

Quantization of diffeomorphism invariant theories of connections with local degrees of freedom

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Quantization of diffeomorphism invariant theories of connections is studied. A solutions of the diffeomorphism constraints is found. The space of solutions is equipped with an inner product that is shown to satisfy the physical reality conditions. This provides, in particular, a quantization of the Husain-Kuchař model. The main results also pave way to quantization of other diffeomorphism invariant theories such as general relativity. In the Riemannian case (i.e., signature +++++), the approach appears to contain all the necessary ingredients already. In the Lorentzian case, it will have to be combined in an appropriate fashion with a coherent state transform to incorporate complex connections.

I. INTRODUCTION

Keeping with the theme of the special issue, this paper will address the problem of quantization of a class of diffeomorphism invariant field theories.

The class can be specified as follows. We will assume that the theory can be cast in a Hamiltonian form. The configuration variable will be a connection 1-form A_a^i on a d -dimensional (“spatial”) manifold and takes values in the Lie algebra of a compact, connected Lie-group. The canonically conjugate momentum, \tilde{E}_i^a , will be a vector field with density weight one (or, equivalently, a $d-1$ form) which takes values in the dual of the Lie algebra. The phase space Γ will thus consist of pairs

(A_a^i, \tilde{E}_i^a) satisfying suitable regularity conditions. Finally, the gauge invariance will be ensured by the Gauss constraint and the (d -dimensional) diffeomorphism invariance, by a vector constraint, such that the entire system is of first class in Dirac’s terminology. Individual theories in this class may have additional features such as specific Hamiltonians or additional constraints. In the main discussion, however, we will ignore such structures and focus only of the features listed above which will be common to all theories in the class.

To make this general setting more concrete, let us list a few illustrative examples of theories which are included in this class. The first is the Husain-Kuchař model [1] which can be thought of as general relativity without the Hamiltonian constraint. Thus, in this model, we only have the Gauss and the (“spatial”) diffeomorphism constraints and the Hamiltonian is a linear combination of them. In this case, we will be able to obtain a complete quantum theory. A second example is provided by Riemannian (i.e., +++++) general relativity, cast in a Hamiltonian framework using self-dual connections. In this case, in addition to the Gauss and the diffeomorphism constraint, there is also the Hamiltonian constraint which dictates “time evolution.” The results of this paper provide only a partial solution to the problem of quantization of this model since the Hamiltonian constraint will not be incorporated. However, as we will indicate in the last section, the general methods employed

appear to be applicable also to the Hamiltonian constraint and the issue is currently being investigated. Next, one can also consider Lorentzian general relativity in terms of a spin connection and its conjugate momentum. Our results will again provide a complete solution to the Gauss and the diffeomorphism constraints. (The Hamiltonian constraint is, however, more difficult to address now. One possible approach is to pass to self-dual connection variables [2] using the coherent state transform of Ref. [3].) Finally, our class allows for Chern-Simons theories whose group is the inhomogeneous version [4] IG of a compact, connected Lie group G . This class includes Riemannian general relativity in 3 space-time dimensions.

From a mathematical physics perspective, one faces two types of problems while quantizing such models. First, the underlying diffeomorphism invariance poses a non-trivial challenge: We have to face the usual field theoretic difficulties that are associated with the presence of an infinite number of degrees of freedom but now *without* recourse to a background space-time geometry. In particular, one must introduce new techniques to single out the quantum configuration space, construct suitable measures on it to obtain Hilbert spaces of states and regulate operators of physical interest. The second set of problems arises because of the presence of constraints. In particular, even after one has constructed a Hilbert space and regularized the constraint operators, one is left with the non-trivial task of *solving* the constraints to isolate the physical states and of introducing an appropriate inner product on them. This is a significant problem even for systems with only a *finite* number of degrees of freedom since, typically, solutions to constraints fail to lie in the initial Hilbert space. Thus, physical states do not even have a natural “home” to begin with! In theories now under consideration, these difficulties become particularly severe: Diffeomorphism invariance introduces an intrinsic non-locality and forces one to go beyond the standard techniques of local quantum field theory.

Our approach to solving these problems is based on two recent developments. The first is the introduction of a new functional calculus on the space of connections modulo gauge transformations which respects the underlying diffeomorphism invariance (see Ref. [5-12]). The second is a new strategy for solving quantum constraints which naturally leads to an appropriate inner product on the physical states (see Ref. [6-13]). Together, the two developments will enable us to complete the general algebraic quantization program [17,18] for the class

of systems under consideration. Thus, we will be able to solve the quantum constraints and introduce the appropriate Hilbert space structure on the resulting space of solutions.

The main ideas underlying these developments can be summarized as follows. Recall first that, in gauge theories, it is natural to use the space \mathcal{A}/\mathcal{G} of connections modulo local gauge transformations as the classical configuration space. In quantum field theories, due to the presence of an infinite number of degrees of freedom, the quantum configuration space is typically an enlargement of its classical counterpart. The enlargement is non-trivial because the measures which define the scalar product tend to be concentrated on “distributional” fields which lie outside the classical configuration space. In gauge theories, if we require that the Wilson loop variables –i.e., the traces of holonomies– should be well-defined also in the quantum theory, a canonical enlargement $\overline{\mathcal{A}/\mathcal{G}}$ of \mathcal{A}/\mathcal{G} becomes available [5]. This space can be thought of as a limit of the configuration spaces of lattice gauge theories for all possible “floating” (i.e., not necessarily rectangular) lattices. Geometric structures on configuration spaces of lattice gauge theories can therefore be used to induce geometric structures on $\overline{\mathcal{A}/\mathcal{G}}$. [6,7,9,11] This enables one to introduce integral and differential calculus on $\overline{\mathcal{A}/\mathcal{G}}$ *without* reference to any background geometry. The calculus can, in turn, be used to introduce measures, Hilbert spaces of square-integrable functions and regulated operators on them.

The strategy of solving quantum constraints, on the other hand, is quite general and not tied to the theories of connections. [15,19] For simplicity, consider the case when there is just one constraint, $C = 0$, on the classical phase space. To quantize the system, as in the standard Dirac procedure, one first ignores the constraint and constructs an auxiliary Hilbert space \mathcal{H}_{aux} , ensuring that the set of “elementary” real functions on the full phase space is represented by self-adjoint operators on \mathcal{H}_{aux} . Thus, \mathcal{H}_{aux} incorporates the “kinematic conditions”. Since the classical constraint is a real function on the phase space, one represents it by a self-adjoint operator \hat{C} on \mathcal{H}_{aux} . The solutions are to be states which are annihilated by \hat{C} , or, alternatively, which are left invariant by the 1-parameter group $U(\lambda) := \exp i\lambda\hat{C}$ generated by \hat{C} . A natural strategy [13,14] to obtain solutions, then, is to begin with a suitable state ϕ in \mathcal{H}_{aux} and average it over the group; formally, $\bar{\phi} := \int d\lambda U(\lambda) \circ |\phi\rangle$ is group invariant. The problem is that, typically, $\bar{\phi}$ does not belong to \mathcal{H}_{aux} ; it is not normalizable. However, it often has a

well-defined action on a dense subset Φ of \mathcal{H}_{aux} in the sense that $\bar{\phi} \cdot \psi := \int d\lambda \langle \phi | U(\lambda) \circ |\psi \rangle$ is well-defined for all $\psi \in \Phi$. That is, $\bar{\phi}$ can be often thought of as an element of the topological dual of Φ (if Φ is equipped with a suitable topology which is finer than the one induced by \mathcal{H}_{aux}). To summarize, group averaging *can* lead to solutions of the quantum constraint but they lie in a space Φ' which is larger than \mathcal{H}_{aux} (if, as is typically the case, zero lies in the continuous part of the spectrum of \hat{C}). Finally, one can introduce an Hermitian inner product on the space of the solutions simply by setting $\langle \bar{\phi}_1 | \bar{\phi}_2 \rangle = \bar{\phi}_1 \cdot \phi_2$. Thus, *if* one can find a dense subspace Φ in \mathcal{H}_{aux} (and equip it with a suitable topology) such that the group averaging procedure maps every element of Φ to a well-defined element of Φ' , one can extract the Hilbert space of physical states. One can show that the resulting physical Hilbert space automatically incorporates the “reality conditions” on physical observables [15,16] *even when they are not known explicitly*.

The purpose of this paper is to use these two developments to obtain the following results for the class of models under consideration:

1. We will construct the *quantum* configuration space $\overline{\mathcal{A}/\mathcal{G}}$ and select the measure μ_0 on it for which $L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_0)$ can serve as the auxiliary Hilbert space \mathcal{H}_{aux} , i.e., can be used to incorporate the kinematical reality conditions of the classical phase space.
2. Introduce the diffeomorphism constraints as well-defined operators on \mathcal{H}_{aux} and demonstrate that there are no anomalies in the quantum theory.
3. Construct a dense subspace Φ of \mathcal{H}_{aux} with the required properties and obtain a complete set of solutions of the diffeomorphism constraints in its topological dual Φ' . We will also characterize the solutions in terms of generalized knots (i.e., diffeomorphism invariance classes of certain graphs) and obtain the Hilbert spaces of physical states by introducing the inner products which ensure that real physical observables are represented by self-adjoint operators.

While the main emphasis of the paper is on presenting a rigorous solution to the diffeomorphism constraint, along the way, we will summarize a number of additional results which are likely to be useful more generally. First, we will exhibit an orthonormal basis in \mathcal{H}_{aux} , introduced by Baez

[20] drawing on spin networks considered by Rovelli and Smolin [21] (see also [22]). Second, we will present a rigorous transform that maps the states in the connection representation (i.e., in \mathcal{H}_{aux}) to functions on the loop space. Furthermore, using the orthonormal basis, we will also present the *inverse* transform [22] from the loop representation [23–25] to the connection representation. Finally, in the case when $d = 3$ and the gauge group is $SU(2)$, using differential calculus on $\overline{\mathcal{A}/\mathcal{G}}$ we will indicate how one can introduce, on \mathcal{H}_{aux} , regulated self-adjoint operators corresponding to areas of 2-surfaces. The spectra of these operators are discrete and provide a glimpse into the nature of quantum geometry that underlies Riemannian quantum general relativity.

The plan of the paper is as follows. Sec.II contains an outline of the general quantization program. Sec.III specifies the precise class of theories considered and presents in greater detail models, mentioned above, that are encompassed by our discussion. Sec.IV recalls the structure of the quantum configuration space $\overline{\mathcal{A}/\mathcal{G}}$. In Sec.V, we construct the auxiliary Hilbert space \mathcal{H}_{aux} and show that a complete set of real-valued functions on the classical phase space is indeed promoted to self-adjoint operators on \mathcal{H}_{aux} . We also present the Baez orthonormal basis and discuss the loop transform and its inverse. The diffeomorphism constraints are implemented in Sec.VI using a series of steps that handle various technical difficulties. Sec.VII summarizes the main results and puts them in a broader perspective.

A number of results which clarify and supplement the main discussion are presented in appendices. Appendix A illustrates some subtleties associated with the group integration procedure in the case when the Poisson algebra of constraints is Abelian. Appendix B summarizes the projective techniques that lie at the heart of the diffeomorphism invariant functional calculus on $\overline{\mathcal{A}/\mathcal{G}}$. Appendix C points out that the requirement of diffeomorphism invariance has certain technical consequences that might not have been anticipated easily. Finally, Appendix D illustrates how one can use the projective techniques to introduce well-defined operators on $\overline{\mathcal{A}/\mathcal{G}}$ which capture geometric notions such as areas of surfaces and volumes of regions. The operators can be made self-adjoint on $L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_0)$ and have discrete spectra. These results provide a glimpse into the nature of quantum geometry.

II. QUANTIZATION OUTLINE

In Ref. [17-18], the Dirac quantization program for constrained systems was extended to incorporate certain peculiarities of diffeomorphism invariant theories such as general relativity. In this section, we will further refine that program using the “group averaging” techniques mentioned in Sec. I. These techniques provide a concrete method for constructing solutions to the quantum constraints *and* for introducing an appropriate scalar product on the space of these solutions.

In the first part of this section, we will spell out the refined version of the program, and in the second, illustrate the various steps involved by applying them to three simple examples.

A. Strategy

Consider a classical system with first class constraints $C_i = 0$ for which the phase space Γ is a real symplectic manifold. The proposal is to quantize this system in a series of steps. (The steps which have been modified from Ref. [17,18] are identified with a prime.)

Step 1. Select a subspace \mathcal{S} of the vector space of all smooth, complex-valued functions on Γ subject to the following conditions:

- a) \mathcal{S} should be large enough so that any sufficiently regular function on the phase space can be obtained as (possibly a suitable limit of) a sum of products of elements in \mathcal{S} .
- b) \mathcal{S} should be closed under Poisson brackets, i.e. for all functions F, G in \mathcal{S} , their Poisson bracket $\{F, G\}$ should also be an element of \mathcal{S} .
- c) Finally, \mathcal{S} should be closed under complex conjugation; i.e. for all F in \mathcal{S} , the complex conjugate F^* should be a function in \mathcal{S} .

Each function in \mathcal{S} is to be regarded as an *elementary classical variable* which is to have an *unambiguous* quantum analog.

Step 2. Associate with each element F in \mathcal{S} an abstract operator \widehat{F} . Construct the free associative algebra generated by these *elementary quantum operators*. Impose on it the canonical commutation relations, $[\widehat{F}, \widehat{G}] =$

$i\hbar\{\widehat{F}, \widehat{G}\}$, and, if necessary, also a set of (anti-commutation) relations that captures the algebraic identities satisfied by the elementary classical variables. Denote the resulting algebra by \mathcal{B}_{aux} .

Step 3. On this algebra, introduce an involution operation \star by requiring that if two elementary classical variables F and G are related by $F^* = G$, then $\widehat{F}^* = \widehat{G}$ in \mathcal{B}_{aux} . Denote the resulting \star -algebra by $\mathcal{B}_{aux}^{(\star)}$.

(Recall that an involution on \mathcal{B}_{aux} is an anti-linear map \star from \mathcal{B}_{aux} to itself satisfying the following three conditions for all A and B in \mathcal{B}_{aux} : i) $(A + \lambda B)^* = A^* + \lambda^* B^*$, where λ is any complex number; ii) $(AB)^* = B^* A^*$; and iii) $(A^*)^* = A$.)

