

Quantum Theory from Quantum Gravity

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November 17, 2003

Abstract

We provide a mechanism by which, from a background independent model with no quantum mechanics, quantum theory arises in the same limit in which spatial properties appear. Starting with an arbitrary abstract graph as the microscopic model of spacetime, our ansatz is that the microscopic dynamics can be chosen so that 1) the model has a low energy limit which reproduces the non-relativistic classical dynamics of a system of N particles in flat spacetime, 2) there is a minimum length, and 3) some of the particles are in a thermal bath or otherwise evolve stochastically. We then construct simple functions of the degrees of freedom of the theory and show that their probability distributions evolve according to the Schrödinger equation. The non-local hidden variables required to satisfy the conditions of Bell's theorem are the links in the fundamental graph that connect nodes adjacent in the graph but distant in the approximate metric of the low energy limit. In the presence of these links, distant stochastic fluctuations are transferred into universal quantum fluctuations.

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1 Introduction

It is often stated that the goal of research in quantum gravity is to find the way in which nature unifies quantum theory with general relativity. One way to try to accomplish this is by a more or less standard quantization of the gravitational field equations, as in loop quantum gravity[1]. Another is to consider some quantum theory in different backgrounds, as in string theory[2].

In loop quantum gravity, one discovers that the appropriate basis states for quantum spatial geometry are spin networks, graphs whose edges are labeled by spins. While a great deal of progress has been achieved, there remain open issues. One is that the theory is not unique. A second is that to test whether a given loop quantum gravity theory is correct, one needs to show how general relativity and flat spacetime arise in the appropriate limit. Spin foams, the path-integral evolution of spin networks, arose as a tool in this effort [3].

In many approaches to quantum gravity, the expectation is that the classical space and time of general relativity are not fundamental but rather they arise as the low-energy approximate description of the fundamental Planck scale theory. Quantum theory, on the other hand, is almost invariably expected to hold unmodified all the way down to Planck distances, although it is often pointed out that a continuous spacetime is already built in quantum theory.

The question we raise in this paper is whether *both* general relativity and quantum theory may only be approximations to the as-yet unknown quantum theory of gravity. We provide a mechanism by which, from a background-independent model with no quantum mechanics, quantum theory arises in the same limit in which spatial properties appear.

More specifically, we start with a simple model of the fundamental theory, based on the adjacency matrix of an abstract graph. We assume that the underlying model has an approximation in which classical physics emerges. In particular, we assume that the theory has a non-relativistic low energy limit in which some of the nodes of the graph correspond to the positions of particles in three dimensional space which we call $x_i^a(t)$, where $1 = 1, \dots, N$ labels the particles and a is a spatial index and that the $x_i^a(t)$ evolve according to Newtonian dynamics with some potential $V(|x_i - x_j|)$. We further ask that *some* of the embedding coordinates are subject to stochastic fluctuations, for example, by being embedded in a heat bath.

We do not discuss here for what choices of dynamics and under what conditions this classical, low energy limit will emerge. Our goal in this paper is different, it is to show that quantum mechanics may also appear in the *same* limit. In particular, we show that quantum theory for all the particles appears only in the limit described above: certain quantities can be defined, functions of both the embedding coordinates and the original graph, whose probability distributions evolve in time according to the Schrödinger equation. Planck's constant then turns out to be a derived quantity.

From the perspective of the issues in the foundations of quantum theory, this is a

stochastic hidden variable model. From the work of Bell [6], we know that any hidden variable theory has to be non-local. However, in all these works and the relevant experimental tests, locality is of course defined with respect to the causal structure of spacetime. If the smooth 3+1 spacetime we live in is only approximate, all kinds of possibilities present themselves. Indeed, in our model, there are two notions of locality. There is a notion of locality in the graph of the fundamental theory: two nodes are nearby if they are connected in the graph. A separate notion of locality holds in the embedding of the graph used in the low energy limit. Two particles, represented by the embedding of two nodes in the graph, are nearby if they are close in the metric of the embedding space.

These two notions of locality will not in general coincide. We shall see that the non-locality needed to derive quantum theory from a deterministic model arises exactly because of this. The non-locality required to recover quantum theory from a non-quantum fundamental theory is non-locality in space. But if space itself is an emergent property, relevant only at a coarse grained level, the fundamental theory can still be local, if by that we mean local in the topology in which the fundamental degrees of freedom are defined. Microscopic locality is a generic property of quantum gravity theories such as spin foams, in which smooth spacetime is expected to be emergent.

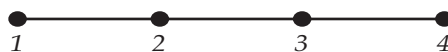
What we present in this article is only an outline of such a mechanism in which quantum theory arises because of the discrepancy between microscopic locality and locality in the emergent spacetime. Much remains to be filled in before one has a complete model. We discuss some possibilities and open issues in the Conclusions.

The outline of this paper is as follows. In the next section we introduce our model and the precise assumptions about its low energy limit. The main method of this paper is stochastic differential equations[4], the elements of which we need are reviewed in section 3. We make use of a formulation of quantum theory due to Nelson, and in section 4 we review the conditions he gives for a stochastic dynamics to reproduce the predictions of quantum theory[5]. In section 5 we present the main result of this paper, which is the derivation of Nelson's conditions from our model. This is followed by a brief statement of conclusions and questions for future work.

2 The model

We start with the fundamental theory given by a very simple model: a graph Γ with a finite set of N nodes. Two nodes can be connected by at most one edge and there are no self-loops (edges from a node to itself).

Two vertices are called *adjacent* if they are connected by an edge. The *adjacency matrix* Q of Γ is an $N \times N$ symmetric matrix, with $Q_{ij} = 1$ if there is an edge in the graph connecting nodes i and j , and 0 otherwise. For example, the adjacency matrix of the graph



is

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The graph can be thought of as a simple model of a universe of N fundamental building blocks, where the only information we are given is the adjacency of these subparts of the universe.

The simplest possible way to set up a correspondence between the graph and a flat 3-dimensional space is to embed the graph in R^3 . The nodes of Γ then acquire coordinates x_i^a ; $a = 1, 2, 3$; $i = 1, \dots, N$. We will not discuss here the mechanism by which the embedding of the graph into R^3 arises. Such mechanisms have been discussed elsewhere in the spin foam literature¹.

