infty$. Similarly the pure $\gamma$ solution becomes the Schwarzschild solution as $\gamma\to 1$, but the geometric behavior is discontinuous as $\gamma\to 1$. It would seem that a collapsing object may not be able to go adiabatically and smoothly to the Schwarzschild solution if its exterior is described by the $n\gamma$ solution. The other limit for the $n\gamma$ solution $n\to 0$, $n\gamma\to$finite, should describe a source which is becoming increasingly cylindrically symmetric. Collapsing cylindrically symmetric sources may form naked singularities.[2] It is interesting to speculate on the fate of a source described by the $n\gamma$ metric with small $n$. Of course the aforestated remarks only suggest that the $n\gamma$ metric is worthy of further study. An understanding of the nature of the collapse of an object whose exterior is initally given by the $n\gamma$ metric can only be reached after an investigation of the dynamical problem.

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[1]D. Papadopoulos, B. Stewart, and L. Witten, Phys. Rev. D 24, 320 (1981). The $\gamma$ metric has a long history, some of which is cited in Ref. 1 of this paper. In the literature it is sometimes called the Voorhees or the Zipoy-Voorhees metric.

[2]B. D. Miller, J. Math. Phys. 22, 1452 (1981).