These steps are the same as in Ref. [17,18]. The main idea in the remaining steps was to use the “reality conditions” –i.e., the requirement that a suitable class of classical observables be represented by self-adjoint operators– to determine the inner product on physical states. This strategy has been successful in a number of examples [18], including a model field theory that mimics several features of general relativity [26]. For the class of theories now under consideration, however, we will refine the remaining steps along the lines of Ref. [13-16].

While we will retain the idea that the classical reality conditions should determine the inner product, we will not need to explicitly display a complete set of classical observables (i.e., functions which Poisson commute with the constraints). Instead, we will work with the complete set of functions (\mathcal{S}) on the *unconstrained* phase space, noting that the reality properties of such functions will determine the reality properties of the observables. The idea is then to implement the reality conditions of operators in $\mathcal{B}_{aux}^{(\star)}$ on an *auxiliary* Hilbert space \mathcal{H}_{aux} from which the physical phase space \mathcal{H}_{phys} will be finally constructed.

Step 4'. Construct a linear \star -representation R of the abstract algebra $\mathcal{B}_{aux}^{(\star)}$ via linear operators on an auxiliary Hilbert space \mathcal{H}_{aux} , i.e. such that

$$R(\widehat{A}^*) = R(\widehat{A})^\dagger \quad (\text{II.1})$$

for all \widehat{A} in $\mathcal{B}^{(\star)}$, where \dagger denotes Hermitian conjugation with respect to the inner product in \mathcal{H}_{aux} .

We now wish to construct the physical Hilbert space \mathcal{H}_{phys} , which will in general *not* be a subspace of \mathcal{H}_{aux} . We proceed as follows.

Step 5'a. Represent the constraints C_i as self-adjoint operators \widehat{C}_i (or, their exponentiated action, representing the finite gauge transformations, as unitary operators \widehat{U}_i) on \mathcal{H}_{aux} .

This step provides a quantum form of the constraints that we will use to define observables and physical states. We will look for solutions of the constraints in terms of *generalized* eigenvectors of \widehat{C}_i which will lie in the topological dual Φ' of some dense subspace $\Phi \subset \mathcal{H}_{aux}$ (see also Ref. [19,27]). Since Φ and Φ' will be used to build the physical Hilbert space, we will consider only physical operators that are well behaved with respect to Φ .

Step 5'b. Choose a suitable dense subspace $\Phi \subset \mathcal{H}_{aux}$ which is left invariant by the constraints \widehat{C}_i and let $\mathcal{B}_{phys}^{(\star)}$ be the \star -algebra of operators on \mathcal{H}_{aux} which commute with the constraints \widehat{C}_i and such that, for $A \in \mathcal{B}_{phys}^{(\star)}$, both A and A^\dagger are defined on Φ and map Φ to itself.

Note that the choice of Φ is subject to two conditions: on the one hand it should be large enough so that $\mathcal{B}_{phys}^{(\star)}$ contains a ‘‘sufficient number’’ of physically interesting operators, and, on the other, it should be small enough so that its topological dual Φ' is ‘‘sufficiently large’’ to serve as a home for physical states. The key idea now is to find an appropriate map $\eta : \Phi \rightarrow \Phi'$ such that $\eta(\phi)$ is a solution to the constraint for all $\phi \in \Phi$. (Note that the natural class of maps from Φ to Φ' is *anti*-linear (c.f., the adjoint map)).

Step 5'c. Find an anti-linear map η from Φ to the topological dual Φ' that satisfies:

- (i) For every $\phi_1 \in \Phi$, $\eta(\phi_1)$ is a solution of the constraints; i.e.,

$$0 = (\widehat{C}_i(\eta\phi_1))[\phi_2] := (\eta\phi_1)[\widehat{C}_i\phi_2] \quad (\text{II.2})$$

for any $\phi_2 \in \Phi$. Here, the square brackets denote the natural action of Φ' on Φ .

- (ii) η is real and positive in the sense that, for all $\phi_1, \phi_2 \in \Phi$,

$$\begin{aligned} (\eta\phi_1)[\phi_2] &= ((\eta\phi_2)[\phi_1])^* & \text{and} \\ (\eta\phi_1)[\phi_1] &\geq 0. \end{aligned} \quad (\text{II.3})$$

- (iii) η commutes with the action of any $A \in \mathcal{B}_{phys}^{(\star)}$ in the sense that

$$(\eta\phi_1)[A\phi_2] = ((\eta A^\dagger \phi_1))[\phi_2] \quad (\text{II.4})$$

for all $\phi_1, \phi_2 \in \Phi$.

(The appearance of the adjoint on the r.h.s. of (II.4) corresponds to the anti-linearity of η .)

Step 5'd. The vectors $\eta\phi$ span a space \mathcal{V}_{phys} of solutions of the constraints. We introduce an inner product on \mathcal{V}_{phys} through

$$\langle \eta\phi_1, \eta\phi_2 \rangle_{phys} = (\eta\phi_2)[\phi_1] \quad (\text{II.5})$$

The requirement (II.3) guarantees that this inner product is well defined and that it is Hermitian and positive definite. Thus, the completion of \mathcal{V}_{phys} with respect to (II.5) is a ‘physical’ Hilbert space \mathcal{H}_{phys} .

(Note that the positions of ϕ_1 and ϕ_2 must be opposite on the two sides of (II.5) due to the anti-linear nature of η .)

At this point, the reader may fear that this list of conditions on η will never be met in practice. That the new step 5' may actually simplify the quantization program follows from the observation of [13,14] (and [15,16] for the case when the Poisson algebra of constraints is Abelian) that a natural candidate for such a map exists.

Let us indicate, heuristically, how this can come about. Assume that the exponentiated form of all constraints \widehat{C}_i defines the unitary action (\widehat{U}) of a group (of gauge transformations) K on \mathcal{H}_{aux} . Then, a natural candidate for the map η is provided by the ‘‘group averaging procedure’’. Set

$$\eta|\phi\rangle := \left(\int_K dk \widehat{U}(k)|\phi\rangle \right)^\dagger = \int_K dk \langle \phi | \widehat{U}^{-1}(k) \rangle, \quad (\text{II.6})$$

where dk denotes a bi-invariant measure on K (or, rather, on the orbit through $|\phi\rangle$), and ignore, for the moment, the issue convergence of the integral in (II.6). Then, it is easy to check that η satisfies properties (i)-(iii) in 5'c. Finally, the expression (II.5) of the scalar product reduces to:

$$\langle \eta\phi_1, \eta\phi_2 \rangle_{phys} = \int_K dk \langle \phi_2 | U^{-1}(k) | \phi_1 \rangle_{aux}. \quad (\text{II.7})$$

Thus, it is intuitively clear that the requirements of step 5 can be met in a large class of examples.

Let us return to the general program. The last step is to represent physical operators on \mathcal{V}_{phys} . This is straightforward because the framework provided by step 5 guarantees that \mathcal{H}_{phys} carries an (anti) \star -representation (see below) of $\mathcal{B}_{phys}^{(\star)}$ as follows:

Step 6'. Operators in $A \in \mathcal{B}_{phys}^{(\star)}$ have a natural action (induced by duality) on Φ' that leaves \mathcal{V}_{phys} invariant. Use this fact to induce densely defined operators A_{phys} on \mathcal{H}_{phys} through

$$A_{phys}(\eta\phi) = \eta(A\phi). \quad (\text{II.8})$$

This leads to an *anti*- \star -representation of $\mathcal{B}_{phys}^{(\star)}$ in the sense that the map (II.8) from $\mathcal{B}_{phys}^{(\star)}$ to the operators on \mathcal{H}_{phys} is an anti-linear \star -homomorphism. Thus, *the reality properties of the physical operators $\mathcal{B}_{phys}^{(\star)}$ on \mathcal{H}_{aux} descend to the physical Hilbert space.*

We conclude this subsection with two remarks. Suppose, first, that for some $A \in \mathcal{B}_{phys}^{(\star)}$ we have $A = A^\dagger$ on \mathcal{H}_{aux} . If the operators $(A \pm i)^{-1}$ are both defined on Φ and preserve Φ , then the range of $A_{phys} \pm i$ contains \mathcal{V}_{phys} and is dense in \mathcal{H}_{phys} . It then follows that A_{phys} is essentially self-adjoint [28] on \mathcal{H}_{phys} . The second remark has to do with our restriction to strong observables, i.e., observables which commute with constraints. On physical grounds, on the other hand, one should deal with more general, weak observables. It is often the case that every weak observable of the system is weakly equivalent to a strong observable. In these cases, our restriction does not lead to a loss of generality. In more general cases, on the other hand, an extension of this procedure to encompass weak observables is needed.

B. Examples

We will now present three examples to illustrate how the group averaging procedure can be carried out in practice. (Parameterized Newtonian particles and some other examples are treated in Ref. [16] and appendix A contains general comments on the case of Abelian constraints.) The non-trivial application of this procedure to diffeomorphism invariant theories will be given in Sec. VI.

Example A

As a first test case, let us consider a non-relativistic particle in three dimensions subject to the classical constraint $p_z = 0$, so that the associated gauge transformations are just translations in the z -direction. Since the interesting classical functions can be built from x, y, z, p_x, p_y, p_z , we let these six functions span the classical subspace \mathcal{S} of step 1 and construct the algebra $\mathcal{B}_{aux}^{(\star)}$ of step 3. We choose the auxiliary Hilbert space to be

$\mathcal{H}_{aux} = L^2(\mathbb{R}^3, dx dy dz)$ and let $\hat{x}, \hat{y}, \hat{z}$ act by multiplication and $\hat{p}_x, \hat{p}_y, \hat{p}_z$ act by ($-i$ times) differentiation so that all six operators are self-adjoint.

Clearly, our physical states will be associated with *generalized* eigenstates of \hat{p}_z . We wish to view such states as distributions that act on some dense subspace $\Phi \subset \mathcal{H}_{aux}$. With our choice of operators, it is natural to take Φ to be the space of smooth functions with compact support. Note that the Fourier transform \tilde{f}_0 of any such function f_0 is smooth. Hence, for any $g_0 \in \Phi$, the distribution $\eta(g_0) := \tilde{g}_0^* \delta(p_z)$ has well defined action on any $f_0 \in \Phi$:

$$\eta(g_0)[f_0] = \int_{\mathbb{R}^3} \tilde{g}_0^*(p) \delta(p_z) \tilde{f}_0(p) dp_x dp_y dp_z, \quad (\text{II.9})$$

where, as before, $*$ denotes complex conjugation. Note that this action may be constructed by averaging over the translation group through

$$\eta(g_0)[f_0] = \int_{\mathbb{R}^3} dx dy dz \int_{\mathbb{R}} dz' g_0^*(x, y, z + z') f_0(x, y, z). \quad (\text{II.10})$$

We now let \mathcal{V}_{phys} be the linear space spanned by such $\eta(g_0)$. This space is annihilated by \hat{p}_z (under the dual action of \hat{p}_z on Φ') and will become a dense subspace of the physical Hilbert space \mathcal{H}_{phys} .

For f, g in \mathcal{V}_{phys} , let f_0 be an element of Φ that maps to f under η . Then, our prescription (II.9) yields the following physical inner product: $\langle f, g \rangle_{phys} = g[f_0]$, where f_0 may be any smooth function $f_0(x, y, z)$ of compact support for which $f(x, y) = \int dz f_0^*(x, y, z)$; i.e., $\eta(f_0) = f$. Thus, the physical inner product is just $\int f^*(x, y) g(x, y) dx dy$. It is Hermitian, positive definite, and independent of the choice of f_0 . The resulting \mathcal{H}_{phys} is just what one would expect on intuitive grounds and, since the observables $\hat{x}, \hat{y}, \hat{p}_x, \hat{p}_y$ act on \mathcal{V}_{phys} by multiplication and ($-i$ times) differentiation, they define self-adjoint operators on \mathcal{H}_{phys} and the reality conditions are satisfied in the usual way.

Finally, note that there is a freedom to scale the map η by a constant: for real positive a , the use of $\eta_a = a\eta$ would simply re-scale the physical inner product by an overall factor and lead to an equivalent physical Hilbert space. This freedom can be traced back to the fact that the Haar measure (dz' in II.10) on a non-compact group is unique only up to a multiplicative factor.

Example B

Our second example (also treated in Ref. [14,16]) will be the massive free relativistic par-

ticle in four-dimensional Minkowski space. Recall that this system may be classically described by a phase space \mathbb{R}^8 with coordinates x^μ, p_ν for $\mu, \nu \in \{0, 1, 2, 3\}$. It is subject to the constraint $p^2 + m^2 = 0$ and has an associated set of gauge transformations which may be loosely interpreted as ‘time reparametrizations.’ Again, these classical functions define the space \mathcal{S} of step 1 and the algebra $\mathcal{B}_{aux}^{(*)}$ of step 3. Thus, we represent them by self-adjoint quantum operators \hat{x}^μ, \hat{p}_ν which act on the auxiliary Hilbert space $L^2(\mathbb{R}^4, d^4x)$ by multiplication and ($-i$ times) differentiation. We will concentrate on the dense space Φ of smooth functions with compact support, so that elements f_0 of Φ have smooth Fourier transforms \tilde{f}_0 .

Let us attempt to apply the group averaging technique and define $\eta(f_0)$ (for $g_0 \in \Phi$) such that, for any $g_0 \in \Phi$,

$$\eta(f_0)[g_0] = \int_{\mathbb{R}^4} d^4x \int_{\mathbb{R}} d\lambda [\exp(i\lambda\hat{C})f_0^*](x)g_0(x), \quad (\text{II.11})$$

where $\hat{C} = \hat{p}^2 + m^2$. By spectral analysis, we know that the Fourier transform \tilde{f} of $f = \eta(f_0)$ is just $\tilde{f}_0^* \delta(p^2 + m^2)$ so that (II.11) does in fact define an element of Φ' .

The span of such f defines the linear space \mathcal{V}_{phys} . Now, for any $f, g \in \mathcal{V}_{phys}$, choose some f_0 such that $f = \eta(f_0)$ and define $\langle f, g \rangle_{phys} = g[f_0]$. Note that the inner product (II.11) is just the integral of $\tilde{f}_0^*(p)\tilde{g}_0(p)$ over the mass shell. This inner product is manifestly positive definite, Hermitian, and independent of the choice of f_0 and g_0 . Thus, the resulting \mathcal{H}_{phys} is the usual Hilbert space associated with the free relativistic particle, except that it contains both the ‘positive and negative frequency parts’ as orthogonal subspaces. While none of the operators \hat{x}^μ, \hat{p}_ν are observables, they can be used to construct observables on \mathcal{H}_{aux} for which the induced operators on \mathcal{H}_{phys} are the familiar Newton-Wigner operators (see Ref. [16, 29]). Again, any of the maps $\eta_a = a\eta$ may be used in this construction.