The argument of this paper assumes the existence of that approximation, and asks what becomes of the information discarded in taking the limit, having to do with the original combinatorial structure. We shall show that under certain mild assumptions, variables which are functions of both the embedding variables and the original combinatorial degrees of freedom evolve quantum mechanically. We do not need to specify the dynamics of the model or the process by which Newtonian dynamics is extracted in a low energy and non-relativistic limit. We need only a few assumptions concerning the limit, which we now specify.

We first require that there is a *minimum length* in the embedding, namely that

$$|x_i^a - x_j^a|_{\min} \sim l, \quad (1)$$

l being that minimum distance. For example, l could be the Planck length. Let us also call L the *average distance* between two nodes:

$$\langle (x_i^a - x_j^a)^2 \rangle =: L^2. \quad (2)$$

The graph is subject to some microscopic rules of evolution (that is, the microscopic model is a spin foam with a single in and out graph and presumably a single interpolating history) such that:

¹For example, in loop quantum gravity and spin foam models, matter fields such as fermion and scalar fields live on spin network nodes and these are expected to be identified as particles in the low energy limit under discussion. Alternatively, it may be the case that part of the approximation procedure involves a process of coarse graining of a spin network or a spin foam, so that what corresponds to classical spacetime is a coarse grained spin network or spin foam. If so, again, we do not need, for the purpose of this paper, to know any details of the coarse graining procedure except that it takes as input a combinatorial structure and gives, for certain states or histories required to describe the non-relativistic limit of general relativity, an embedding of that structure in R^3 . Similar remarks apply to other discrete formulations of quantum gravity including causal sets and dynamical triangulations. If the theory has a low energy approximation that recovers general relativity, it will have a further approximation which recovers Newtonian dynamics of point particles in R^3 .

1. The model possesses a low energy limit, for $L \gg l$, in which the node coordinates evolve according to Newtonian mechanics as if they were massive particles²:

$$S_l = \int dt \left(\sum_{i=1}^N \frac{m}{2} (\dot{x}_i^a)^2 - V(x) \right). \quad (3)$$

For simplicity, we take all the masses to be the same.

2. *Some* of the node positions are subject to a Brownian motion, namely, in addition to eq. (3), some of the $\{x_i^a\}$ obey the stochastic differential equation

$$dx_i^a(t) = b_i^a(x(t), t)dt + dw_i^a(t), \quad (4)$$

where the $dw(t)$ are Gaussian with mean 0, mutually independent, and

$$\langle dw_i^a(t) dw_j^b(t) \rangle = 2\nu_i^x dt \delta^{ab} \delta_{ij}, \quad (5)$$

where ν_i is the diffusion coefficient for x_i^a . We will give the details of the fluctuations of the x 's in section 3.1.

This stochastic part of the evolution of some of the node coordinates could be due either to the corresponding nodes being in a thermal bath (i.e., there are hot regions in the embedded model), or some other source of uncertainty in assigning coordinates to the nodes (i.e., the coordinates could be a coarse-grained description of several of the underlying nodes).

3. There is no requirement that the adjacency of the graph translates into locality in the embedding. That is, in the graph the nearest neighbours of node i are the nodes in Γ that can be reached from i by traveling along a single edge. Similarly, the next-nearest neighbours are those nodes that can be reached using two edges, and so on. In the embedding description, given the coordinate $\{x_i^a\}$ of the same node, we may ask for its neighbours up to distance r from $\{x_i^a\}$, namely all the nodes which lie inside a ball with center $\{x_i^a\}$ and radius r . Clearly, it is possible for two nodes to be nearest neighbours in the graph and arbitrarily far apart in the embedding (see fig.1).
4. Finally, we will assume for the purposes of this paper that the edges of the graph, and hence the elements of the adjacency matrix Q_{ij} do not evolve in time.

We may summarize the features of this model as: a fundamental finiteness, a Newtonian limit in which an external time parameter can be identified and stochastic fluctuations when the spatio-temporal description is used. We will see that the very surprising feature is that quantum theory is also contained in this model, in fact, we will derive it from it! That is, quantities can be defined that are simple functions of the graph and the embedding geometry which, as we will show, satisfy the Schrödinger equation.

²It suffices for *some* nodes to evolve according to eq. (3), but for calculational simplicity we take the sum in (3) over all N nodes.

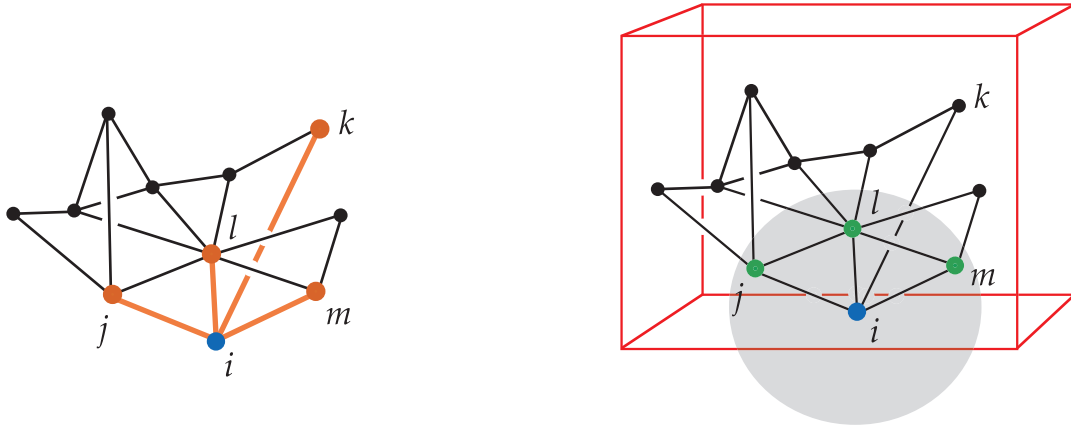


Figure 1: A sketch of how locality in the graph may not be preserved in the embedding. In the abstract graph on the left, the nearest neighbours of i are j, k, l and m . In the embedded graph on the right, a local neighborhood of i defined by the metric of R^3 is drawn, and k is far outside it.

