Nonperturbative dynamics for abstract (p,q) string networks

Fotini Markopoulou* and Lee Smolin†
Center for Gravitational Physics and Geometry Department of Physics, The Pennsylvania State University, University Park, Pennsylvania 16802
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We describe abstract (p,q) string networks which are the string networks of Sen without the information about their embedding in a background spacetime. The non-perturbative dynamical formulation invented for *spin* networks, in terms of causal evolution of dual triangulations, is applied on them. The formal transition amplitudes are sums over discrete causal histories that evolve (p,q) string networks. The dynamics depend on two free $SL(2,Z)$ invariant functions which describe the amplitudes for the local evolution moves. [S0556-2821(98)05020-6]

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I. INTRODUCTION

In a very interesting paper [1], Sen introduces the notion of a network of (p,q) strings and hypothesizes that they might play a role in a non-perturbative formulation of string theory analogous to the role $SU(2)$ spin networks play in non-perturbative quantum general relativity [2,3]. These are composed of (p,q) strings that meet at trivalent vertices introduced by Schwarz [4] and shown to be Bogomol'nyi-Prasad-Sommerfield (BPS) states in [5]. This suggestion has been explored in several recent papers [6,7]. In this note we take up Sen's suggestion and show that the (p,q) string networks fit naturally into a class of generalizations of spin networks that have been studied as possible non-perturbative quantum theories of gravity [8]. The dynamics of spin networks that one of us suggested in [9] may be applied directly to (p,q) string networks to yield a formal non-perturbative background independent formulation of their dynamics. This formulation is in turn closely related to path integral formulations of non-perturbative quantum gravity based on deformations of topological quantum field theories [10–15].

Penrose originally invented spin networks to provide a simple realization of the idea that the quantum geometry of spacetime must be discrete and based only on algebra and combinatorics, with all continuum notions arising in the continuum limit [16]. As introduced by Sen in [1], (p,q) string networks depend on an embedding in a flat background $(9+1)$-dimensional spacetime. However if they are to serve as the basis of a non-perturbative formulation of string theory the (p,q) string networks should be removed from this context and postulated to exist as background independent combinatorial entities from which the continuum description is to be derived as a suitable approximation. The possibility of this is suggested by the striking fact that, just as in Penrose's original formulation [16], angles for the embedding of the (p,q) string network in flat space are derived purely from the combinatorics of the network [1].

This raises the question of how the dynamics and observable algebra of the theory are to be defined in the absence of a background manifold. In [17,9,8] we have studied this problem for non-embedded extensions of spin-networks. This led to the introduction of a general theory with two basic features: (1) The spin networks are extended to 2-dimensional labeled surfaces and the space of states is constructed algebraically. (2) Dynamics is expressed in terms of local moves that generate histories as combinatorial structures which share several of the properties of Lorentzian spacetimes including causal structure and many-fingered time.

As shown by Schwarz [4] and elaborated for networks by Krogh and Lee [6], (p,q) strings can be understood as labeled 2-dimensional surfaces. This leads directly to the application of the dynamical formulation introduced in [8,9] to string networks.

We must emphasize that we have not yet shown whether the dynamics we propose has any connection with the dynamics of (p,q) string networks embedded in background manifolds. The dynamics in the form proposed here is specified by two free $SL(2,Z)$ invariant functions. In the future, these may be determined by the condition that they match the dynamics predicted by M theory, but this has not yet been done.

In the next section we introduce the abstract (p,q) string networks, which obey the combinatorics of the (p,q) networks of Sen [1] but are not embedded in any background manifold. In Sec. III we describe non-perturbative dynamics for these abstract (p,q) networks.

II. ABSTRACT (p,q) STRING NETWORKS

Spin networks (and their extensions to general quantum groups) can be understood in the general framework first developed for conformal field theory and topological field theory [19–21] in which states associated to various manifolds are constructed from algebraic operations. It is easy to see that (p,q) string networks can be formulated in the same framework.

We consider sets $\{(p_i,q_i)\}$ consisting of n relatively prime pairs of integers. Each pair may be visualized as the first homology classes of a T^2 as in [6], but the only role this will play in the abstract formalism is that the torus parameter τ may come into the $SL(2,Z)$ invariant functions that specify

*Email address: fotini@phys.psu.edu
†Email address: smolin@phys.psu.edu

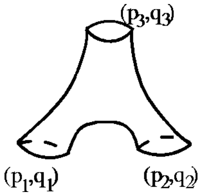

FIG. 1. The representation of the product (1) as a three string vertex.