Example C

Finally, we consider what we will call the massive free relativistic particle on a globally hyperbolic, curved four dimensional space-time \mathcal{M} with metric $g_{\mu\nu}$. We will allow an arbitrary space-time for which the wave operator $\nabla_\mu \nabla^\mu$ is essentially self-adjoint when acting on the Hilbert space $L^2(\mathcal{M}, dv)$, where dv is the space-time volume element.

We take the classical phase space to be $\Gamma = T^*M$, but subject our system to the constraint $g^{\mu\nu}(x)p_\mu p_\nu + m^2 = 0$. Here, p_μ is the four-momentum and this constraint generates an associated group of gauge symmetries. We choose smooth functions on \mathcal{M} and $V^\mu p_\mu$ for complete vector fields V^μ on \mathcal{M} to generate the subspace \mathcal{S} and the algebra $\mathcal{B}_{aux}^{(*)}$. It is then natural to choose \mathcal{H}_{aux} to be $L^2(\mathcal{M}, dv)$ and to represent real functions on \mathcal{M} by self-adjoint operators that act by multiplication. Similarly, real complete vector fields V^μ are represented by the self-adjoint differential operators $(-i)V^\mu \partial_\mu - \frac{i}{2} \text{div}(V)$, where $\text{div}(V)$ denotes the divergence of v with respect to the space-time metric; $\mathcal{L}_V dv = \text{div}(V)dv$. The constraint is promoted to the unique self-adjoint extension \hat{C} of the wave operator on $L^2(\mathcal{M}, dv)$. (The freedom to add a multiple of the scalar curvature of $g_{\mu\nu}$ does not affect the discussion that follows.)

It is again natural to take Φ to be the space of smooth functions on \mathcal{M} with compact support. We then define the map $\eta : \Phi \rightarrow \Phi'$ by

$$(\eta f_0)(x) = \left(\int_{\mathbb{R}} d\lambda e^{i\lambda\hat{C}} f_0^* \right)(x) \quad (\text{II.12})$$

and take \mathcal{V}_{phys} to be its image. Here we appeal to Gel'fand spectral theory [27] to show that the resulting generalized eigenstates lie in the topological dual Φ' of Φ . As before, the natural concept is in fact the family of maps $\eta_a = a\eta$ for $a \in \mathbb{R}^+$. The physical Hilbert space \mathcal{H}_{phys} is the completion of \mathcal{V}_{phys} in the inner product $\langle f, g \rangle_{phys} = g[f_0]$ where f_0 satisfies $f = \eta_1(f_0)$. This inner product is independent of the particular choice of f_0 , is Hermitian and positive definite, and self adjoint operators A on \mathcal{H}_{aux} which preserve Φ and commute with \hat{C} induce symmetric, densely defined operators A_{phys} on \mathcal{H}_{phys} .

The construction of \mathcal{H}_{phys} may come as a surprise to some readers as it seems to violate the accepted idea that there is no well-defined notion of a single relativistic quantum particle in a non-stationary space-time. The ‘resolution’ is that the quantum theory defined above does not exhibit the properties that one would require for it to describe a ‘physical’ free particle. In particular, it contains no notion of a conserved probability associated with Cauchy surfaces, as our particle appears to ‘scatter backwards in time’ when it encounters a lump of space-time curvature. (Re-collapsing cosmologies [16] illustrate a similar effect). In addition, this framework cannot be used as the one-particle Hilbert space to build a relativistic field theory. Recall that an essential element in the

construction of a quantum field from a one particle Hilbert space is that the inner product on the Hilbert space be compatible with the symplectic structure on the space of classical solutions (which is given by the Klein-Gordon inner product). That this is not the case for our inner product may be seen from the fact that it contains no notion of a conservation law associated with Cauchy surfaces.

III. THE CLASS OF THEORIES

In this section, we spell out in some detail the class of theories to be considered and discuss various features which will be used in subsequent sections. The section is divided into three parts. We present the general framework in the first, some illustrative examples of theories satisfying our assumptions in the second, and in the third, a set of functions on the phase spaces of these theories which will serve as elementary variables in the quantization program.

A. General framework

Let suppose that the underlying “space-time” M is a $d + 1$ dimensional manifold with topology $M = \mathbb{R} \times \Sigma$ where Σ is an orientable, real analytic, d dimensional manifold. We wish to consider field theories on M which admit a Hamiltonian formulation with following features:

a) The phase space consists of canonical pairs (A_a^i, \tilde{E}_i^a) where A_a^i is a connection 1-form on Σ taking values in the Lie algebra of a compact, connected gauge group G , and \tilde{E}_i^a , its conjugate momentum, is a vector density of weight one on Σ which takes values in the dual of the Lie algebra of G . The fundamental Poisson brackets will be:

$$\{A_a^i(t, x), \tilde{E}_j^b(t, y)\} = \delta(x, y)\delta_j^i\delta_a^b. \quad (\text{III.1})$$

b) The theory is a constrained dynamical system subject to (at least) the following two constraints:

$$G_i := (\partial_a \tilde{E}^a + [A_a, \tilde{E}^a])_i = 0 \quad \text{and} \quad (\text{III.2})$$

$$V_a := \text{tr}[F_{ab}\tilde{E}^b] = 0 \quad (\text{III.3})$$

where F is the curvature of A . The first of these will be referred to as the *Gauss constraint* and the second as the *vector* or the *diffeomorphism constraint*. A given theory in the class may well have other constraints.

It is easy to check that the canonical transformations generated by the Gauss constraint correspond to local gauge transformations associated

with G while those associated with (a suitable combination of the Gauss and) the vector constraint correspond to diffeomorphisms of Σ . The constraint algebra formed by these two constraints is of first class. The action of these theories will have the general form:

$$S = \frac{1}{c^2} \int_{\mathbb{R}} dt \int_{\Sigma} d^d x (\text{tr}[\dot{A}_a \tilde{E}^a] - [-\text{tr}[\Lambda G] + N^a V_a + \text{other terms}]), \quad (\text{III.4})$$

where c is a coupling constant, Λ^i, N^a are associated Lagrange multipliers and “other terms” could contain additional constraints. (For simplicity, we have left out possible boundary terms.) We will assume that the full system of constraints is of first class and that the Hamiltonian is (weakly) invariant under the canonical transformations generated by *all* constraints.

In the following sections, for most part, we will focus just on the Gauss and the vector constraints.

B. Example theories

In this section, we will provide several examples to illustrate the type of theories that are encompassed by our analysis.

A) The Husain-Kuchař model

This is perhaps the simplest non-trivial example. Here, the gauge group G is $SU(2)$ and the manifold Σ is 3-dimensional. As mentioned in the Introduction, it has no further constraints and the Hamiltonian is a linear combination of the two constraints. Somewhat surprisingly, the model does arise from a manifestly covariant, 4-dimensional action [1]. Although it is not of direct physical significance, this model is interesting from a mathematical physics perspective because it has all the features of general relativity except the Hamiltonian constraint.

B) Riemannian general relativity

A second model is provided by 4-dimensional general relativity with metrics of signature $(++++)$. Again, at least at first sight, this model is not of direct physical interest. However, since it contains all the conceptual non-trivialities of Lorentzian general relativity, it provides an excellent arena to test various quantization strategies. Furthermore, there are some indications that, if one were to solve this model completely, one may be able to pass to the quantum theory of Lorentzian general relativity by a “generalized Wick rotation” which would map suitably regular functions of the Euclidean

self-dual connections to holomorphic functions of the Lorentzian self-dual connections.

Since this model is not discussed in the literature, we will write down the basic equations governing it. We will, however be brief since the Lorentzian counterpart of this case has been analyzed in detail in Ref.[2,17]. The key idea here is to use a Palatini-type of action, however with *self-dual* connections. Thus, we begin with:

$$S(^4A, e) = \int_M d^4x(e) e_I^a e_J^b ({}^4F)_{ab}^{IJ} \quad (\text{III.5})$$

where the a, b, c are the four-dimensional tensor indices, $I, J, K = 1, \dots, 4$ are the ‘‘internal’’ $SO(4)$ indices, e_I^a is a tetrad (for a positive definite metric), e its determinant, ${}^4A_a^{IJ}$, a *self-dual* connection and ${}^4F_{ab}^{IJ}$, its curvature. Although we are using self-dual connections, the variation of this action provides precisely the vacuum Einstein’s equations.

For simplicity, let us assume that the 3-manifold Σ is compact. (The asymptotically flat case requires more care but can be treated in an analogous fashion [2,17].) Then, if we perform a 3+1 decomposition, let t^a be the ‘‘time-evolution’’ vector field, and use a suitable basis in the 3-dimensional self-dual sub-algebra of the $SO(4)$ Lie-algebra, we can cast the action in the form:

$$S = \int_{\mathbb{R}} dt \int_{\Sigma} d^3x [\tilde{E}_i^a \mathcal{L}_t A_a^i - \{ -A_t^i G_i + N^a V_a + \frac{1}{2} \tilde{N} C \}]. \quad (\text{III.6})$$

Here indices a, b, \dots refer to the tangent space of Σ and i, j, \dots to the self-dual ($SU(2)$) Lie algebra; $A_t := t^a {}^4A_a$, N^a and \tilde{N} are Lagrange multipliers; and, (A_a^i, \tilde{E}_i^a) are the canonical variables. Thus, symplectic structure is given by

$$\{A_a^i(x), \tilde{E}_j^b(y)\} = \delta_a^b \delta_j^i \delta^{(3)}(x, y). \quad (\text{III.7})$$

The variation of the action with respect to the Lagrange multipliers yields, as usual, the first class constraints of Riemannian general relativity:

$$\begin{aligned} G_i &:= \mathcal{D}_a \tilde{E}_i^a \equiv \partial_a \tilde{E}_i^a + \epsilon_{ijk} A_a^j \tilde{E}_k^a = 0, \\ V_a &:= F_{ab}^i \tilde{E}_i^b = 0, \text{ and} \\ C &:= F_{abi} \tilde{E}_j^a \tilde{E}_k^b \epsilon^{ijk} = 0. \end{aligned} \quad (\text{III.8})$$

These are, respectively, the Gauss, the vector and the scalar constraint. Thus, in the Hamiltonian form, the theory is similar to the Husain-Kuchař model except for the presence of the additional scalar constraint.

How do we make contact with the more familiar Hamiltonian form of the theory in terms of metrics and extrinsic curvatures? The two are related simply by a canonical transformation. Regard \tilde{E}_i^a as a triad on Σ with density weight one and denote by Γ_a^i the spin-connection defined by it. Define K_a^i via: $K_a^i = \Gamma_a^i - A_a^i$. Then, $(A_a^i, \tilde{E}_i^a) \mapsto (\tilde{E}_i^a, K_a^i)$ is a canonical transformation. \tilde{E}_i^a determines the 3-metric q_{ab} on Σ via $\tilde{E}_i^a \tilde{E}^{bi} = qq^{ab}$, and K_a^i determines the extrinsic curvature K_a^b via $\sqrt{q} K_a^b = K_a^i \tilde{E}_i^b$, where q is the determinant of q_{ab} . Note, however, that, while the constraints (III B) are all low order polynomials in terms of the connection variables, they become non-polynomial in terms of the metric variables. Hence, if one uses the metric formulation, it is much more difficult to promote them to well-defined operators on an auxiliary Hilbert space.

C) Lorentzian general relativity in the spin connection formulation

In the Lorentzian signature, self-dual connections are complex. Therefore, the formulation of the Lorentzian theory in terms of self-dual connections [2,17] falls outside the scope of this paper. However, as in the Euclidean case, one can consider the real fields (\tilde{E}_i^a, K_a^i) as a canonical pair. By a contact transformation, one can replace the triad \tilde{E}_i^a by the spin-connection Γ_a^i and K_a^i by the momentum \tilde{P}_i^a conjugate to Γ_a^i . In the new canonical pair, the configuration variable is a $SU(2)$ connection whence the framework falls in the class of theories considered here. One can show that the Gauss and the vector constraints retain their form; A_a^i and \tilde{E}_i^a in (III.7) are simply replaced by Γ_a^i and \tilde{P}_i^a respectively. Therefore, in this formulation, the theory belongs to the class under consideration.

Unfortunately, however, the remaining, scalar constraint seems unmanageable in terms of Γ_a^i and \tilde{P}_i^a . Hence, this formulation is not directly useful beyond the Gauss and the vector constraints [30]. As mentioned in the Introduction, to handle the Hamiltonian constraint, one would have to use a different strategy, e.g., the one involving a coherent state transform [3] and pass to the (Lorentzian) self-dual representation.

D) Chern-Simons theories

Let G may be any compact, connected Lie group. Then, one can construct a natural ‘‘inhomogeneous version’’ IG of G . As a manifold, IG is isomorphic to the cotangent bundle over G and, as a group, it is a semi-direct product of G with an Abelian group which has the same dimension as G . If G is chosen to be the rotation group, $SO(3)$, then IG

is the Euclidean group in three dimensions. (For details, see Ref. [4,31].) Let us now set the dimension d of Σ to be 2 and consider the Chern-Simons theory based on IG . (If G is chosen to be $SU(2)$, this theory is equivalent to 3-dimensional Riemannian general relativity.) It is straightforward to check that all our assumptions from Sec. III A are satisfied.

We can also consider a more sophisticated enlargement $I_\Lambda G$ of G which is parametrized by a real number Λ (see Ref. [31]). In the case when G is $SU(2)$, the Chern-Simons theory based on $I_\Lambda G$ is the same as Riemannian general relativity with a cosmological constant. (Curiously, the theory that results from $G = SU(2)$ and Λ negative is also isomorphic, in an appropriate sense, with the Lorentzian, 3-dimensional gravity with a *positive* cosmological constant.) All these theories also fall in the class under consideration. Note however that, in general, the Chern-Simons theories based on compact gauge groups G –rather than IG or $I_\Lambda G$ – fall outside this class since these theories do not have canonical variables of the required type.