3 Stochastic dynamics

Let us define the matrix

$$M^a = X^a + lQ^a \quad (6)$$

where $Q^a = Q$ and X^a is the $N \times N$ matrix with diagonal elements $X_{ii}^a = x_i^a$, the positions of the N particles in the a component, and zeros elsewhere. As before, l is the minimal length. The matrix M evolves in time according to the dynamics in eq. (3), which affects only the first term, X .

This is the simplest function that contains both the information about the graph, i.e., the graph's adjacency matrix, and the information of the embedding of the graph, i.e., the coordinates of the nodes. The physical intuition is that the eigenvalues λ_i^a of M^a represent corrections to the positions x_i^a arising from the nodes which are adjacent in the graph but non-local with respect to the embedding. The motivation for this stems from the following result, which we will prove in the next sections:

Let n be the average valence of a node in Q . When the x_i^a 's evolve according to classical mechanics, and when $\sqrt{n} \frac{l^2}{L^2} \ll 1$, the evolution of the probability distributions for the λ_i^a 's is given, to leading order in $\sqrt{n} \frac{l^2}{L^2}$, by the Schrödinger equation.

The first step in showing this is to use the standard formulas from perturbation theory to express λ_i^a to leading order as,

$$\lambda_i^a = x_i^a + l^2 \sum_{j \neq i} \frac{(Q_{ij}^a)^2}{x_i^a - x_j^a} + \dots \quad (7)$$

One can check that the second, fluctuating term in the above expression is of order

$$\Delta\lambda_i^a := l^2 \sum_{j \neq i} \frac{(Q_{ij}^a)^2}{x_i^a - x_j^a} \sim \sqrt{n} \frac{l^2}{L}, \quad (8)$$

since it is a sum of n terms of random signs. Thus,

$$\left\langle \frac{\Delta\lambda}{\lambda} \right\rangle \approx \sqrt{n} \frac{l^2}{L^2}, \quad (9)$$

where $\langle \rangle$ means averaged over the nodes and the thermal ensemble.

We will work in the regime

$$\sqrt{n} \frac{l^2}{L^2} \ll 1. \quad (10)$$

Note that when $\sqrt{n} \frac{l^2}{L^2} \rightarrow 0$, $M^a \rightarrow X^a$ and $\lambda_i^a \rightarrow x_i^a$, so that the dynamics becomes classical. We will study the leading order corrections around this limit. We will find that quantum mechanics can be understood in this sense as giving the leading order correction in $\sqrt{n} \frac{l^2}{L^2}$ around the classical limit.

3.1 The thermal fluctuations of the $\{x_i^a\}$

Recall that we allow for some of the $\{x_i^a\}$ to be subject to stochastic fluctuations. The Brownian motion of the $\{x_i^a\}$ is described by the stochastic differential equation[4, 5]

$$dx_i^a(t) = b_i^a(x(t), t)dt + dw_i^a(t), \quad (11)$$

for $dt \geq 0$, where the $dw(t)$ are Gaussian with mean 0, mutually independent, and

$$\langle dw_i^a(t) dw_j^b(t) \rangle = 2\nu^x_i dt \delta^{ab} \delta_{ij}. \quad (12)$$

ν_i is the diffusion coefficient for x_i^a . The $dw(t)$ are independent of the $x(s)$ for $s \leq t$, so b_i^a is the mean forward velocity

$$b_i^a(x(t), t) = Dx_i^a(t), \quad (13)$$

where the mean forward derivative D is defined by

$$Dx(t) = \lim_{\Delta t \rightarrow 0^+} \left\langle \frac{x(t + \Delta t) - x(t)}{\Delta t} \right\rangle. \quad (14)$$

The process is asymmetrical in time, so for $dt \leq 0$ we have

$$dx_i^a(t) = b_{*i}^a(x(t), t)dt + dw_{*i}^a(t), \quad (15)$$

where w_* has the same properties as w except that the dw_* are independent of the $x(s)$ with $s \geq t$. Similar to the forward case,

$$b_{*i}^a(x(t), t) = D_*x_i^a(t) \quad (16)$$

is the mean backward velocity, where the mean backward derivative D_* is given by³

$$D_*x(t) = \lim_{\Delta t \rightarrow 0^+} \left\langle \frac{x(t) - x(t - \Delta t)}{\Delta t} \right\rangle. \quad (17)$$

We define the *average diffusion coefficient* of the x 's to be

$$\nu^x := \langle \nu^x_i \rangle. \quad (18)$$

3.2 The resulting fluctuations of the $\{\lambda_i^a\}$

It is not surprising that the thermal fluctuations of the x 's lead to fluctuations of the corrected values, the λ 's. It is also not surprising that even if a particular x_i is not fluctuating, the corresponding λ_i does, because of the second term in eq.(7). What will be surprising is that the fluctuations of the λ 's have a very different character than those of the x 's, and this is the subject of our paper. Let us first calculate the fluctuations on the corrected positions.

We take the stochastic derivative of the perturbative expansion (7) to find that

$$d\lambda_i^a = dx_i^a - l^2 \sum_{j \neq i} \frac{(Q_{ij}^a)^2}{(x_i^a - x_j^a)^2} (dx_i^a - dx_j^a). \quad (19)$$

Using (11), we rewrite this as

$$d\lambda_i^a = \left[b_i^a - l^2 \sum_{j \neq i} \frac{(Q_{ij}^a)^2}{(x_i^a - x_j^a)^2} (b_i^a - b_j^a) \right] dt + dw_i^a - l^2 \sum_{j \neq i} \frac{(Q_{ij}^a)^2}{(x_i^a - x_j^a)^2} (dw_i^a - dw_j^a). \quad (20)$$

Thus, the λ 's evolve by the stochastic differential equation

$$d\lambda_i^a = \beta_i^a dt + dy_i^a, \quad (21)$$

where β_i^a is the mean forward velocity of λ_i^a ,

$$\beta_i^a(\lambda, t) = D\lambda_i^a, \quad (22)$$

and the fluctuating part can be read off eq.(20) to be

$$dy_i^a = dw_i^a \left[1 - l^2 \sum_{j \neq i} \frac{(Q_{ij}^a)^2}{(x_i^a - x_j^a)^2} \right] + l^2 \sum_{j \neq i} \frac{(Q_{ij}^a)^2}{(x_i^a - x_j^a)^2} dw_j^a. \quad (23)$$

³If $x(t)$ is differentiable, then $Dx(t) = D_*x(t) = dx/dt$, but this is not the case for the thermal motion of the nodes.