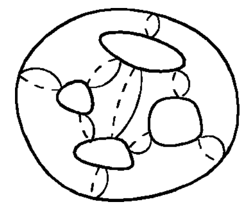

FIG. 2. A genus 4 surface cut into six trinions B_I^3 by circles c_α.

the dynamics. We may note that the relatively prime pairs (p,q) also label the rational numbers or, alternatively, the projective representations of the maps of the circle to itself.[1] As a result, there is a natural multiplication defined on them, which is

$$(p_1,q_1)\otimes(p_2,q_2)=(p_1+p_2,q_1+q_2)=r(p_3,q_3)\rightarrow(p_3,q_3) \tag{1}$$

where r, a positive integer, is the greatest common factor of p_1+p_2 and q_1+q_2. Associativity follows from the uniqueness of prime factorization.

The basic idea of a categorical construction of a topological quantum field theory (TQFT) or conformal field theory [19–21] is that sets of labels, which are representations of some algebra, are mapped to manifolds such that the product is represented by cobordisms. In this case we will associate the sets $\{(p_i,q_i)\}$ to sets of circles so that the algebra they generate is mapped to cobordisms of two manifolds. The product (1) is symbolized by a trinion, or 3-punctured sphere, with an object (p,q) associated with each incoming puncture and $(-p,-q)$ associated with the one outgoing puncture as in Fig. 1. When $r\neq 0$ we say there exists a morphism from $(p_1,q_1)\otimes(p_2,q_2)$ to (p_3,q_3). The product (1) is thus the algebraic counterpart of the three string vertex, which when the strings are represented as tubes becomes the trinion. Under reversal of orientation of a puncture we require $(p,q)\rightarrow(-p,-q)$. Taking all the orientations consistently a trinion is labeled by three relatively prime pairs (p_i,q_i) such that,

$$\sum_{i=1}^{3} p_i=\sum_{i=1}^{3} q_i=0. \tag{2}$$

A trinion with labels (p_i,q_i) satisfying Eq. (2) will be called a good trinion.

An abstract (p,q) string network is then described by a compact surface $\mathcal{S}$ with m punctures constructed from sewing good trinions together along the punctures, as shown in Fig. 2. The result is a surface $\mathcal{S}$ with a set of circles c_α which decompose it into a set of good trinions, with labels $(p,q)_\alpha$ on the punctures c_α of the trinions. A closed (p,q) string network is one without free punctures.

A non-degenerate string network will be one that has a trinion decomposition in which no two trinions meet on more than one circle. We may note that the string networks of Sen [1] are non-degenerate because the strings are straight lines in flat spacetime. We will thus restrict ourselves to the consideration of non-degenerate abstract string networks.

To the set of non-degenerate abstract (p,q) string networks we associate a Hilbert space $\mathcal{H}^{SN}$ constructed as follows. Given each surface $\mathcal{S}$ with its circles c_α there is a space $\mathcal{V}_{c_\alpha}^{\mathcal{S}}$ spanned by a basis of states $|\mathcal{S},c_\alpha;(p,q)_\alpha\rangle$. We choose the inner product $\langle|\rangle$ so that these are orthonormal [8]. We then define

$$\mathcal{H}^{SN}=\oplus\mathcal{V}_{c_\alpha}^{\mathcal{S}} \tag{3}$$

where the sum is over all labeled surfaces $(\mathcal{S},c_\alpha)$.[2]

What distinguishes theories of the type we are about to propose from 3-d topological quantum field theory is that there are operators which act on $\mathcal{H}^{SN}$ to change the topology of the surface $\mathcal{S}$.

For example, consider Υ^1 and Υ^2 to be two open (p,q) string nets, each with n, free punctures, with labels $(p,q)_k$, $k=1,\ldots,p$. Thus Υ^1 and Υ^2 have the same boundary, but are otherwise different. There is then an operator $\mathcal{C}_{\Upsilon^1,\Upsilon^2}$ which acts on each state in the basis $|\mathcal{S},c_\alpha;(p,q)_\alpha\rangle$ by looking for components of $(\mathcal{S},(p,q)_\alpha)$ which are isomorphic to $(\Upsilon^1,(p,q)_k)$, and replaces it by a different 2-surface $(\Upsilon^2,(p,q)_k)$, which has *the same boundary*, but otherwise differs.