C. An (over)complete set of gauge invariant functions

In this section, for simplicity of presentation, we will focus on the case $d = 3$ and $G = SU(2)$. Generalizations to higher dimensions and other compact, connected groups is, however, straightforward. For simplicity, we will solve the Gauss constraint classically. (See, however, the first part of Sec. VII.) Therefore, it is natural to regard the space \mathcal{A}/\mathcal{G} of (sufficiently well-behaved) connections on Σ modulo (sufficiently regular) gauge transformations as the effective configuration space. Phase space is then the cotangent bundle on \mathcal{A}/\mathcal{G} . Our aim is to single out a convenient set of functions on this phase space which can be used as “elementary classical variables” in the quantization program of Sec. II.

Wilson loop functions are the obvious candidates for configuration variables. These will be associated with piecewise analytic loops on Σ , i.e., with piecewise analytic maps $\alpha : S^1 \mapsto \Sigma$. (Thus, the loops do not have a preferred parameterization, although in the intermediate stages of calculations, it is often convenient to choose one.) The Wilson loop variables $T_\alpha(A)$ are given by:

$$T_\alpha(A) := \text{tr } h_\alpha[A] \equiv \text{tr} \mathcal{P} \exp \int_\alpha A, \quad (\text{III.9})$$

where the trace is taken in the fundamental repre-

sentation. As defined, these are functions on the space of connections. However, being gauge invariant, they project down naturally to \mathcal{A}/\mathcal{G} . The momentum observables, T_S are associated with piecewise analytic strips S , i.e., ribbons which are foliated by a 1-parameter family of loops. For technical reasons, it is convenient to begin with piecewise analytic embeddings $S : (1, 1) \times S^1 \mapsto \Sigma$ and use them to generate more general strips. Set

$$T_S(A) := \int_S dS^{ab} \eta_{abc} T_{\alpha_\tau}^c(\sigma, \tau), \quad \text{where} \\ T_{\alpha_\tau}^c(\sigma, \tau) := \text{tr}(h_{\alpha_\tau}(\sigma, \tau)[A] \tilde{E}^c(\sigma, \tau)), \quad (\text{III.10})$$

σ, τ are coordinates on S (with τ labeling the loops within S and σ running along each loop α_τ), η_{abc} denotes the Levi-Civita tensor density on Σ , and, as before h_{α_τ} denotes the holonomy along the loop α_τ . Again, the functions T_S are gauge invariant and hence well-defined on the cotangent bundle over \mathcal{A}/\mathcal{G} . They are called “momentum variables” because they are linear in \tilde{E}_i^a .

Properties of these variables are discussed in some detail in Ref. [17]. Here we recall only the main features. First they constitute a complete set in the sense that their gradients span the cotangent space almost everywhere on the phase space over \mathcal{A}/\mathcal{G} . However, they are not all independent. Properties of the trace operation in the fundamental representation of $SU(2)$ induce relations between them. These algebraic relations have to be incorporated in the quantum theory. It is interesting that the Poisson brackets can be expressed in terms of simple geometric operations between loops and strips. We will illustrate this by writing out one of these Poisson brackets which will be needed in the subsequent sections:

$$\{T_\alpha, T_S\} = \sum_i \text{sgn}_i(S, \alpha) [T_{S \circ_i \alpha} - T_{S \circ_i \alpha^{-1}}] \quad (\text{III.11})$$

where the sum is over transverse intersections i between the the loop α and the strip S , $\text{sgn}_i(S, \alpha)$ takes values $0, \pm 1$ depending on the orientation of the tangent vector of α and the tangent plane of S at the i -th intersection point and $S \circ_i \alpha$ is a loop obtained by composing the loop in the strip S passing through the intersection point with the loop α . (Note that the same geometric point in Σ may feature in more than one intersection i .) Thus, in particular, the Poisson bracket vanishes unless the loop α intersects the strip S .

The Poisson bracket between two strip functionals also vanishes unless the two strips intersect. If

they do, the bracket is given by a sum of slightly generalized strip functionals. The generalization consists only of admitting certain strip maps $S : (1, 1) \times S^1 \mapsto \Sigma$ which is not necessarily embeddings, and integrating in (III.10) over a suitable sub-manifold I without boundary, $I \subset (0, 1) \times S^1$, such that for every loop α_τ , $\alpha_\tau \cap S(I)$ is a closed loop. The Poisson bracket between these more general strips closes. We did not simply begin with these more general strips because, in quantum theory, it is easier to begin with the embedded strips and let them generate more general ones.

In Sec. V, we will use these loop and strip functionals as elementary classical variables to construct the auxiliary Hilbert space.

IV. QUANTUM CONFIGURATION SPACE

To complete the first four steps in the quantization program, it is convenient to proceed in two stages. First, one focuses on just the configuration variables T_α and constructs representations of the corresponding “holonomy algebra.” This naturally leads to the notion of a quantum configuration space. By introducing suitable geometric structures on this space, one can then represent the momentum operators corresponding to T_S .

We will begin, in this section, by isolating the quantum configuration space. In the second part, we will present three convenient characterizations of this space. A number of constructions used in the subsequent sections depend on these characterizations. In the third part, we introduce elements of calculus on this space which will lead to the definition of the momentum operators in Sec. V.

A. $\overline{\mathcal{A}/\mathcal{G}}$ a completion of \mathcal{A}/\mathcal{G}

In the classical theory, \mathcal{A}/\mathcal{G} serves as the gauge invariant configuration space for the class of theories under consideration. We will now show that, in the passage to quantum theory, one is led to enlarge this space [5]. Recall that an enlargement also occurs in, for example, scalar quantum field theory [32,33].

Let us begin by constructing the Abelian algebra of configuration operators. This algebra is, of course, generated by finite linear combinations of functions T_α on \mathcal{A}/\mathcal{G} with complex coefficients. By construction, it is closed under the operation of taking complex conjugation. Thus, it is a \star -subalgebra of the algebra of complex-valued, continuous bounded functions on \mathcal{A}/\mathcal{G} . It separates

the points of \mathcal{A}/\mathcal{G} in the sense that, if $[A_1] \neq [A_2]$ (i.e., if the gauge equivalence classes of A_1 and A_2 in \mathcal{A} do not coincide), there exists a loop α such that: $T_\alpha(A_1) \neq T_\alpha(A_2)$. Thus, as indicated in Sec. III C, the set of configuration variables is sufficiently large. This algebra is called the *holonomy algebra* and denoted by $\mathcal{H}\mathcal{A}$. To obtain a greater degree of control, it is convenient to introduce on it a norm and convert it into a C^* algebra.

Let us therefore set

$$\|f\| = \sup_{[A] \in \mathcal{A}/\mathcal{G}} |f([A])| \quad (\text{IV.1})$$

and complete $\mathcal{H}\mathcal{A}$ with respect to this norm we obtain a commutative C^* -algebra $\overline{\mathcal{H}\mathcal{A}}$. (This algebra is equipped with identity, given by T_\emptyset , where \emptyset is the trivial, i.e., point loop.) We are now in a position to apply the powerful representation theory of C^* -algebras.

The first key result we will use is the Gel’fand-Naimark theorem, that every C^* algebra with identity is isomorphic to the C^* -algebra of all continuous bounded functions on a compact Hausdorff space called the *spectrum* of the algebra. The spectrum can be constructed directly from the algebra: it is the set of all \star -homomorphisms from the given C^* -algebra to the \star -algebra of complex numbers. We will denote the spectrum of $\overline{\mathcal{H}\mathcal{A}}$ by $\overline{\mathcal{A}/\mathcal{G}}$. It is easy to show that \mathcal{A}/\mathcal{G} is densely embedded in $\overline{\mathcal{A}/\mathcal{G}}$; thus, $\overline{\mathcal{A}/\mathcal{G}}$ can be regarded as a completion of \mathcal{A}/\mathcal{G} .

Recall that, since $\overline{\mathcal{H}\mathcal{A}}$ is the C^* -algebra of configuration variables, our primary objective here is to construct its representations. Now, a key simplification occurs because one has a great deal of control on the representation theory. Let $\rho : \overline{\mathcal{H}\mathcal{A}} \rightarrow B(\mathcal{H})$ denote a cyclic representation of $\overline{\mathcal{H}\mathcal{A}}$ by bounded operators on some Hilbert space \mathcal{H} . Let Γ be the “vacuum expectation value functional”:

$$\Gamma(f) = \langle \rho(f)\Omega, \Omega \rangle, \quad (\text{IV.2})$$

where Ω is a cyclic vector and f any element of $\overline{\mathcal{H}\mathcal{A}}$. Clearly, Γ is a positive linear functional on $\overline{\mathcal{H}\mathcal{A}}$. Since $\overline{\mathcal{H}\mathcal{A}}$ is isomorphic with the C^* -algebra of continuous functions on $\overline{\mathcal{A}/\mathcal{G}}$, Γ can be regarded as a positive linear functional also on $C^0(\overline{\mathcal{A}/\mathcal{G}})$. Now, since $\overline{\mathcal{A}/\mathcal{G}}$ is compact, the Riesz representation theorem ensures that there is a unique regular Borel measure μ on $\overline{\mathcal{A}/\mathcal{G}}$ such that

$$\Gamma(f) = \int_{\overline{\mathcal{A}/\mathcal{G}}} d\mu([A]) \tilde{f}([A]), \quad (\text{IV.3})$$

where $\tilde{f} \in C^0(\overline{\mathcal{A}/\mathcal{G}})$ corresponds to f in $\overline{\mathcal{H}\mathcal{A}}$. This immediately implies that any cyclic representation

of $\overline{\mathcal{HA}}$ is unitarily equivalent to a “connection representation” given by

$$\begin{aligned} \overline{\mathcal{HA}} &\rightarrow B(L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu)) \\ (\rho(f)\psi)([A]) &= \tilde{f}([A])\psi([A]) , \end{aligned} \quad (\text{IV.4})$$

where the measure μ is defined through (IV.3). Therefore the set of regular measures on $\overline{\mathcal{A}/\mathcal{G}}$ is one-to-one correspondence with the set of cyclic representations of $\overline{\mathcal{HA}}$.

To summarize, in *any* cyclic representation of $\overline{\mathcal{HA}}$, quantum states can be thought of as (square-integrable) functions on $\overline{\mathcal{A}/\mathcal{G}}$ (for some choice of measure. Recall that cyclic representations are the basic “building blocks” of general representations.) Hence, $\overline{\mathcal{A}/\mathcal{G}}$ can be identified with the *quantum configuration space*. The enlargement from \mathcal{A}/\mathcal{G} to $\overline{\mathcal{A}/\mathcal{G}}$ is non-trivial because, typically, \mathcal{A}/\mathcal{G} is contained in a set of zero measure [9].

We will conclude this subsection with a general remark. In the construction of the quantum configuration space, we have avoided the use of the non-gauge invariant affine structure of the space \mathcal{A} of connections and worked instead directly on \mathcal{A}/\mathcal{G} . (For earlier works in the same spirit, see [34].) This is in contrast with the gauge fixing strategy that is sometimes adopted in constructive quantum field theory [32,33] which then faces global problems associated with Gribov ambiguities.

B. Characterizations of $\overline{\mathcal{A}/\mathcal{G}}$

Since $\overline{\mathcal{A}/\mathcal{G}}$ is the domain space of quantum states, it is important to understand its structure. In this subsection, therefore, we will present three characterizations of this space, each illuminating its structure from a different perspective.

Denote by $\mathcal{L}_{x_0}\Sigma$ the space of continuous piecewise analytic loops on Σ based at an arbitrarily chosen but fixed point x_0 . Two loops α, β are said to be holonomically equivalent if for every $A \in \mathcal{A}$ we have

$$H(\alpha, A) = H(\beta, A) . \quad (\text{IV.5})$$

The corresponding equivalence classes are called *hoops*. For notational simplicity we will use lower case greek letters to denote these classes as well. The set of all hoops forms a group called the hoop group which is denoted by \mathcal{HG}_{x_0} . A smooth connection $A \in \mathcal{A}$ defines a homomorphism from \mathcal{HG}_{x_0} to $SU(2)$, which is smooth in a certain sense [35]

$$H(., A) : \mathcal{HG}_{x_0} \rightarrow SU(2) . \quad (\text{IV.6})$$

We can now present the first characterization: $\overline{\mathcal{A}/\mathcal{G}}$ is naturally isomorphic to the set of *all* homomorphisms from \mathcal{HG}_{x_0} to $SU(2)$ modulo conjugation [6]. (The conjugation serves only to eliminate the freedom to perform gauge transformations at the base point. Note that the homomorphism here need not even be continuous.) This result makes it possible to show further that $\overline{\mathcal{A}/\mathcal{G}}$ is a limit of configuration spaces of gauge theories living in arbitrary lattices for which the space of connections modulo gauge transformations coincides with finite products of copies of $SU(2)$ modulo conjugation [9–12].

The second characterization is in terms of these limits. To introduce it, let us begin with the notion of independent hoops [6]. Hoops $\{\beta_1, \dots, \beta_n\}$ will be said to be independent if loop representatives exist such that each contains an open segment that is traced exactly once and which intersects other representatives at most at a finite number of points. Let now $S_n(\beta_1, \dots, \beta_n)$ denote the subgroup of \mathcal{HG}_{x_0} generated by a set of independent hoops $\{\beta_1, \dots, \beta_n\}$. The space $H(S_n)$ of all homomorphisms (modulo conjugation) from S_n to $SU(2)$ is homeomorphic to $SU(2)^n/Ad$, which in turn can be thought of as the configuration space of the “floating” (i.e., non-rectangular) lattice formed by $\{\beta_1, \dots, \beta_n\}$. Now, if we consider a larger subgroup $S_m \supset S_n$ of the hoop group, we have a natural projection map $p_{S_n S_m}$, where

$$\begin{aligned} p_{S_n S_m} : H(S_m) &\rightarrow H(S_n) \\ p_{S_n S_m}(h) &= h|_{S_n} . \end{aligned} \quad (\text{IV.7})$$

In the lattice picture, the projection is obtained simply by restricting the configurations on the larger lattice to the smaller lattice.