As in the case of the x 's, we may also write

$$d\lambda_i^a = \beta_{*i}^a dt + dy_{*i}^a, \quad (24)$$

where β_{*i}^a is the mean backward velocity of λ_i^a

$$\beta_{*i}^a(\lambda, t) = D_* \lambda_i^a. \quad (25)$$

One reason that some of the x 's could be subject to thermal fluctuations is if in the low energy limit $\sqrt{n} \frac{l^2}{L^2} \ll 1$ there are regions in the universe at finite temperature. In this case we may expect that x 's that are near each other in the embedding geometry will be at a similar temperature. As we shall note in the conclusion, this is not the only possible source of such local stochastic fluctuations, but it may serve to give intuition.

Let us now relate the diffusion of the λ 's to that of the x 's. What is of interest in the present paper is nodes whose x 's are not fluctuating. If a given x_i is not fluctuating, i.e., $\nu_i = 0$, what is the fluctuation of the corrected λ_i ?

The diffusion constant of λ_i can be computed from the above equation:

$$\begin{aligned} 2\nu^{\lambda_i} dt &= \langle (dy_i^a)^2 \rangle \\ &= l^4 \left\langle \left(\sum_{j \neq i} \frac{(Q_{ij}^a)^2}{(x_i^a - x_j^a)^2} dw_j^a \right)^2 \right\rangle \\ &= l^4 \left\langle \sum_{j \neq i} \frac{(Q_{ij}^a)^4}{(x_i^a - x_j^a)^4} 2\nu_j^x dt \right\rangle \end{aligned} \quad (26)$$

Let us now assume that the graph connects each λ_i^a to nodes which are uniformly distributed in the embedding space. This allows us to deduce that,

$$\nu^{\lambda_i} = l^4 \left\langle \sum_{j \neq i} \frac{(Q_{ij}^a)^4}{(x_i^a - x_j^a)^4} 2\nu_j^x \right\rangle \sim n \nu^x \frac{l^4}{L^4}. \quad (27)$$

This tells us that each λ is subject to a Brownian motion, *even if its local environment (in the embedding) is at zero temperature*. This is because the origin of the brownian motion of the λ_i^a 's is noise in the x_i^a 's which are distant in space, but near in the graph. This Brownian motion is *irreducible*, in that it cannot be decreased by changing the local conditions, and it is *universal* in that it applies to all the λ 's. We will see in the next section that the consequence of these fluctuations is that the eigenvalues evolve as if quantum theory is true.

4 Nelson's derivation of quantum mechanics

We have assumed that the original coordinates x are subject to stochastic fluctuations, possibly due to hot regions in the universe. We then interpreted the eigenvalues λ of the

embedded spin foam, given by the matrix M , as the corrected positions of the nodes of the graph. In the previous section we saw that this results in a Brownian motion for *all* of the λ 's, transferred from the hot regions to all the λ 's via edges in the original graph (entries in Q_{ij}) that are non-local connections with respect to the embedding.

Nelson, in his important work [5] considered the stochastic evolution of a particle in position x , with probability distribution $\rho(x, t)$ and current velocity $v_i^a(x, t)$, and showed that it will evolve in a way equivalent to a solution of the time independent Schrödinger equation, so long as three conditions are satisfied. In this section, we state Nelson's conditions. In the next section, instead of the position of a particle we consider the corrected positions λ of the nodes of our graph. We will show that Nelson's equations are satisfied for the λ 's in a certain approximation which amounts to specific scaling relations between l, L , the valence of the graph n and the total number of nodes N . Under these conditions, the nodes evolve according to quantum mechanics!

Nelson's conditions for a particle in position \mathbf{x} , with probability distribution $\rho(\mathbf{x}, t)$ and current velocity $\mathbf{v}(\mathbf{x}, t)$, are [5]:

1. The particle undergoes an irreducible and universal Brownian motion, with diffusion constant ν inversely proportional to the mass, m . The proportionality defines Planck's constant \hbar , by

$$\nu = \frac{\hbar}{m}. \quad (28)$$

2. Even though the particle's evolution is fluctuating, its probability density and current must evolve according to laws that are time reversible.
3. The current velocity has to be irrotational, namely, it is proportional to a scalar $S(\mathbf{x})$,

$$\mathbf{v}(\mathbf{x}, t) = \frac{\partial S(\mathbf{x}, t)}{\partial \mathbf{x}}. \quad (29)$$

Nelson shows that, when these conditions are satisfied, one can define the function

$$\Psi(\mathbf{x}, t) = \sqrt{\rho(\mathbf{x}, t)} e^{\frac{i}{\hbar} S(\mathbf{x}, t)}, \quad (30)$$

and show that the coupled non-linear equations for ρ and \mathbf{v} reduce to a single linear equation:

$$i\hbar \frac{d\Psi}{dt} = \left[-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) \right] \Psi. \quad (31)$$

5 Satisfying Nelson's conditions

We will now show that Nelson's conditions are satisfied in our system in the regime $\sqrt{n} \frac{l^2}{L^2} \ll 1$.

5.1 The kinematics of the λ 's

The evolution of the λ 's in time describes the evolution of the nodes in the embedded graph. In place of Nelson's particle at position \mathbf{x} , we will check for Nelson's conditions for a node i of the graph, its position given by λ_i^a .

We have already seen that the λ 's are subject to an irreducible, universal Brownian motion, governed by the stochastic equations (21) and (24). We now need to describe the process in some more detail. Let us choose a particular x_i^a whose corresponding $\nu_i = 0$. We can say that that node is "cold". We study the probability distribution $\rho(\lambda, t)$ of the corresponding $\lambda_i^a(t)$.