Now, there may be more than one places in $(\mathcal{S},(p,q)_\alpha)$ where $(\Upsilon^1,(p,q)_k)$ is recognized as a subset. There are then several maps

$$r_I:(\Upsilon^1,(p,q)_k)\rightarrow(\mathcal{S},(p,q)_\alpha). \tag{4}$$

For each I the map r_I picks out a set of n non-intersecting circles c_k^I, $k=1,\ldots,n$ in $\mathcal{S}$. Cutting $\mathcal{S}$ on these circles decomposes it into the two pieces $r_I(\Upsilon^1)$ and $(\mathcal{S}-r_I(\Upsilon^1))$. We may then sew Υ^2 onto $(\mathcal{S}-r_I(\Upsilon^1))$ along the circles c_k^I as they are also identical with the set of punctures of Υ^2. We may call this state $|(\mathcal{S}-r_I(\Upsilon^1))\cup\Upsilon^2\rangle$ with the dependence on the circles and labelings understood. The operator $\mathcal{C}_{\Upsilon^1,\Upsilon^2}$ is then defined by

[1] We thank Louis Crane for this observation.

[2] All of this can be expressed compactly by saying that there is a functor from the cobordism category *Cob* whose objects are sets of circles and whose morphisms are two manifolds with boundary to the category of relatively prime pairs of integers with the product (1). The general formulation of this kind of correspondence is described in [20,19,21].

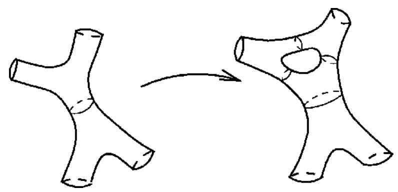

FIG. 3. A local substitution move.

$$\mathcal{C}_{\Upsilon^1,\Upsilon^2}|\mathcal{S}\rangle=\sum_{I} |(\mathcal{S}-r_I(\Upsilon^1))\cup\Upsilon^2\rangle. \tag{5}$$

A slightly more complicated definition can be made when $\mathcal{S}$ is not compact. An example of such a substitution move is given in Fig. 3. These operators are local as they only change local pieces of a string network which are isomorphic to a given piece. We now show that a particular set of such operators may be used to formulate the dynamics of the abstract (p,q) string networks.

III. DYNAMICS

Following the proposal made for causal evolution of spin networks in [9] we may give a general form for dynamics of (p,q) string networks that is locally causal. We note that in a background independent formalism the notion of locality must be based on the states themselves.

For the $2+1$ case it is actually more convenient to describe the dynamics in the dual picture [9]. An abstract trivalent string network $|\mathcal{S},j_\alpha\rangle$ may be equivalently described by a labeled triangulation $\mathcal{T}$, (see Fig. 4) in which the sides of the triangles carry the (p,q) labels and the condition (2) is satisfied by the three (p,q) pairs around each triangle. If the string network is closed, the corresponding triangulated surface is compact. We may also note that non-degeneracy of the string network implies that the triangulation is non-degenerate in the sense that no two triangles share more than one edge. The issue of the evolution of a network is now translated to the evolution of a 2-dimensional triangulation. Now, the generating *local* evolution moves from a non-degenerate 2-dimensional triangulation to another one, which leave invariant the topology of the triangulated surface, are the Pachner moves ([22], which were defined originally for *PL* manifolds). These have already been used to describe

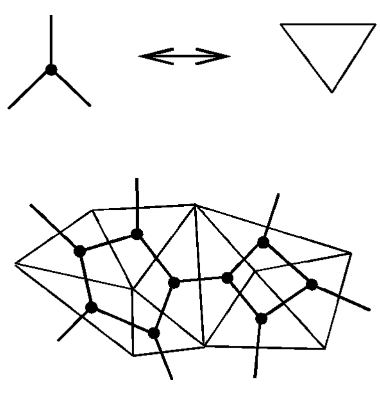

FIG. 4. Triangulations dual to trivalent networks.

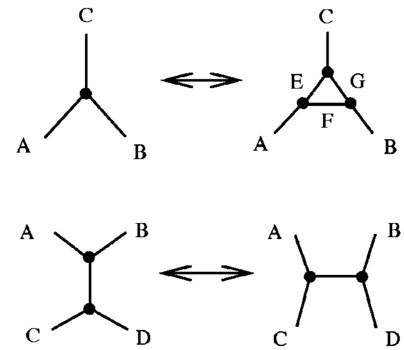

FIG. 5. The $(2+1)$ dimensional Pachner moves.

causal spacetime histories of spin networks [9] (and in the Euclidean case they can be found in [11,14]). Let us use them here to write down a formal path-integral evolution for abstract (p,q) networks.