The family $(H(S_n), p_{S_n S_m})$ is called a *projective family* labeled by the subgroups S_n of the hoop group (see appendix B). Since the theory for a larger lattice contains more information, it is desirable to consider larger and larger lattices, i.e., bigger and bigger subgroups of the hoop group. Unfortunately the projective family itself does not have a “largest element” from which one can project to any other. However, such an element can in fact be obtained by a standard procedure called the “projective limit.”. Now, given the space $\overline{\mathcal{A}/\mathcal{G}}$, we have a surjective projection p_{S_n} to $H(S_n)$ for *any* subgroup S_n of the hoop group:

$$\begin{aligned} p_{S_n} : \overline{\mathcal{A}/\mathcal{G}} &\rightarrow SU(2)^n/Ad \\ [A] &\mapsto ([A(\beta_1)], \dots, [A(\beta_n)]) \end{aligned} \quad (\text{IV.8})$$

where the brackets $[,]$ on the right hand side denote conjugacy classes. This suggests $\overline{\mathcal{A}/\mathcal{G}}$ may be

the projective limit of the family $(H(S_n), p_{S_n S_m})$. Detailed considerations show that this is indeed the case [9].

This characterization of $\overline{\mathcal{A}/\mathcal{G}}$ as a limit of finite dimensional spaces allows the introduction of integral calculus [6–8,10,12] on $\overline{\mathcal{A}/\mathcal{G}}$ using integration theory on finite dimensional spaces. Roughly, measures on lattice configuration spaces $H(S_n)$ which are compatible with the projections $P_{S_n S_m}$ from larger lattices to the smaller ones induce measures on the projective limit $\overline{\mathcal{A}/\mathcal{G}}$. In particular, this strategy was first used in [6] to construct a natural, faithful, diffeomorphism invariant measure μ_0 on $\overline{\mathcal{A}/\mathcal{G}}$ from the induced Haar measures on the configuration spaces $H(S_n)$ of lattice theories. More precisely, μ_0 is defined by:

$$p_{S_n \star} \mu_0 = p_{Ad \star} \mu_H \otimes \dots \otimes \mu_H . \quad (\text{IV.9})$$

where, μ_H denotes the Haar measure on $SU(2)$, p_{Ad} denotes the quotient map

$$p_{Ad} : SU(2) \times \dots \times SU(2) \rightarrow SU(2) \times \dots \times SU(2)/Ad , \quad (\text{IV.10})$$

and $f_\star \mu$ denotes the push-forward of the measure μ with respect to the map f .

This description uses hoops as the set of “probes” for the generalized connections. A related approach, developed by Baez, [7] relies on the (gauge dependent) probes defined by analytic edges. This strategy provides a third characterization of $\overline{\mathcal{A}/\mathcal{G}}$, again as a projective limit, but of a projective family labeled by graphs rather than hoops. It is this characterization that is best suited for developing differential calculus [8,11]. Since it is used in the subsequent sections, we will discuss it in greater detail.

Let us begin with the set \mathcal{E} of all oriented, unparametrized, embedded, analytic intervals (edges) in Σ . We introduce the space $\overline{\mathcal{A}}$ of (generalized) connections on Σ as the space of all maps $A : \mathcal{E} \rightarrow SU(2)$, such that

$$A(e^{-1}) = [A(e)]^{-1}, \text{ and } A(e_2 \circ e_1) = A(e_2)A(e_1) \quad (\text{IV.11})$$

whenever two edges $e_2, e_1 \in \mathcal{E}$ meet to form an edge. Here, $e_2 \circ e_1$ denotes the standard path product and e^{-1} denotes e with opposite orientation. The group $\overline{\mathcal{G}}$ of (generalized) gauge transformations acting on $\overline{\mathcal{A}}$ is the space of all maps $g : \Sigma \rightarrow SU(2)$ or equivalently the Cartesian product group

$$\overline{\mathcal{G}} := \times_{x \in \Sigma} SU(2) . \quad (\text{IV.12})$$

A gauge transformation $g \in \overline{\mathcal{G}}$ acts on $A \in \overline{\mathcal{A}}$ through

$$[g(A)](e_{p_2, p_1}) = (g_{p_2})^{-1} A(e_{p_2, p_1}) g_{p_1} , \quad (\text{IV.13})$$

where e_{p_2, p_1} is an edge from $p_1 \in \Sigma$ to $p_2 \in \Sigma$ and g_{p_i} is the group element assigned to p_i by g . The group $\overline{\mathcal{G}}$ equipped with the product topology is a compact topological group. Note also that $\overline{\mathcal{A}}$ is a closed subset of the Cartesian product of all \mathcal{A}_e ,

$$\overline{\mathcal{A}} \subset \times_{e \in \mathcal{E}} \mathcal{A}_e , \quad (\text{IV.14})$$

where the space \mathcal{A}_e of all maps from the one point set $\{e\}$ to $SU(2)$ is homeomorphic to $SU(2)$. $\overline{\mathcal{A}}$ is then compact in the topology induced from this product.

The space $\overline{\mathcal{A}}$ (and also $\overline{\mathcal{G}}$) can also be regarded as the projective limit of a family labeled by graphs in Σ in which each member is homeomorphic to a finite product of copies of $SU(2)$. [9,12] Let us now briefly recall this construction as it underlies the introduction of calculus on $\overline{\mathcal{A}/\mathcal{G}}$.

Definition 1 *A graph on Σ is a finite subset $\gamma \subset \mathcal{E}$ such that (i) two different edges, $e_1, e_2 : e_1 \neq e_2$ and $e_1 \neq e_2^{-1}$, of γ meet, if at all, only at one or both ends and (ii) if $e \in \gamma$ then $e^{-1} \in \gamma$.*

The set of all graphs in Σ will be denoted by $\text{Gra}(\Sigma)$. In $\text{Gra}(\Sigma)$ there is a natural relation of partial order \geq ,

$$\gamma' \geq \gamma \quad (\text{IV.15})$$

whenever every edge of γ is a path product of edges associated with γ' . Furthermore, for any two graphs γ_1 and γ_2 , there exists a γ such that $\gamma \geq \gamma_1$ and $\gamma \geq \gamma_2$, so that $(\text{Gra}(\Sigma), \geq)$ is a directed set.

Given a graph γ , let \mathcal{A}_γ be the associated space of assignments ($\mathcal{A}_\gamma = \{A_\gamma | A_\gamma : \gamma \rightarrow SU(2)\}$) of group elements to edges of γ , satisfying $A_\gamma(e^{-1}) = A_\gamma(e)^{-1}$ and $A_\gamma(e_1 \circ e_2) = A_\gamma(e_1)A_\gamma(e_2)$, and let $p_\gamma : \overline{\mathcal{A}} \rightarrow \mathcal{A}_\gamma$ be the projection which restricts $A \in \overline{\mathcal{A}}$ to γ . Notice that p_γ is a surjective map. For every ordered pair of graphs, $\gamma' \geq \gamma$, there is a naturally defined map

$$p_{\gamma\gamma'} : \mathcal{A}_{\gamma'} \rightarrow \mathcal{A}_\gamma, \text{ such that } p_\gamma = p_{\gamma\gamma'} \circ p_{\gamma'} . \quad (\text{IV.16})$$

With the same graph γ , we also associate a group \mathcal{G}_γ defined by

$$\mathcal{G}_\gamma := \{g_\gamma | g_\gamma : V_\gamma \rightarrow SU(2)\} , \quad (\text{IV.17})$$

where V_γ is the set of *vertices* of γ ; that is, the set V_γ of points of Σ lying at the ends of edges of

γ . There is a natural projection $\overline{\mathcal{G}} \rightarrow \mathcal{G}_\gamma$ which will also be denoted by p_γ and is again given by restriction (from Σ to V_γ). As before, for $\gamma' \geq \gamma$, p_γ factors into $p_\gamma = p_{\gamma\gamma'} \circ p_{\gamma'}$ to define

$$p_{\gamma\gamma'} : \mathcal{G}_{\gamma'} \rightarrow \mathcal{G}_\gamma . \quad (\text{IV.18})$$

Note that the group \mathcal{G}_γ acts naturally on \mathcal{A}_γ and that this action is equivariant with respect to the action of $\overline{\mathcal{G}}$ on $\overline{\mathcal{A}}$ and the projection p_γ . Hence, each of the maps $p_{\gamma\gamma'}$ projects to new maps also denoted by

$$p_{\gamma\gamma'} : \mathcal{A}_{\gamma'}/\mathcal{G}_{\gamma'} \rightarrow \mathcal{A}_\gamma/\mathcal{G}_\gamma . \quad (\text{IV.19})$$

We collect the spaces and projections defined above into a (triple) projective family $(\mathcal{A}_\gamma, \mathcal{G}_\gamma, \mathcal{A}_\gamma/\mathcal{G}_\gamma, p_{\gamma\gamma'})$. It is not hard to see that $\overline{\mathcal{A}}$ and $\overline{\mathcal{G}}$ as introduced above are just the projective limits of the first two families. Finally, the quotient of compact projective limits is the projective limit of the compact quotients, [12]

$$\overline{\mathcal{A}/\mathcal{G}} = \overline{\mathcal{A}}/\overline{\mathcal{G}} . \quad (\text{IV.20})$$

This concludes our third characterization of $\overline{\mathcal{A}/\mathcal{G}}$. (Note that the projections $p_{\gamma\gamma'}$ in (IV.16), (IV.18) and (IV.19) are different from each other and that the same symbol $p_{\gamma\gamma'}$ is used only for notational simplicity; the meaning should be clear from the context.)

Using again the normalized Haar measure on $SU(2)$, the construction (IV.9,IV.10) may be repeated for this projective family [7]. This leads to a natural (“Haar”) measure μ'_0 defined on $\overline{\mathcal{A}}$ via

$$\mu'_0 = \{\mu_\gamma = \mu_H \otimes \dots \otimes \mu_H\} . \quad (\text{IV.21})$$

Under the natural projection map to $\overline{\mathcal{A}/\mathcal{G}}$, the push forward of this measure yields μ_0 of (IV.9).

C. Differential calculus on $\overline{\mathcal{A}/\mathcal{G}}$

We now recall from Ref. [11] some elements of calculus on $\overline{\mathcal{A}/\mathcal{G}}$ defined using calculus on finite dimensional spaces and the representation of $\overline{\mathcal{A}/\mathcal{G}}$ as a projective limit. This framework will allow us, in the next section, to represent \widehat{T}_S as operators on $L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_0)$.

Although our primary interest is $\overline{\mathcal{A}/\mathcal{G}}$, it will be convenient to introduce geometric structures on $\overline{\mathcal{A}}$. Vector fields and other operators that are invariant under the action of $\overline{\mathcal{G}}$ on $\overline{\mathcal{A}}$ will descend to $\overline{\mathcal{A}/\mathcal{G}} = \overline{\mathcal{A}}/\overline{\mathcal{G}}$ and provide us with differential geometry on the quotient.

Let us begin by introducing the space of C^n cylindrical functions on $\overline{\mathcal{A}}$ (for details, see appendix B):

$$\text{Cyl}^n(\overline{\mathcal{A}}) = \bigcup_{\gamma \in \text{Gra}(\Sigma)} (p_\gamma)^* C^n(\mathcal{A}_\gamma) \quad (\text{IV.22})$$

where $p_\gamma^* f = f \circ p_\gamma$ is the pull-back to $\overline{\mathcal{A}}$ of the C^n function f on the manifold \mathcal{A}_γ . The sub-space of $\overline{\mathcal{G}}$ -invariant functions in $\text{Cyl}^n(\overline{\mathcal{A}})$ constitutes the space $\text{Cyl}^n(\overline{\mathcal{A}/\mathcal{G}})$ of C^n cylindrical functions on $\overline{\mathcal{A}/\mathcal{G}}$. Although any one element of $\text{Cyl}^n(\overline{\mathcal{A}/\mathcal{G}})$ knows only about the restriction of the A to a graph γ , since we allow all possible graphs, the space $\text{Cyl}^n(\overline{\mathcal{A}})$ is in fact quite large. In the application of the quantization program, $\text{Cyl}^\infty(\overline{\mathcal{A}/\mathcal{G}})$ will serve as the space Φ , i.e., the analog of the space of C^∞ functions of compact support used in the examples in Sec. II.B.

Let us now consider vector fields. These can be regarded as derivations of the algebra $\text{Cyl}^\infty(\overline{\mathcal{A}})$, i.e.

$$X : \text{Cyl}^\infty(\overline{\mathcal{A}}) \rightarrow \text{Cyl}^\infty(\overline{\mathcal{A}}) \quad (\text{IV.23})$$

$$X(fg) = X(f)g + fX(g) . \quad (\text{IV.24})$$

A natural way to construct these vector fields is via consistent families of vector fields (X_γ) on \mathcal{A}_γ . This correspondence is given by the natural measure μ'_0 on $\overline{\mathcal{A}}$ and

$$\int_{\mathcal{A}_\gamma} \overline{g}_\gamma X_\gamma(f_\gamma) d\mu_\gamma^H = \int_{\overline{\mathcal{A}}} \overline{g} X(f) d\mu'_0 , \quad (\text{IV.25})$$

for all $f_\gamma, g_\gamma \in C^1(\mathcal{A}_\gamma)$, where $f = p_\gamma^* f_\gamma$ and $g = p_\gamma^* g_\gamma$. The family (X_γ) is (μ'_0) -consistent in the sense that for all $\gamma' \geq \gamma$, and for all $f_\gamma, g_\gamma \in C^1(\mathcal{A}_\gamma)$,

$$\int_{\mathcal{A}_{\gamma'}} \overline{p_{\gamma\gamma'}^* g_\gamma} X_{\gamma'}(p_{\gamma\gamma'}^* f_\gamma) d\mu_{\gamma'}^H = \int_{\mathcal{A}_\gamma} \overline{g}_\gamma X_\gamma(f_\gamma) d\mu_\gamma^H . \quad (\text{IV.26})$$

The cylindrical vector fields take a particularly simple form if there exists a γ_0 such that

$$(p_{\gamma\gamma'})_* X_{\gamma'} = X_\gamma \quad (\text{IV.27})$$

for all $\gamma' \geq \gamma \geq \gamma_0$. These vector fields were introduced and studied in detail in Ref.[11]. They will play an important role in the next section for the representation of \widehat{T}_S as operators.

More general cylindrical operators

$$B : \text{Cyl}^\infty(\overline{\mathcal{A}}) \rightarrow \text{Cyl}^\infty(\overline{\mathcal{A}}) \quad (\text{IV.28})$$

can be associated with families (B_γ) of operators acting on $C^\infty(\mathcal{A}_\gamma)$ and satisfying the same consistency conditions as vector fields in (IV.26). Examples of such operators are Laplacians [3,11] on $\overline{\mathcal{A}}$ and the geometric operators discussed in Appendix D.