First of all, as ρ is a probability,

$$\int \lambda_i^a \rho(\lambda, t) = 1 \quad (32)$$

must hold. Next, as a consequence of (21) and (24) it follows from the theory of stochastic differential equations that ρ satisfies the forward Fokker-Planck equation

$$\dot{\rho} = -\frac{\partial (\beta_i^a \rho)}{\partial \lambda_i^a} + \nu^\lambda \nabla \rho, \quad (33)$$

and the backward Fokker-Planck equation

$$\dot{\rho} = -\frac{\partial (\beta_{*i}^a \rho)}{\partial \lambda_i^a} - \nu^\lambda \nabla \rho. \quad (34)$$

The average of the above two equations yields the equation of continuity

$$\dot{\rho} = -\frac{\partial (\rho v_i^a)}{\partial \lambda_i^a}, \quad (35)$$

where we define v_i^a by

$$v_i^a(\lambda, t) := \frac{1}{2} (\beta_i^a + \beta_{*i}^a). \quad (36)$$

From the equation of continuity, we identify v_i^a as the *current velocity* [5].

Subtracting the forward Fokker-Planck from the backward, we find

$$\beta_{*i}^a = \beta_i^a - 2\nu^\lambda \frac{1}{\rho} \frac{\partial \rho}{\partial \lambda_i^a}. \quad (37)$$

We define the *osmotic velocity* $u_i^a(\lambda, t)$ as

$$u_i^a(\lambda, t) := \frac{1}{2} (\beta_i^a - \beta_{*i}^a). \quad (38)$$

Then eq. (37) becomes

$$u_i^a = \nu_i^\lambda \frac{\partial \ln \rho(\lambda, t)}{\partial \lambda_i^a}. \quad (39)$$

5.2 Conditions 1 and 2

We now return to the first two of Nelson's conditions. In postulating (3) as the original, uncorrected evolution in the low-energy limit, we have assumed that in that limit the nodes behave as if they have the same mass m . Then, to satisfy the first condition above, we simply need to take eq.(28) to be the definition of \hbar .

We now come to the second condition. We will follow the way that is implemented by Nelson in [5]. The trajectories of the λ 's are non-differentiable, but one can use the mean forward and mean backward derivatives D and D_* defined, in equations (14) and (17) to define the average stochastic acceleration of an eigenvalue λ_i^a by

$$a_i^a(\lambda, t) := \frac{1}{2} (DD_* + D_*D) \lambda_i^a. \quad (40)$$

This definition is time reversible. Nelson requires that Newton's laws hold for this averaged acceleration, namely, that it is proportional to the gradient of a potential:

$$a_i^a = -\frac{1}{m} \frac{\partial V(\lambda)}{\partial \lambda_i^a}. \quad (41)$$

To check this second condition in our model, we will compute the stochastic acceleration a in eq.(40) directly from (7). The terms we get by computing (40) from (7) can be written as

$$a_i^a = \alpha_i^a + \Delta\alpha_i^a, \quad (42)$$

where

$$\alpha_i^a(x, t) := \frac{1}{2} (DD_* + D_*D) x_i^a \quad (43)$$

and $\Delta\alpha_i^a$ is similarly the symmetrized derivative of the correction term $\Delta\lambda_i^a$ in eq.(8).

We already have, from eq.(3), that

$$\alpha_i^a(x, t) = -\frac{1}{m} \frac{\partial V(x)}{\partial x_i^a}. \quad (44)$$

This can be expanded as

$$\frac{1}{m} \frac{\partial V(x)}{\partial x_i^a} \sim \frac{1}{m} \frac{\partial V(\lambda)}{\partial \lambda_i^a} - \Delta\lambda_j^b \frac{1}{m} \frac{\partial^2 V(\lambda)}{\partial \lambda_i^a \partial \lambda_j^b} + \mathcal{O}((\Delta\lambda)^2). \quad (45)$$

Recall that $\Delta\lambda_i^a \sim \sqrt{n}l^2/L^2$ (eq.(8)).

For the second term $\Delta\alpha$ in (42), we apply the symmetrized derivative on (8). We use

$$Df(x, t) = \left(\frac{d}{dt} + b^{ai} \frac{\partial}{\partial x^{ai}} + \sum_{ai} \nu_i \frac{\partial^2}{\partial x^{ai} \partial x^{ai}} \right) f(x, t) \quad (46)$$

$$D_* f(x, t) = \left(\frac{d}{dt} + b_*^{ai} \frac{\partial}{\partial x^{ai}} - \sum_{ai} \nu_i \frac{\partial^2}{\partial x^{ai} \partial x^{ai}} \right) f(x, t) \quad (47)$$

from which it follows that $Dx_i^a = b_i^a$ and $D_* x_i^a = b_{*i}^a$:

$$\begin{aligned} \Delta \alpha_i^a &= -\frac{l^2}{2} \sum_{k \neq i} \frac{(Q_{ik}^a)^2}{(x_i^a - x_k^a)^2} (\alpha_i^a - \alpha_k^a) \\ &+ 2l^2 \sum_{k \neq i} \frac{(Q_{ik}^a)^2}{(x_i^a - x_k^a)^3} (b_i - b_k)^a (b_{*i} - b_{*k})^a \\ &+ l^2 \sum_{k \neq i} \frac{\nu_k^x (Q_{ik}^a)^2}{(x_i^a - x_k^a)^2} \left[12 \frac{v_i^a - v_k^a}{(x_i^a - x_k^a)^2} + 2 \frac{\partial_{ak}(v_i^a - v_k^a)}{(x_i^a - x_k^a)} + \partial_{ak}^2 (v_i^a - v_k^a) \right] \\ &+ 24 \sum_{k \neq i} \frac{(\nu_k^x)^2 l^2 (Q_{ik}^a)^2}{(x_i^a - x_k^a)^5} \end{aligned} \quad (48)$$

We now make a few physical assumptions, which allow us to bound the correction terms. First, we assume that the non-local connections are distributed sufficiently uniformly. This means that for large n there is a single quantity R such, that for all i and a ,

$$l^2 \sum_{k \neq i} \frac{Q_{ik}^2}{(\lambda_i^a - \lambda_k^a)^2} = Rn \frac{l^2}{L^2} [1 + \Delta R_i^a] \quad (49)$$

where

$$\Delta R_i^a < \frac{1}{\sqrt{n}} \quad (50)$$

As a result, we can estimate,

$$\Delta \alpha_i^a \sim \sqrt{n} \frac{l^2}{L^2} \left[\langle \ddot{x}^a \rangle + \frac{\langle (\dot{x})^2 \rangle}{L} + \frac{\nu^x \sqrt{\langle (\dot{x})^2 \rangle}}{L^2} + \frac{(\nu^x)^2}{L^3} \right]. \quad (51)$$