The Pachner moves are transitions from an initial (p,q) triangulation $\mathcal{T}_i$ to a final one $\mathcal{T}_f$. They are labelled tetrahedra having the triangles of $\mathcal{T}_i$ and $\mathcal{T}_f$ as their boundaries. The different moves are the different ways of gluing a tetrahedron on n connected triangles of $\mathcal{T}_i$ and get $4-n$ connected triangles in $\mathcal{T}_f$, $n\leqslant 3$. The new edges which are created by the move must be labeled consistently with Eq. (1) given the labelings of the remaining edges. See Fig. 5. For (p,q) networks, these moves are particularly simple. The $1\rightarrow 3$ move is always possible. The inverse, $3\rightarrow 1$ requires that there are three triangles in $\mathcal{T}_i$ around a single vertex, as in Fig. 5. A $2\rightarrow 2$ move is only possible when labels of the four external sides add up to (0,0).

By iterating the moves one constructs a sequence of string networks $\Gamma_0,\Gamma_1,\ldots=\{\Gamma_i\}$. As in general relativity in which local evolution of initial data generates a spacetime, any such sequence generates a *history*. Also as in general relativity, a given history may be generated many ways by local evolution moves. For this reason it is best to define it abstractly, as follows.

A history $\mathcal{M}$ between an initial string network Γ_0 and a final string network Γ_f is then a three dimensional simplicial complex consisting of N tetrahedra τ_I such that $\partial\mathcal{M}=\Gamma_0\cup\Gamma_f$ (where we identify the string net with its dual triangulation). Each edge of $\mathcal{M}$ is labeled by a pair (p,q) such that the conditions (2) are satisfied on every face. The faces of each history make up a causal set[3] that is defined as follows. The faces of each tetrahedra are divided into a past set $\mathcal{P}$ and a future set, $\mathcal{F}$ so that for two faces $f_1\in\mathcal{P}$ and $f_2\in\mathcal{F}$ of the same tetrahedron we have $f_1<f_2$ (where $<$ indicates the partial ordering). We then extend this to all faces in $\mathcal{M}$ so that $f_1<f_r$ if there is a sequence of r faces f_i such that $f_i<f_{i+1}$ within some tetrahedron. Thus each history has a discrete causal structure.

[3]A causal set is a partially ordered set, where the order relation stands for causality, which has no closed causal loops. The formulation of discrete quantum theories of gravity with such causal structures has been studied by Sorkin and collaborators [23] and 't Hooft [24].

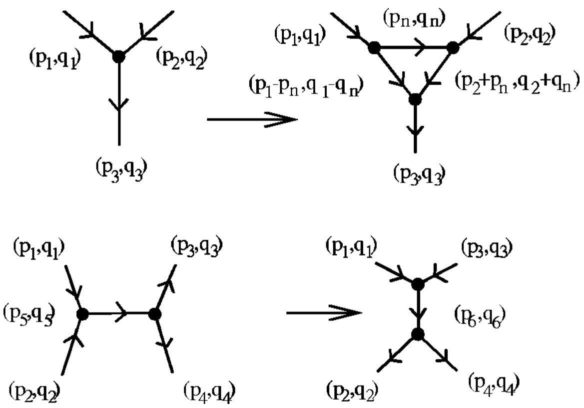

FIG. 6. The allowed Pachner moves for (p,q) string networks.

Given the causal structure of a history $\mathcal{M}$ there is a natural definition of a spacelike surface. A discrete spacelike surface, Δ, is a collection of triangles in $\mathcal{M}$ without 2 boundary and with no two triangles causally related. Because the causal structure is local, each history $\mathcal{M}$ can be foliated in many different ways into sequences of spacelike surfaces [abstract (p,q) networks], $\mathcal{T}_i, \Delta_1, \ldots, \Delta_I, \ldots, \mathcal{T}_f$. If a history was generated by a sequence Γ_i of string networks each of these is a spacelike slice, but the history $\mathcal{M}$ they construct will have many spacelike slices not in the list $\{\Gamma_i\}$. Thus, the theory has a discrete analogue of the multi-fingered time of general relativity.

To each tetrahedron T in $\mathcal{M}$ we can associate an amplitude, which is the amplitude for a local transition between two string networks. The amplitude $\mathcal{A}_{T_I}[\tau,(p_i,q_i)_I]$ will then depend on the labels of the edges of that tetrahedron. To be consistent with string theory it should be some $SL(2,Z)$ invariant function of the the string theory parameter τ and the (p,q) integers on its faces. There will also be in general different amplitudes for different causal structures on the tetrahedron, T_I, this is indicated by the dependence on T_I in the form of the amplitude $\mathcal{A}_{T_I}[\tau,(p_i,q_i)_I]$.