V. QUANTUM KINEMATICS

We are now ready to apply the algebraic quantization of program of Sec. II to the class of theories under consideration. In this section, we will complete the first four steps in the program. We begin by introducing the auxiliary Hilbert space \mathcal{H}_{aux} which incorporates the reality conditions on the loop-strip functions and then analyze some of its structure.

A. Auxiliary Hilbert space and reality conditions

Let us use the vector space generated by finite linear combinations (with constant coefficients) of the loop and strip functionals of Sec. III C as the space \mathcal{S} of elementary classical variables and denote by $\mathcal{B}_{aux}^{(\star)}$ the resulting \star -algebra. Our job now is to find a \star -representation of this algebra by operators on a Hilbert space \mathcal{H}_{aux} .

Let us choose for \mathcal{H}_{aux} the space $L^2(\overline{\mathcal{A}/\overline{\mathcal{G}}}, d\mu_0)$, where μ_0 is the faithful, diffeomorphism invariant measure on $\overline{\mathcal{A}/\overline{\mathcal{G}}}$ induced by the Haar measure on the gauge group. The discussion of Sec. IV A tells us that the configuration operators \hat{T}_α should act by multiplication:

$$(\hat{T}_\alpha \circ \psi)([A]) := T_\alpha([A])\psi([A]) \quad (\text{V.1})$$

for all $\psi \in L^2(\overline{\mathcal{A}/\overline{\mathcal{G}}}, d\mu_0)$. By construction, these operators are (bounded and) self-adjoint; the reality conditions on the configuration variables are thus incorporated. Note that this would have been the case for any choice of measure; it is not essential to choose μ_0 at this stage.

The condition that \hat{T}_S be represented by self-adjoint operators, on the other hand, does restrict the measure significantly. Since T_S is linear in momentum, one would expect it to be represented by the Lie derivative along a vector field on $\overline{\mathcal{A}/\overline{\mathcal{G}}}$. This expectation is essentially correct. The detailed definition of \hat{T}_S is, however, somewhat complicated.

Let us begin by introducing a simpler operator from which \hat{T}_S will be constructed. Consider an analytic loop α . we can think of it as a graph with

just one edge. Fix a point p on α and a $d-2$ -dimensional subspace W of the tangent space at p . (Recall that the underlying manifold Σ is d dimensional.) Then, given a graph $\gamma \geq \alpha$, and a function F_γ on \mathcal{A}_γ , we wish to define the action of a vector field $X_{\alpha,W}$ on F_γ . The key idea is to exploit the fact that, if γ has n edges, (e_1, \dots, e_n) , then \mathcal{A}_γ is isomorphic with $(SU(2))^n$ and can be coordinatized by n group valued coordinates (g_1, \dots, g_n) . Using this fact, we set:

$$X_{\alpha,W} \circ F_\gamma := \text{tr}(h_\alpha \tau_j) k^{ij} \sum_{e \in \gamma} [k^-(e) X_{e,i}^L + k^+(e) X_{e,i}^R] \circ F_\gamma \quad (\text{V.2})$$

where

$$k^\pm(e) := \begin{cases} 0 & \text{if } e^\pm \neq p \\ \frac{1}{4}[\text{sgn}(\dot{e}^\pm, \dot{\alpha}^\pm, W) + \text{sgn}(\dot{e}^\pm, \dot{\alpha}^\mp, W)] & \text{if } e^\pm = p \end{cases}$$

Here, h_α is the (generalized) holonomy function on \mathcal{A}_γ associated with the loop α , τ_i are the Pauli matrices, k^{ij} , the metric in the Lie algebra of $SU(2)$, $X_{e,i}^R$ and $X_{e,i}^L$ are the right and the left invariant vector fields on the copy of the group associated with the edge e which point in the i -th direction at the identity of the group, e^\pm refers to the two ends of the edges, $\text{sgn}(\dot{e}^\pm, \dot{\alpha}^\pm, W)$ is $0, \pm 1$ depending on the relative orientation of the vectors involved and the subspace W , and α^+ (respectively, α^-) is the outgoing (incoming) segment of α at p . While the definition of this vector field seems complicated at first, it is in fact straightforward to calculate its action on functions on \mathcal{A}_γ . In particular, what counts is only the dependence of the function F_γ on the group elements corresponding to the edges which pass through p for which the orientation factor is non-zero.

For each $\gamma \geq \alpha$, we now have a vector field on \mathcal{A}_γ . One can check that these vector fields satisfy the compatibility conditions (IV.27) and thus provides a vector field (X_γ) on $\overline{\mathcal{A}}$ which we will again denote by $X_{\alpha,W}$. The definition then immediately implies that this vector field is invariant under $\overline{\mathcal{G}}$. Hence it has a well-defined action on the space $\text{Cyl}^1(\overline{\mathcal{A}/\overline{\mathcal{G}}})$ on $\overline{\mathcal{A}/\overline{\mathcal{G}}}$ of differential cylindrical functions on $\overline{\mathcal{A}/\overline{\mathcal{G}}}$ and a well defined divergence with respect to μ_0 . [11] A direct calculation shows that

$$\text{div} X_{\alpha,W} = 0. \quad (\text{V.3})$$

We are now ready to define the strip operators. Given a strip S which is *analytically* embedded in Σ , let us set

$$\hat{T}_S := -i\hbar \sum_{x \in S} X_{\alpha_x, W_x} \quad (\text{V.4})$$

where W_x is any $(d-2)$ plane through x which is transversal to the loop α_x in the strip passing through x and tangent to the strip. Although there is an uncountably infinite number of loops involved in this definition (V.4), the action of \hat{T}_S is nonetheless well-defined on cylindrical functions since, in this action, only a finite number of terms give non-zero contributions. The simplest cylindrical functions are the traces of holonomies. On these, the action of \hat{T}_S reduces simply to:

$$(\hat{T}_S \circ T_\beta)([A]) = -i\hbar \sum_i \text{sgn}_i(S, \beta) [T_{S \circ_i \beta}[A] - T_{S \circ_i \beta^{-1}}[A]], \quad (\text{V.5})$$

where we have used the same notation as in (III.11). This is action that one would have expected on the basis of the Poisson bracket III.11, so that the commutators between \hat{T}_β and \hat{T}_S are the required ones. Finally, using the fact that each vector field $X_{\alpha, W}$ is divergence-free, one can show that \hat{T}_S is essentially self-adjoint. Thus, the representation of these elementary operators does incorporate all the reality conditions.

We will conclude with two remarks.

1. Our strip operators have been directly defined only for analytically embedded strips. Since more general strip functionals were generated by Poisson brackets of the analytically embedded ones, the corresponding operators are obtained by taking commutators between the ‘‘basic’’ strip operators. 2. In the above discussion, we first set $\mathcal{H}_{aux} = L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_0)$, introduced loop and strip operators on it, and argued that the resulting representation of \mathcal{B}_{aux} satisfies the reality conditions. There is in fact a stronger result. One can begin with cylindrical functions on $\overline{\mathcal{A}/\mathcal{G}}$ and define \hat{T}_α and \hat{T}_S as above. Then, μ_0 is the only non-trivial measure on $\overline{\mathcal{A}/\mathcal{G}}$ for which the reality conditions can be satisfied. (The qualification ‘‘non-trivial’’ is necessary because, as was pointed out in Sec. III, the loop-strip variables are complete everywhere except at the flat connections with trivial holonomies and one can introduce another measure which is concentrated just at that point of $\overline{\mathcal{A}/\mathcal{G}}$ which will also incorporate the reality conditions.) Thus, the overall situation is similar to that in ordinary quantum mechanics where the Lebesgue measure is uniquely picked out by the reality conditions once we specify the standard representation, $-i\hbar \vec{\nabla}$ of the momentum operator.

B. Spin networks and the (inverse) loop transform

In this subsection, we recall [20] that \mathcal{H}_{aux} admits a convenient basis and point out the relation between the connection and the loop [23,24] representations.

Let us begin with the notion of ‘‘spin-networks’’ as formulated by Baez [20] (see also Ref. [21, 22]).

The geometrical object called *spin-network* is a triple $(\gamma, \vec{\pi}, \vec{c})$ consisting of

- (i) a graph γ ,
- (ii) a labeling $\vec{\pi} := (\pi_1, \dots, \pi_n)$ of edges e_1, \dots, e_n of that graph γ with irreducible representations π_i of G ,
- (iii) a labeling $\vec{c} = (c_1, \dots, c_m)$ of the vertices v_1, \dots, v_m of γ with *contractors* c_j (see below).

Each contractor c_j is an intertwining operator from the tensor product of the representations corresponding to the incoming edges at a vertex v_j to the tensor product of the representations labeling the outgoing edges. Because the group G is compact, the vector space of all possible contractors c_j associated with a given vector $\vec{\pi}$ and vertex v_j is finite dimensional.

To (i–iii) we add a fourth ‘non-degeneracy’ condition,

(iv) for every edge e the representation π_e is non-trivial and γ is a ‘minimal’ graph in the sense that if another graph γ' occupies the same set of points in Σ , then each edge of γ' is contained in an edge of γ . (Equivalently, γ' can always be built by subdividing the edges of γ , but γ cannot be so built from γ' .)

A *spin-network state* is simply a C^∞ cylindrical function on $\overline{\mathcal{A}/\mathcal{G}}$ (a \mathcal{G} invariant function on $\overline{\mathcal{A}}$) constructed from a spin-network,

$$T_{\gamma, \vec{\pi}, \vec{c}}[A] := \text{tr}[\otimes_{i=1}^n \pi_i(h_{e_i}(A)) \cdot \otimes_{j=1}^m c_j]. \quad (\text{V.6})$$

for all $A \in \overline{\mathcal{A}}$, where, as before, $h_{e_i}(A) = A(e_i)$ is an element of G associated with an edge e_i and ‘ \cdot ’ stands for contracting, at each vertex v_j of γ , the upper indices of the matrices corresponding to all the incoming edges and the lower indices of the matrices assigned to all the outgoing edges with all the indices of c_j .

Using the spin-network states it is easy to construct an orthonormal basis in \mathcal{H}_{aux} . To begin, given a pair $\gamma, \vec{\pi}$, consider the vector space $\mathcal{H}^{\gamma, \vec{\pi}}$ spanned by the spin-network states $T_{\gamma, \vec{\pi}, \vec{c}}$ given by all the possible contractors \vec{c} associated with $\gamma, \vec{\pi}$ as above. Note, that

$$\mathcal{H}_{aux} = \bigoplus_{\gamma, \vec{\pi}} \mathcal{H}_{aux}^{\gamma, \vec{\pi}} \quad (\text{V.7})$$

where $\gamma, \vec{\pi}$ ranges over all the pairs of minimal graphs and labelings by irreducible, non-trivial representations, the sum is orthogonal and the spaces $\mathcal{H}_{aux}^{\gamma, \vec{\pi}}$ are finite dimensional. Thus, we need only choose an orthonormal basis in each $\mathcal{H}_{aux}^{\gamma, \vec{\pi}}$. An explicit construction is given in Ref. [20,22].

We now turn to loop transforms. This discussion will be brief because it is not used in the rest in the rest of the paper. Given any measure μ on $\overline{\mathcal{A}/\mathcal{G}}$ we can perform the integrals

$$\chi(\alpha_1, \dots, \alpha_r) := \int_{\overline{\mathcal{A}/\mathcal{G}}} d\mu_0(A) T_{\alpha_1}(A) \dots T_{\alpha_r}(A) \quad (\text{V.8})$$

to obtain a function of multi-loops. In the case when $G = SU(n)$, Mandelstam identities enable us to express finite products of traces of holonomies in terms of sums of products involving r or less traces where r is the rank of the group. Hence, in the loop representation, we have to deal only with functions of r or less loops. On the other hand, by the Riesz-Markov theorem, any positive linear functional on $C^0(\overline{\mathcal{A}/\mathcal{G}})$ that satisfies the conditions induced by the Mandelstam identities is the loop transform χ of a regular measure supported on $\overline{\mathcal{A}/\mathcal{G}}$. Thus, there is a one to one correspondence between regular measures μ and their characteristic functions χ . This result is analogous to the Bochner theorem that is used in the framework of constructive quantum field theory [32]. In fact, the loop transform can be thought of as a precise analog of the Fourier transform for a quantum field theory with a linear quantum configuration space.

We will now indicate how one can explicitly recover the finite joint distributions of the measure μ from its characteristic functional. (Details will appear elsewhere [22].) This reconstruction of the measure can be regarded as the inverse loop transform. Given a measure μ , choose an orthonormal basis of spin-network states $T_{\gamma, \vec{\pi}, \vec{c}_I}$ and define the associated spin-network characteristic function to be the analog of (V.8), namely

$$\chi(\gamma, \vec{\pi}, \vec{c}_I) := \langle T_{\gamma, \vec{\pi}, \vec{c}_I} \rangle . \quad (\text{V.9})$$

We will say that the characteristic functional is absolutely summable if and only if, for any finitely generated graph γ , the series

$$\sum_{\vec{\pi}} \sum_{\vec{c}_I = \vec{c}_I(\vec{\pi})} |\chi(\gamma, \vec{\pi}, \vec{c}_I)| < \infty \quad (\text{V.10})$$

is absolutely convergent. We can now state the theorem [22] in question

Theorem V.1 *Let the loop transform of a measure be such that the characteristic functional is absolutely summable. Then the associated family of compatible measures on \mathcal{A}_γ is given by:*

$$d\mu_\gamma(g_1, \dots, g_n) = \sum_{\vec{\pi}} \sum_{\vec{c}_I} T_{\vec{\pi}, \vec{c}_I}^*(g_1, \dots, g_n) \times \chi(\gamma, \vec{\pi}, \vec{c}_I) d\mu_H(g_1, \dots, g_n) . \quad (\text{V.11})$$

This is a precise analogue of the inverse Fourier transform in the linear case.

VI. THE HILBERT SPACE OF DIFFEOMORPHISM INVARIANT STATES

Our discussion in sections 4 and 5 has served to introduce and study the auxiliary Hilbert space $\mathcal{H}_{aux} = L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_0)$. As this space carries a \star -representation of the algebra (III.10) defined by the loop and strip operators (\hat{T}_α and \hat{T}_S), we have implemented steps 1-4 of the refined algebraic quantization program (see section II A). In the present section, we will complete the remaining steps (5 and 6) and construct the Hilbert space of diffeomorphism invariant states. For simplicity, we assume throughout this section that the underlying manifold Σ is \mathbb{R}^3 (although the results on \mathbb{R}^n are identical).