We now consider the ratio of this to the classical acceleration α_i^a . We have

$$\frac{\Delta \alpha_i^a}{\alpha_i^a} \sim \sqrt{n} \frac{l^2}{L^2} \left[1 + \frac{\langle (\dot{x})^2 \rangle}{\ddot{x}_i^a L} + \frac{\nu^x \sqrt{\langle (\dot{x})^2 \rangle}}{\ddot{x}_i^a L^2} + \frac{(\nu^x)^2}{\ddot{x}_i^a L^3} \right]. \quad (52)$$

However the ratio

$$\frac{\langle (\dot{x})^2 \rangle}{\ddot{x}_i^a L} \quad (53)$$

is on average proportional to twice the ratio of an average kinetic energy to an average potential energy (because on average the potentials are long ranged forces so that with L

a typical interparticle distance, $\ddot{x}_i^a L \sim V/m$). Assuming that the system is in equilibrium we know from the virial theorem of statistical physics that this ratio is of order unity.

The remaining terms are order unity or less, assuming that on average

$$\nu^x < \frac{\ddot{x}_i^a L^2}{\sqrt{\langle (\dot{x})^2 \rangle}}, \quad (\nu^x)^2 < \ddot{x}_i^a L^3 \quad (54)$$

These tell us that on average the random motion of the $x_i^{a'}$'s is less important than their bulk motions.

As a result, the ratio (52) is order $\sqrt{n} l^2/L^2$, so that the terms in $\Delta\alpha_i^a$ can be neglected compared to α_i^a .

As a result we have that to leading order, the stochastic acceleration is dominated by the classical forces so that,

$$a_i^a(\lambda, t) \approx - \frac{1}{m} \frac{\partial V(\lambda)}{\partial \lambda_i^a}. \quad (55)$$

Hence Nelson's second condition is satisfied.

We note that the conditions have been satisfied only to leading order. We expect corrections of order $\sqrt{n} l^2/L^2$. We may note that \hbar itself is by eqs. (28,27) proportional to $m\nu^x n l^4/L^4$. This tells us that we must assume that $m\nu^x$ is large for small n , so that \hbar is order unity.

5.3 Time-independent Schrödinger equation

At this point, and before considering the third condition, we are already able to satisfy the time-independent Schrödinger equation for the simple case in which the probability distribution is static

$$\dot{\rho} = 0. \quad (56)$$

In this case, condition 3 is trivially satisfied since, by eq. (35),

$$v_i^a = 0. \quad (57)$$

We apply D and D_* to β_i^a and β_{*i}^a to find that, in the general case, the stochastic acceleration is given by

$$a_i^a = \dot{u}_i^a + v_j^b \frac{\partial v_i^a}{\partial \lambda_j^b} - u_j^b \frac{\partial u_i^a}{\partial \lambda_j^b} - \nu_i^\lambda \nabla^2 u_i^a. \quad (58)$$

For the static case this reduces to

$$a_i^a = -u_j^b \frac{\partial u_i^a}{\partial \lambda_j^b} - \nu_i^\lambda \nabla u_i^a. \quad (59)$$

In our approximation in which the acceleration is given by eq. (41), and using the definition of \hbar in eq. (??), this can be rewritten as

$$\frac{\partial}{\partial \lambda_i^a} \left[\frac{1}{2} u^2 + \frac{\hbar}{m} \frac{\partial u_j^b}{\partial \lambda_j^b} \right] = \frac{1}{m} \frac{\partial}{\partial \lambda_i^a} V. \quad (60)$$

We may integrate this to obtain

$$\frac{1}{2} u^2 + \frac{\hbar}{m} \frac{\partial u_j^b}{\partial \lambda_j^b} = \frac{1}{m} (V - E), \quad (61)$$

where E is a constant with dimensions of energy.

Equation (60) is nonlinear in u . However, with a change of variables⁴

$$\Psi(\lambda) = \sqrt{\rho(\lambda)}, \quad (62)$$

in eq. (39), it is equivalent to a linear equation, the time-independent Schrödinger equation

$$\left[-\frac{\hbar^2}{2m} \nabla^2 + V - E \right] \Psi = 0. \quad (63)$$

for real Ψ .

5.4 The time dependent Schrödinger equation

We now go on to consider the general case in which the probability distribution evolves in time. To do this we need to study the third condition.

We need to show that the current velocity $V_i^a(\lambda)$ has vanishing curl, to the same order of approximation that the time independent Schrodinger equation holds

$$\frac{\partial v_i^a}{\partial \lambda_m^d} - \frac{\partial v_m^d}{\partial \lambda_i^a} < \sqrt{n} \frac{l^2}{L^2}. \quad (64)$$

We can show that the curl of the current velocity of the λ 's vanishes *when* the curl of the *classical* probability current can be neglected. The latter quantity is defined as,

$$V_i^a(x) = \frac{1}{2} (b_i^a(x, t) + b_{*i}^a(x, t)). \quad (65)$$

We then assume that

$$\frac{\partial V_i^a}{\partial x_m^d} - \frac{\partial V_m^d}{\partial x_i^a} < \sqrt{n} \frac{l^2}{L^2} \quad (66)$$

⁴Note that with eq. (39) $\frac{1}{2} u^2 + \nu \frac{\partial u_j^b}{\partial \lambda_j^b} = 2\nu^2 \Psi^{-1} \nabla^2 \Psi$.

Let us start by expressing the current velocity $v_i^a(\lambda)$ for the eigenvalues in terms of the current velocity $V_i^a(x)$ for the diagonal elements.