If the history $\mathcal{M}$ contains N tetrahedra T_I, the amplitude for $\mathcal{M}$ is local when it is the product

$$\mathcal{A}[\mathcal{M}] = \prod_{I=1}^{N} \mathcal{A}_{T_I}[\tau,(p_i,q_i)_I]. \tag{6}$$

For the $1\rightarrow 3$ Pachner move, $\mathcal{A}_{1\rightarrow 3}[\tau,p_i,q_i]$ must be an $SL(2,Z)$ invariant function of the parameter τ, the three incoming labels (p_i,q_i), $i=1,2,3$, symmetric in the i, and an internal pair (p_n,q_n) that runs around the internal loop, subject to Eq. (2). (See Fig. 6.) The amplitude for the inverse $3\rightarrow 1$ moves is by Hermiticity the complex conjugate of the $1\rightarrow 3$ moves. For the $2\rightarrow 2$ move the amplitude $\mathcal{A}_{2\rightarrow 2}[\tau,p_i,q_i]$ is a real function of τ and the four external edges (p_i,q_i), subject to the constraint that $\Sigma_{i=1}^{4}(p_i,q_i) = (0,0)$. (See Fig. 6.) An invariant choice of $\mathcal{A}_{1\rightarrow 3}[\tau,p_i,q_i]$ and $\mathcal{A}_{2\rightarrow 2}[\tau,p_i,q_i]$ defines a non-perturbative (p,q) network theory. Using the operators of the last section, a Hermitian operator $\hat{H}$ on $\mathcal{H}^{SN}$ can be defined [8] such that $\langle \mathcal{T}^2|\hat{H}|\mathcal{T}^1\rangle$ vanishes unless $\mathcal{T}^2$ differs from $\mathcal{T}^1$ by the action of one Pachner move, in which case it is equal to the corresponding amplitude, $\mathcal{A}_{T_I}[\tau,(p_i,q_i)_I]$.

The result of the action $\hat{H}|\mathcal{T}_i\rangle$ is then the linear combination of the states reached by the possible Pachner moves from $|\mathcal{T}_i\rangle$ each multiplied by the corresponding amplitude $\mathcal{A}_{T_I}[\tau,(p_i,q_i)_I]$.[4] Given $\mathcal{T}_i$ and $\mathcal{T}_f$ then, the transition amplitude between the two is

$$\mathcal{A}_{\mathcal{T}_i \rightarrow \mathcal{T}_f} = \langle \mathcal{T}_i | e^{\imath t H} | \mathcal{T}_f \rangle. \tag{7}$$

Expansion of (7) gives the amplitude as a sum over histories $\mathcal{M}$. t is a formal parameter which scales the amplitudes and has no interpretation as a physical time. (One measure of physical time is the number, N, of tetrahedra in each history, as this is related, in the continuum limit, to an integral of a positive function over the spacetime history.) We note that formally the evolution defined by Eq. (7) is unitary.

We do not know anything more about the choices of the amplitudes. It is possible that consistency with perturbative string theory plus $SL(2,Z)$ invariance will constrain or determine them.[5] Alternately, one may try to search the space of amplitudes for choices that lead to critical behavior, resulting in a continuum limit. The appropriate critical behavior should be related to directed percolation, for reasons explained in [17]. There is also a result that suggests that perturbative string theories may be derived from the perturbation theory of theories of the kind we have considered here, in the case that they have a good continuum limit [18].

IV. CONCLUSION

What we have described here is a proposal for a formal path integral evolution of abstract (p,q) networks based on two steps. The first is the construction of a space of states $\mathcal{H}^{SN}$ composed of abstract graphs with the same labels and combinatorics as a large class of BPS states which are networks of (p,q) strings embedded in $(9+1)$-dimensional Minkowski space. This fits into the categorical framework behind much recent work in non-perturbative quantum gravity and topological field theory. The second step is a proposal for the dynamics of the theory based on a set of local moves. The exact dynamics of the theory is not prescribed, but its choice reduces to the specification of two $SL(2,Z)$ invariant functions.

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[4]For the details of this construction in the general case, see [8].

[5]We thank Shyamoli Chaudhuri and Djordje Minic for conversations about this.

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