A key step in our construction will involve an appropriate averaging of spin-network states over the diffeomorphism group. This averaging procedure was considered, independently, by John Baez [36] as a tool for constructing a rich variety of diffeomorphism invariant measures on $\overline{\mathcal{A}/\mathcal{G}}$.

A. Formulation of the diffeomorphism constraint

Recall that the diffeomorphism constraint is given by:

$$V_a(x) := \text{tr}[F_{ab}(x) \tilde{E}^b(x)] = 0 . \quad (\text{VI.1})$$

Let us consider the smeared version of this constraint,

$$V_N := \int_{\mathbb{R}^3} N^a(x) V_a(x) d^3x = 0 , \quad (\text{VI.2})$$

where N^a are complete analytic vector fields on Σ . (We require analyticity because the edges of our graphs are assumed to be analytic. See Sec. IV and V.) Denote by φ_t the 1-parameter family

of diffeomorphisms generated by N^a on Σ . Now, as shown in Appendix C, V_N has a natural action on the space of smooth functions on \mathcal{A}/\mathcal{G} which can be used to define a 1-parameter family $U(t)$ of unitary operators on \mathcal{H}_{aux} , providing us a faithful, unitary representation of the group φ_t . On spin network states, the action of the operator U_φ corresponding to φ is given by:

$$U_\varphi \circ (T_{\alpha, \vec{\pi}, \vec{c}}) = T_{\alpha, \vec{\pi}, \vec{c}} \circ \varphi = T_{\varphi\alpha, \varphi\vec{\pi}, \varphi\vec{c}}, \quad (\text{VI.3})$$

where $\varphi\alpha$ is the image of the graph α under the analytic diffeomorphism and $\varphi\vec{\pi}$ and $\varphi\vec{c}$ are the corresponding vector of representations and contraction associated with the new graph $\varphi\alpha$.

Thus, as needed in the group averaging procedure, each constraint V_N is promoted to a 1-parameter family of unitary operators. Varying N^a , we obtain, on \mathcal{H}_{aux} , a unitary representation of the group of diffeomorphisms on Σ generated by complete analytic vector fields. Thus, there are no anomalies. Note that this is *not* a formal argument; the operators $U(t)$ corresponding to V_N are rigorously defined on a proper Hilbert space, and they are unitary because the measure μ_0 is diffeomorphism invariant.

Note that U_φ preserves the space $\text{Cyl}^\infty(\overline{\mathcal{A}/\mathcal{G}})$ of smooth cylindrical functions. Since $\text{Cyl}^\infty(\overline{\mathcal{A}/\mathcal{G}})$ is also preserved by our algebra of elementary quantum operators (generated by \hat{T}_α and \hat{T}_S), it is natural to take $\text{Cyl}^\infty(\overline{\mathcal{A}/\mathcal{G}})$ to be the dense subspace $\Phi \subset \mathcal{H}_{aux}$ of step 5'b of the refined algebraic quantization program. Finally, we need to specify a topology on Φ . Finite dimensional examples suggest that we let one of the standard nuclear topologies of the $C^\infty(\mathcal{A}_\gamma) \cong C^\infty(SU^n(2))$ induce the required topology on $\text{Cyl}^\infty(\overline{\mathcal{A}/\mathcal{G}})$.

We will seek ‘solutions of the constraints’ in the topological dual Φ' , the space of cylindrical distributions. Diffeomorphisms have a natural action on $\overline{\phi} \in \Phi'$ by duality and we will say that $\overline{\phi} \in \Phi'$ is a solution of the diffeomorphism constraints if

$$\overline{\phi}(U_\varphi \circ \phi) = \overline{\phi}(\phi) \quad \text{for all } \varphi \in \text{Diff}(\Sigma) \text{ and } \phi \in \Phi. \quad (\text{VI.4})$$

Many such distributions exist. For example, given any spin-network state $|\alpha, \vec{\pi}, \vec{c}\rangle$ we may define a distribution $\mu_{\alpha, \vec{\pi}, \vec{c}}$ through its action on any $\phi \in \text{Cyl}^\infty(\overline{\mathcal{A}/\mathcal{G}})$:

$$\mu_{\alpha, \vec{\pi}, \vec{c}}[\phi] := \sum_{|\alpha_2, \vec{\pi}_2, \vec{c}_2\rangle \in [|\alpha, \vec{\pi}, \vec{c}\rangle]} \langle \alpha_2 \vec{\pi}_2, \vec{c}_2 | \phi \rangle \quad (\text{VI.5})$$

where $[|\alpha, \vec{\pi}, \vec{c}\rangle]$ is the set of all spin-network states $|\alpha_2, \vec{\pi}_2, \vec{c}_2\rangle$ such that $U_\varphi |\alpha, \vec{\pi}, \vec{c}\rangle = |\alpha_2, \vec{\pi}_2, \vec{c}_2\rangle$ for

some $\varphi \in \text{Diff}(\Sigma)$. To see that this sum converges and $\mu_{\alpha, \vec{\pi}, \vec{c}}$ is a well defined element of Φ' , write ϕ as

$$\phi = \sum_{\gamma', \vec{\pi}'} f_{\gamma', \vec{\pi}'} \quad (\text{VI.6})$$

where $f_{\gamma', \vec{\pi}'}$ is the orthogonal projection of ϕ onto the space $\mathcal{H}_{aux}^{\gamma', \vec{\pi}'}$. The sum ranges over all the vector spaces of the orthogonal decomposition, however, since ϕ is cylindrical, there are contributions only for $\gamma' \leq \gamma$ for some graph γ . Substitute (VI.6) into (VI.5). On the right hand side, the products vanish unless $(\alpha_2, \vec{\pi}_2) = (\gamma', \vec{\pi}')$. Thus, there are only a finite number of nonzero terms; the right hand side of (VI.6) is finite and defines an element of Φ' . Note that, heuristically, we have invoked the idea of group averaging to construct these distributions, using a discrete measure on the orbit of $|\alpha, \pi, c\rangle$ under $\text{Diff}(\Sigma)$.

B. The issue of independent sectors

Having identified a suitable dense subspace $\Phi \subset \mathcal{H}_{aux}$ and having seen that its topological dual Φ' is large enough to contain diffeomorphism invariant distributions, we now wish to construct a map $\eta : \Phi \rightarrow \Phi'$ that completes step 5'c in our program. This will, however, be more complicated than for the examples in Sec.II due to the fact that each state $|\phi\rangle \in \Phi$ has an infinite ‘isotropy group’ of diffeomorphisms that leave $|\phi\rangle$ invariant. Thus, the sum in (VI.5) was not over the entire diffeomorphism group, but only over the *orbit* of the state $|\alpha, \pi, c\rangle \in \Phi$.

Because the sum in (VI.5) itself depends on the state $|\alpha, \vec{\pi}, \vec{c}\rangle$, our definition of the inner product on \mathcal{V}_{diff} will have to take into account the fact that the orbit size is state-dependent. While the infinite size of the orbits would appear to make this difficult, a simplification will occur as the presence of ‘infinitely different’ isotropy groups will imply that $L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_0)$ carries a *reducible* representation of the algebra of observables. In fact, we show below that \mathcal{H}_{aux} can be written as a direct sum of subspaces such that, on each subspace, the sizes of orbits are ‘comparable’. This will allow us to give a well defined averaging procedure by treating each such subspace separately in section VI C. A similar situation is discussed in appendix A.

In order to classify these isotropy groups, let us consider for each spin-network state $|\alpha, \vec{\pi}, \vec{c}\rangle$ the collection \mathcal{E}_α of analytic edges of the graph α . For technical reasons, we shall focus on graphs for which, given any edge $e \in \mathcal{E}_\alpha$, there is an analytic

real function f which vanishes on the maximal analytic curve \tilde{e} that extends e , but nowhere else. We shall call such graphs (and their associated curves) ‘type I’, while all others are ‘type II.’ Note that the collection of \tilde{e} defined by the type I graph α intersect at most a countable number of times and so define a graph $\tilde{\alpha}$ with countably many edges.

Now, given any n type I maximal analytic curves (i.e., curves which cannot be analytically extended) in \mathbb{R}^3 and any distinct maximal analytic curve \tilde{e} (not necessarily of type I), there is a multi-parameter family of analytic diffeomorphisms that preserves the n type I curves but does not preserve \tilde{e} . To see this, begin with any constant vector field X_0 on \mathbb{R}^3 which is not everywhere tangent to \tilde{e} on \tilde{e} . Let f_i be the real analytic function that vanishes exactly on the i th maximal type I curve. Then the product f of the f_i is a real analytic function that vanishes exactly on the union of these curves. Thus, the complete analytic vector field $X = fe^{-f^2}X_0$ exponentiates to a one parameter family of analytic diffeomorphisms that preserves the n maximal type I curves, but does not preserve \tilde{e} .

Thus, for two spin-network states $|\alpha_1, \vec{\pi}_1, \vec{c}_1\rangle$ and $|\alpha_2, \vec{\pi}_2, \vec{c}_2\rangle$, (with α_1 of type I) either $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are identical or there are infinitely many diffeomorphisms φ which preserve one of these (say, $\tilde{\alpha}_1$) but move the other (say, $\tilde{\alpha}_2$).

As in section II, we consider the algebra $\mathcal{B}_{phys}^{(*)}$ of operators A on \mathcal{H}_{aux} that *i*) are defined on Φ and map Φ into itself, *ii*) have adjoints A^\dagger defined on Φ which map Φ into itself, and *iii*) commute with the action of all diffeomorphisms φ . (Note that the last condition implies that A^\dagger also commute with constraints.) Let $|\phi_1\rangle = |\alpha_1, \vec{\pi}_1, \vec{c}_1\rangle$ and $|\phi_2\rangle = |\alpha_2, \vec{\pi}_2, \vec{c}_2\rangle$ be the spin-network states above, so that there are infinitely many diffeomorphisms φ which move $\tilde{\alpha}_2$ but move no edge of $\tilde{\alpha}_1$. Then, for such a φ , the matrix elements $\langle\phi_1|A|\phi_2\rangle_{aux}$ of any $A \in \mathcal{B}_{phys}^{(*)}$ must satisfy

$$\langle\phi_1|A|\phi_2\rangle_{aux} = \langle\phi_1|\varphi A|\phi_2\rangle_{aux} = \langle\phi_1|A\varphi|\phi_2\rangle_{aux} \quad (\text{VI.7})$$

while $\langle\phi_2|\varphi|\phi_2\rangle_{aux} = 0$. Thus, either $\langle\phi_1|A|\phi_2\rangle_{aux} = 0$ or the vector $A^\dagger|\phi_1\rangle$ has an infinite number of equal components. However, $|\phi_1\rangle \in \Phi$ lies in the domain of A^\dagger so that $A^\dagger|\phi_1\rangle$ is normalizable, whence $\langle\phi_1|A|\phi_2\rangle_{aux}$ must vanish. Since the adjoint of A is also in $\mathcal{B}_{phys}^{(*)}$, $\langle\phi_2|A|\phi_1\rangle_{aux}$ vanishes as well. We thus have a ‘super-selection rule’ between states associated with the graphs $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ and the representation of $\mathcal{B}_{phys}^{(*)}$ on \mathcal{H}_{aux} is reducible.

Note in particular that we have a superselection rule between states associated with graphs of type I and type II. We may thus decompose the representation of $\mathcal{B}_{phys}^{(*)}$ on $L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_0)$ as $\mathcal{H}_{aux} = \mathcal{H}_{aux}^I \oplus \mathcal{H}_{aux}^{II}$. Because the subspace \mathcal{H}_{aux}^I is technically much simpler, we will focus on this sector and ignore \mathcal{H}_{aux}^{II} in what follows. (This division into two types would be unnecessary if we could replace the analytic loops by smooth ones in the beginning of our construction. See Sec. VII.)

Our discussion above implies that \mathcal{H}_{aux}^I is in fact a direct sum of representations, each acting in a subspace $\mathcal{H}_{aux}^{\tilde{\alpha}}$ associated with a given (maximally analytically extended) graph $\tilde{\alpha}$. On the other hand, a diffeomorphism φ will map one such subspace $\mathcal{H}_{aux}^{\tilde{\alpha}}$ to another ($\mathcal{H}_{aux}^{\varphi\tilde{\alpha}}$). It is therefore convenient to consider the class $[\tilde{\alpha}]$ of all maximally extended analytic graphs $\tilde{\alpha}_1$ which can be mapped onto $\tilde{\alpha}$ by an analytic diffeomorphism. This gives a decomposition of \mathcal{H}_{aux} through

$$\mathcal{H}_{aux}^I = \bigoplus_{[\tilde{\alpha}]} \mathcal{H}_{aux}^{[\tilde{\alpha}]}, \quad (\text{VI.8})$$

where

$$\mathcal{H}_{aux}^{[\tilde{\alpha}]} = \bigoplus_{\tilde{\alpha}_1 \in [\tilde{\alpha}]} \mathcal{H}_{aux}^{\tilde{\alpha}_1} \quad (\text{VI.9})$$

where both direct sums are implicitly over only graphs of type I. The sectors $\mathcal{H}_{aux}^{[\tilde{\alpha}]}$ are truly independent in the sense that they are not mixed by any physical operators $A \in \mathcal{B}_{phys}^{(*)}$ or any diffeomorphism φ . Thus, from now on, we will treat each $\mathcal{H}_{aux}^{[\tilde{\alpha}]}$ individually.

C. A Family of Maps

We now wish to implement step 5’ of the program separately within each ‘independent sector’ $\mathcal{H}_{aux}^{[\tilde{\alpha}]}$ of $L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_0)$, associated with the class of maximally analytically extended graphs $\tilde{\alpha}_1$ which are mapped onto $\tilde{\alpha}$ by analytic diffeomorphisms. Thus, for each $\mathcal{H}_{aux}^{[\tilde{\alpha}]}$, we introduce the dense subspace $\Phi^{[\tilde{\alpha}]} \subset \mathcal{H}_{aux}^{[\tilde{\alpha}]}$ of functions that are (C^∞) cylindrical over graphs associated with $[\tilde{\alpha}]$ and the corresponding topological dual $\Phi'^{[\tilde{\alpha}]}$. We will identify a vector space $\mathcal{V}_{diff}^{[\tilde{\alpha}]}$ and impose an inner product to define a Hilbert space $\mathcal{H}_{diff}^{[\tilde{\alpha}]}$ of diffeomorphism invariant states. As before, $\mathcal{V}_{diff}^{[\tilde{\alpha}]}$ will be the image of a family of maps $\eta_a^{[\tilde{\alpha}]} : \Phi^{[\tilde{\alpha}]} \rightarrow \Phi'^{[\tilde{\alpha}]}$ and

will contain only diffeomorphism invariant distributions. (Here, $a = a_{[\tilde{\alpha}]} \in \mathbb{R}^+$. For simplicity of notation, we will not make the dependence of a on $[\tilde{\alpha}]$ explicit.)