It is straightforward to show that

$$v_i^a(\lambda) = V_i^a(x) - \sum_{j \neq i} \frac{l^2 Q_{ij}^2}{(x_i^a - x_j^a)^2} (V_i^a(x) - V_j^a(x)) + \dots \quad (67)$$

Using eq. (7), we can write this as

$$v_i^a(\lambda) = V_i^a(\lambda) - l^2 \sum_{j \neq i} \frac{Q_{ij}^2}{(\lambda_i^a - \lambda_j^a)^2} (V_i^a(\lambda) - V_j^a(\lambda)) + \dots - l^2 \sum_k \sum_{l \neq k} \frac{Q_{kl}^2}{\lambda_k^b - \lambda_l^b} \frac{\partial V_i^a(\lambda)}{\partial \lambda_k^b} + \dots \quad (68)$$

Under the assumptions (66-50), it is straightforward to compute the curl of the current velocity for the eigenvalues

$$\frac{\partial v_i^a}{\partial \lambda_m^d} - \frac{\partial v_m^d}{\partial \lambda_i^a}. \quad (69)$$

It contains several terms, but they can all be shown to be smaller than the terms we neglected in the derivation of the time independent Schrödinger equation. The details of the calculation are given in Appendix A. Thus, we have

$$\frac{\partial v_i^a}{\partial \lambda_m^d} - \frac{\partial v_m^d}{\partial \lambda_i^a} < \sqrt{n} \frac{l^2}{L^2} \quad (70)$$

and we have verified equation (64). Then there exists an $S(\lambda)$ such that

$$V_i^a = \frac{1}{m} \frac{\partial S(\lambda)}{\partial \lambda_i^a} + \mathcal{O}\left(\sqrt{n} \frac{l^2}{L^2}\right) \quad (71)$$

By following the same logic as led to the time independent Schrödinger equation, we can follow Nelson's derivation [5]. Using (71), we construct the wavefunction (30). Using Nelson's stochastic modification of Newton's laws, which follow to order $\sqrt{n}l^2/L^2$, from the conditions we have demonstrated, we can show that (30) satisfies the Schrodinger equation, (31) up to $\mathcal{O}\left(\sqrt{n} \frac{l^2}{L^2}\right)$.

6 Conclusions

In this paper, we started with a graph-based model of a quantum theory of gravity which we assumed has a simple kind of low energy limit in which the graph is embedded in R^3 , the graph nodes acquire coordinate positions, and they evolve like Newtonian particles with mass m and potential V . We then showed that in the regime

$$0 < \sqrt{n} \frac{l^2}{L^2} \ll 1 \quad (72)$$

and when: 1) some of the x 's are in thermal equilibrium, so that the averaged diffusion coefficient ν^x is non-vanishing and the conditions of the virial theorem are satisfied ($\frac{\langle \dot{x}^2 \rangle}{\dot{x}_i^a} L \sim 1$), and 2) the temperature in the region of particle i vanishes, so that the diffusion constant due to local influences is zero, then the evolution of the probability distribution for the eigenvalues λ_i^a is described by a solution to the time-independent Schrödinger equation with mass m and potential V and \hbar given by $m\nu_\lambda$.

We have also assumed that the non-local connections are distributed uniformly throughout the system. This allows us to use eqs's (49,50) as well as to deduce (27).

Furthermore, under the additional assumption that the curl of the probability current for the classical variables $V(x^a)$ may be neglected (eq. (66)), the time-dependent Schrödinger equation follows, to the same order of approximation.

We close with a few comments about future work.

The model is not intended to be realistic. Much more work needs to be done to understand whether or not the basic strategy uncovered here can lead to a real physical theory. Among the things that need to be done are:

1. While we show that the Schrödinger equation is satisfied to a certain approximation by our probability distributions, we do not show that *all* solutions to the Schrödinger equation arise this way. This is due partly to limitations in Nelson's approach. For this reason it may be interesting to recast the model in the framework of Adler's approach to quantum theory as emergent from a dynamics of matrices[9].
2. Is quantum mechanics recovered beyond the non-relativistic approximation? Nelson's stochastic quantum theory has been extended to relativistic field theory [7], so it is possible that the present results can also be extended.
3. The corrections to the approximation in which quantum theory emerges can be studied. It is possible that they lead to non-linear corrections to the Schrödinger equation. We may note that to keep quantum effects large compared to terms that have been neglected we must keep the product $m\nu_x$ large as $\hbar \sim m\nu_x n l^4 / L^4$.
4. If we calculate the fluctuations of λ_i^a 's for whom the corresponding x_i^a 's are subject to thermal noise, is the result the correct finite-temperature quantum theory?
5. The same phenomena may occur in other discrete, combinatorial theories of quantum gravity, such as causal sets [8]. These share with spin foam models the problem that the conditions necessary so that the information in combinatorial states or histories are well represented by embedding them in low dimensional manifolds are in general very hard to satisfy. The result presented here opens up the possibility that these conditions do not need to be satisfied. Instead the obstructions that prevent a good matching between the two notions of locality involved in embedding a combinatorial structure in a low dimensional manifold so as to match metric relations may lead instead to the discovery of the source of the non-local hidden variables necessary for a realistic formulation of quantum theory.

6. Different hypotheses can be considered regarding the origin of the stochastic fluctuations of the x_i^a 's. Connecting them with temperature may, in a more realistic model, lead to predictions of a time dependence of \hbar that could falsify such a theory. Other possibilities include an intrinsic uncertainty in the embedding coordinate due to the fact that the embedding is an emergent property of an underlying fundamental theory whose degrees of freedom are combinatorial.

Acknowledgements

We would like to thank our colleagues at PI and especially Antony Valentini and Olaf Dreyer for comments and encouragement. We are grateful also to Steven Adler for detailed comments on the manuscript.

A Vanishing of the antisymmetric derivatives of the current velocity

We show here the details of the calculations that establish (70). We begin with the relationship between the two current velocities, V_i^a for the x_i^a and v_i^a for the λ_i^a .

$$v_i^a = V_i^a - l^2 \sum_{k \neq i} \frac{Q_{ik}^2}{(\lambda_i^a - \lambda_k^a)^2} (V_i^a - V_k^a) \quad (73)$$

We use

$$\frac{\partial V_i^a}{\partial \lambda_j^b} = \frac{\partial V_i^a}{\partial x_l^c} \frac{\partial x_l^c}{\partial \lambda_j^b} \quad (74)$$

together with

$$x_i^c = \lambda_i^c - l^2 \sum_{m \neq i} \frac{Q_{ml}^2}{\lambda_l^c - \lambda_m^c} + \dots \quad (75)$$

to find

$$\frac{\partial V_i^a}{\partial \lambda_j^b} = \frac{\partial V_i^a}{\partial x_j^b} \left(1 + l^2 \sum_{m \neq j} \frac{Q_{mj}^2}{(\lambda_j^b - \lambda_m^b)^2} \right) - l^2 \sum_{l \neq j} \frac{\partial V_i^a}{\partial x_j^b} \frac{Q_{jl}^2}{(\lambda_l^b - \lambda_j^b)^2} \quad (76)$$

Combining this with (73) we have

$$\begin{aligned} \frac{\partial v_i^a}{\partial \lambda_j^b} &= \frac{\partial V_i^a}{\partial \lambda_j^b} \left[1 - l^2 \sum_{k \neq i} \frac{Q_{ik}^2}{(\lambda_i^a - \lambda_k^a)^2} \right] - l^2 \sum_{k \neq i} \frac{Q_{ik}^2}{(\lambda_i^a - \lambda_k^a)^2} \frac{\partial V_k^a}{\partial \lambda_j^b} \\ &\quad + 2l^2 \delta_b^a \delta_i^j \sum_{k \neq i} \frac{Q_{ik}^2}{(\lambda_i^a - \lambda_k^a)^2} (V_i^a - V_k^a) - 2l^2 \delta_b^a \frac{Q_{ij}^2}{(\lambda_i^a - \lambda_j^a)^2} (V_i^a - V_j^a) \end{aligned} \quad (77)$$

It is easiest to consider separately the different cases. Beginning with $i = j$, we find,

$$\begin{aligned} \frac{\partial v_i^a}{\partial \lambda_i^b} - \frac{\partial v_i^b}{\partial \lambda_i^a} &= \frac{\partial V_i^a}{\partial x_i^b} - \frac{\partial V_i^b}{\partial x_i^a} \\ &- \frac{\partial V_i^a}{\partial x_i^b} (l^2 \sum_{k \neq i} Q_{ik}^2 \frac{1}{(\lambda_i^a - \lambda_k^a)^2}) + \frac{\partial V_i^b}{\partial x_i^a} (l^2 \sum_{k \neq i} Q_{ik}^2 \frac{1}{(\lambda_i^b - \lambda_k^b)^2}) \\ &- l^2 \sum_{k \neq i} Q_{ik}^2 \left[\frac{1}{(\lambda_i^a - \lambda_k^a)^2} \frac{\partial V_k^a}{\partial x_i^b} - \frac{1}{(\lambda_i^b - \lambda_k^b)^2} \frac{\partial V_k^b}{\partial x_i^a} \right] \end{aligned} \quad (78)$$

We now impose conditions (66-50) to find that

$$\frac{\partial v_i^a}{\partial \lambda_i^b} - \frac{\partial v_i^b}{\partial \lambda_i^a} < \sqrt{n} \frac{l^2}{L^2} \quad (79)$$

Similarly, for $i \neq j$ and $a \neq b$ we have

$$\begin{aligned} \frac{\partial v_i^a}{\partial \lambda_j^b} - \frac{\partial v_j^b}{\partial \lambda_i^a} &= \frac{\partial V_i^a}{\partial x_j^b} - \frac{\partial V_j^b}{\partial x_i^a} \\ &- \frac{\partial V_i^a}{\partial x_j^b} (l^2 \sum_{k \neq i} Q_{ik}^2 \frac{1}{(\lambda_i^a - \lambda_k^a)^2}) + \frac{\partial V_j^b}{\partial x_i^a} (l^2 \sum_{k \neq j} Q_{jk}^2 \frac{1}{(\lambda_j^b - \lambda_k^b)^2}) \\ &- l^2 \sum_{k \neq i} Q_{ik}^2 \frac{1}{(\lambda_i^a - \lambda_k^a)^2} \frac{\partial V_k^a}{\partial x_j^b} + l^2 \sum_{k \neq j} Q_{jk}^2 \frac{1}{(\lambda_j^b - \lambda_k^b)^2} \frac{\partial V_k^b}{\partial x_i^a} \end{aligned} \quad (80)$$

Under the same conditions, it follows directly that,

$$\frac{\partial v_i^a}{\partial \lambda_j^b} - \frac{\partial v_j^b}{\partial \lambda_i^a} < \sqrt{n} \frac{l^2}{L^2} \quad (81)$$

Finally, the reader can check the case $a = b$ and $i \neq j$. After a similar calculation we find that

$$\frac{\partial v_i^a}{\partial \lambda_j^a} - \frac{\partial v_j^a}{\partial \lambda_i^a} < \sqrt{n} \frac{l^2}{L^2} \quad (82)$$

References

- [1] C. Rovelli, "Loop Quantum Gravity", Living Rev. Rel. 1 (1998) 1 (gr-qc/9710008); "Quantum Gravity", CUP, to appear, draft available at <http://www.cpt.univ-mrs.fr/~rovelli/rovelli.html>
- [2] M. B. Green, J. H. Schwarz and E. Witten, "Superstring Theory" Cambridge University Press (1987); J. Polchinski, "String Theory", Vol. 1 and 2, Cambridge University Press (1998).

- [3] A. Perez, "Spin Foam Models for Quantum Gravity", Topical Review in Class. Quant. Grav. **20** (2003) R43 (gr-qc/0301113).
- [4] S. Chandrasekhar, "Stochastic problems in physics and astronomy", Rev. Mod. Phys. **15** (1943) 1.
- [5] E. Nelson, "Derivation of the Schrödinger equation from Newtonian mechanics", Phy. Rev., **150**, 1079 (1969); "Quantum Fluctuations", Princeton Series in Physics, Princeton University Press (1985).
- [6] J. S. Bell, "On The Einstein-Podolsky-Rosen Paradox," Physics **1**, 195 (1964).
- [7] M. Pavon, "On the stochastic mechanics of the free relativistic particle", J. Math. Physics **42** (2001), 4846-4856, (quant-ph/0108139).
- [8] L. Bombelli, J. Lee, D. Meyer and R. D. Sorkin, "Spacetime as a causal set", Phys. Rev. Lett. **59** (1987) 521.
- [9] S. Alder, *Statistical dynamics of global unitary invariant matrix models as pre-quantum mechanics*, hep-th/0206120.