To construct the map $\eta_a^{[\tilde{\alpha}]}$, let us first give its action on functions $|f\rangle \in \bigoplus_{\tilde{\pi}} \mathcal{H}_{aux}^{\gamma, \tilde{\pi}}$ associated with some fixed graph γ with $\tilde{\gamma} \in [\tilde{\alpha}]$. The action of $\eta_a^{[\tilde{\alpha}]}$ on general states $|\phi\rangle \in \Phi$ then follows by (finite) anti-linearity. To construct this map, we will need to consider the ‘isotropy’ group $Iso(\tilde{\gamma})$ of diffeomorphisms that map $\tilde{\gamma}$ to itself and the ‘trivial action’ subgroup $TA(\tilde{\gamma})$ of $Iso(\tilde{\gamma})$ which preserves each edge of $\tilde{\gamma}$ separately. We will also need the quotient group $GS(\tilde{\gamma}) = Iso(\tilde{\gamma})/TA(\tilde{\gamma})$ of ‘graph symmetries’ of $\tilde{\gamma}$ and some set $S(\tilde{\gamma})$ of analytic diffeomorphisms which has the property that, for all $\tilde{\beta} \in [\tilde{\gamma}]$ there is exactly one $\varphi \in S(\tilde{\gamma})$ that maps $\tilde{\gamma}$ to $\tilde{\beta}$. The appropriate maps are then given by

$$\eta_a^{[\tilde{\alpha}]}|f\rangle = a \left(\sum_{\varphi_1 \in S(\tilde{\gamma})} \sum_{[\varphi_2] \in GS(\tilde{\gamma})} \varphi_1 \varphi_2 |f\rangle \right)^\dagger \quad (\text{VI.10})$$

where, in the second sum, φ_2 is any diffeomorphism in the equivalence class $[\varphi_2]$. For the reader who feels that this definition has been ‘pulled out of a hat,’ we will provide a heuristic ‘derivation’ below in section VID by ‘renormalizing’ the map given by naive group averaging.

In order to show that $\eta_a^{[\tilde{\alpha}]}|f\rangle$ does in fact define an element of Φ' , note that its action on any state $|g\rangle \in \bigoplus_{\tilde{\pi}} \mathcal{H}_{aux}^{\beta, \tilde{\pi}}$ is given by

$$(\eta_a^{[\tilde{\alpha}]}|f\rangle)[|g\rangle] = \delta_{[\tilde{\gamma}], [\tilde{\beta}]} \sum_{\varphi_2 \in GS(\tilde{\gamma})} \langle f | \varphi_2 \varphi_0 | g \rangle \quad (\text{VI.11})$$

where φ_0 is any diffeomorphism that maps $\tilde{\gamma}$ to $\tilde{\beta}$. Because $\tilde{\gamma}$ may have an infinite number of edges, $GS(\tilde{\gamma})$ may be infinite as well. Nonetheless, we will now show that the above sum contains only a finite number of nonzero terms.

First, note that if there are any nonzero terms at all, we may take $\varphi_0 \gamma = \beta$ without loss of generality. In this case, a term in (VI.11) is nonzero only if the associated φ_2 preserves that graph γ . The key point is to note that, since $\tilde{\gamma}$ may be constructed by analytically extending the edges of the graph γ , the action of any analytic diffeomorphism on the edges of γ determines the action of this diffeomorphism on every edge in the *extended* graph $\tilde{\gamma}$. Thus, the diffeomorphisms $\varphi_2 \in S(\tilde{\gamma})$ that preserve $\tilde{\gamma}$ must rearrange the edges of γ in distinct ways. Since γ contains only a finite number of edges, it follows that there can be at most a finite set of diffeomorphisms φ_2 in $GS(\tilde{\gamma})$ that preserve γ . There are

thus only finitely many terms in (VI.11). The fact that (VI.10) defines an element of Φ' then follows by (finite) linearity.

The space $\mathcal{V}_{diff}^{[\tilde{\alpha}]}$ is then defined to be the image of $\eta_a^{[\tilde{\alpha}]}$. It is clear from the form of the sum (VI.11) that $\eta_a^{[\tilde{\alpha}]}$ is real and positive so that the inner product (VI.11) is well-defined, Hermitian, and positive definite. We may therefore complete each $\mathcal{V}_{diff}^{[\tilde{\alpha}]}$ to define a Hilbert space $\mathcal{H}_{diff}^{[\tilde{\alpha}]}$.

Furthermore, $\eta_a^{[\tilde{\alpha}]}$ commutes with all A in $\mathcal{B}_{phys}^{(*)}$ in the sense that

$$\begin{aligned} (\eta_a^{[\tilde{\alpha}]} \phi_1)[A \phi_2] &= \sum_{\varphi \in GS(\tilde{\alpha})} \langle \phi_1 | \varphi A | \phi_2 \rangle \\ &= (\eta_a^{[\tilde{\alpha}]}(A^\dagger \phi_1))[\phi_2]. \end{aligned} \quad (\text{VI.12})$$

(Here, without loss of generality, we take ϕ_1, ϕ_2 cylindrical over $\tilde{\alpha}$.) It follows that the map $A \mapsto A_{phys}$ (where $A_{phys}(\eta\phi) = \eta(A\phi)$) defines an anti \star -representation of $\mathcal{B}_{phys}^{(*)}$ on $\mathcal{H}_{diff}^{[\tilde{\alpha}]}$. Thus, the ‘reality conditions’ on physical observables have been incorporated.

D. Some final Heuristics

For those who are interested, we now present a short heuristic ‘derivation’ of (VI.11) in which we first average over the entire group of diffeomorphisms (in analogy with [13,14] and section IIB) and then ‘renormalize’ the resulting distribution by canceling (infinite) volumes of isotropy groups. Because a sum of the form $\sum_{\varphi \in \text{Diff}(\Sigma)} \varphi |\phi\rangle$ diverges (even as an element of Φ'), we attempt to remove this divergence by comparing the inner product of two distributions $\bar{\psi}$ and $\bar{\psi}$ in $\mathcal{V}_{diff}^{[\tilde{\alpha}]}$ with the norm of some reference distribution $\bar{\rho}$ which lies in the same vector space $\mathcal{V}_{diff}^{[\tilde{\alpha}]}$. Let us suppose that these ‘heuristic distributions’ are obtained by averaging $|\phi\rangle, |\psi\rangle$, and $|\rho\rangle \in \Phi^{[\tilde{\alpha}]}$ over the diffeomorphism group. For convenience, we will also fix some particular extended analytic graph $\tilde{\alpha}$ and assume that $|\phi\rangle, |\psi\rangle$, and $|\rho\rangle$ lie in $\mathcal{H}_{aux}^{\tilde{\alpha}}$. Then, the *ratio* of the inner product of $\bar{\phi}$ and $\bar{\psi}$ to the norm of $\bar{\rho}$ is

$$\frac{\langle \bar{\phi} | \bar{\psi} \rangle_{diff}}{\langle \bar{\rho} | \bar{\rho} \rangle_{diff}} = \frac{\sum_{\varphi \in \text{Diff}(\Sigma)} \langle \psi | \varphi | \phi \rangle}{\sum_{\varphi \in \text{Diff}(\Sigma)} \langle \rho | \varphi | \rho \rangle} \quad (\text{VI.13})$$

so that a given diffeomorphism φ contributes to this sum only if it preserves $\tilde{\alpha}$. That is, we need only sum over the isotropy group $ISO(\tilde{\alpha})$. Note that we may rewrite the sums over diffeomorphisms in (VI.13) as sums over the cosets $GS(\tilde{\alpha})$

(which give a finite result by the above discussion) multiplied by the (infinite) size of the trivial action subgroup $TA(\tilde{\alpha})$. Formally canceling these infinite factors in the numerator and denominator, we arrive at

$$\frac{\langle \bar{\phi} | \bar{\psi} \rangle_{diff}}{\langle \bar{\rho} | \bar{\rho} \rangle_{diff}} = \frac{\sum_{[\varphi] \in GS(\tilde{\alpha})} \langle \psi | \varphi | \phi \rangle}{\sum_{[\varphi] \in GS(\tilde{\alpha})} \langle \rho | \varphi | \rho \rangle}, \quad (\text{VI.14})$$

where the sum is over the equivalence classes in $GS(\tilde{\alpha})$ and φ is an arbitrary representative of $[\varphi]$. This motivates the definition (VI.10) of the maps $\eta_a^{[\tilde{\alpha}]}$ and the inner product (VI.11).

E. Subtleties

We have seen that the Hilbert space \mathcal{H}_{diff} that results from solving the (Gauss and the) diffeomorphism constraints can be decomposed as a direct sum of Hilbert spaces $\mathcal{H}_{diff}^{[\tilde{\alpha}]}$, each of which carries a representation of the algebra $\mathcal{B}_{phys}^{(*)}$ of physical operators. Thus, elements of $\mathcal{B}_{phys}^{(*)}$ —observables which strongly commute with constraints—do not mix states from distinct Hilbert spaces that feature in the direct sum. Recall, however, that the physical observables have to commute with constraints only weakly and there may well exist weak observables which connect distinct Hilbert spaces. From a physical viewpoint, therefore, we need to focus on the irreducible representations of the algebra of weak observables. If there are no further constraints, (as in the Husain-Kuchař model), these irreducible sectors are properly thought of as separate and, in the standard jargon, *super-selected*. Since by assumption the Hamiltonian operator commutes with all constraints, dynamics will leave each sector invariant. Indeed, no physical observable can map one out of a superselected sector. Thus, a physical realization of the system will involve only one such sector and just which sector arises must be determined by experiment. Unfortunately, as the matter stands, we do not have a manageable characterization of these sectors because we focussed only on strong observables. (In general, weak observables do not satisfy (II.4).)

If the diffeomorphism group represents only a sub-group of the full gauge group (as in the case of general relativity), then there can be a further complication and the situation becomes quite subtle. On the one hand, because we have more constraints, one expects there to be fewer observables. On the other hand, commutator of the an operator with the diffeomorphism constraints may be equal

to one of the new constraints. Then, while the operator would not be an observable of the partial theory that ignores the additional constraints, it *would* be an observable of the full theory. Curiously, this is precisely what happens in the case of 3-dimensional, Riemannian general relativity (i.e., the $ISU(2)$ Chern-Simons theory). The Wilson loop operators \hat{T}_α fail to be weak observables if we consider only the diffeomorphism constraint but they *are* weak observables of the full theory. Furthermore, they mix the independent sectors which are super-selected with respect to diffeomorphisms. We expect that the situation will be similar in 4-dimensional general relativity. Thus, we expect that the physical states of this theory will not be confined to lie in just one $\mathcal{H}_{diff}^{[\tilde{\alpha}]}$; as far as general relativity is concerned, one should not think of these sectors as being *physically* super-selected.

Finally, note that we have asked that the physical states be invariant only under diffeomorphisms generated by vector fields. Large diffeomorphisms are unitarily implemented in the physical Hilbert space; they are symmetries of the theory but not gauge. One *may* wish to treat them as gauge and ask that the “true” physical states be invariant under them as well. If so, one can again apply the group averaging procedure, now treating the modular group as the gauge group. In the case of 3-dimensional Riemannian general relativity on a torus, for example, this procedure is successful and yields a Hilbert space of states that are invariant under all diffeomorphisms.

VII. DISCUSSION

In this paper, we have presented solutions to the Gauss and the diffeomorphism constraints for a large class of theories of connections. The reader may be concerned that we did not apply the quantization program of Sec. II to the Gauss constraint but instead solved it classically. However, we chose this avenue only for brevity; it is straightforward to first use the program to solve the Gauss constraint and then face the diffeomorphism constraint. In this alternate approach, one begins with the space $\overline{\mathcal{A}}$ of generalized connections (see section 4.2) as the classical configuration space and lets the auxiliary Hilbert space be $L^2(\overline{\mathcal{A}}, d\mu'_0)$, where μ'_0 is the induced Haar measure on $\overline{\mathcal{A}}$ (see Ref. [7,12]). Next, one introduces the Gauss constraints as operators on the new auxiliary Hilbert space. The resulting unitary operators just implement the action of the group $\overline{\mathcal{G}}$ of generalized gauge transformations on the Hilbert space. Since $\overline{\mathcal{G}}$ is compact,

the resulting group averaging procedure is straightforward and leads us to $L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_0)$ as the space of physical states with respect to the Gauss constraints. One is now ready to use Sec. VI to implement the diffeomorphism constraints.

The final picture that emerges from our results can be summarized as follows. To begin with, we have the auxiliary Hilbert space \mathcal{H}_{aux} . While it does not appear in the final solution, it does serve three important purposes. First, it ensures that real, elementary functions on the classical phase space are represented by self-adjoint operators, so that the “kinematical reality conditions” on the full phase space are incorporated in the quantum theory. Second, it enables us to promote constraints to *well-defined* operators thereby making the analysis of potential anomalies mathematically sound. Finally the space Φ , whose topological dual Φ' is the “home” of physical quantum states, is extracted as a dense sub-space of \mathcal{H}_{aux} . The physical states $\bar{\phi} \in \Phi'$ are obtained by “averaging” states $\phi \in \Phi$ over the orbits of the diffeomorphism group appropriately. Care is needed because the orbits themselves have an infinite volume and because, in general, different orbits have different isotropy groups. These features lead to diff-superselected sectors. Each sector is labeled by the diffeomorphism class $[\tilde{\alpha}]$ of “maximally extended” (type I) graphs $\tilde{\alpha}$. Operators on \mathcal{H}_{aux} which leave Φ invariant have an induced action on the topological dual, Φ' of Φ . If they commute with the diffeomorphism operators on \mathcal{H}_{aux} , they descend to the space \mathcal{V}_{diff} of (diff-)physical states. The sectors are diff-superselected in the sense that each of them is left invariant by operators on Φ' which descend from observables –i.e., self-adjoint operators which commute with the diffeomorphism operators– on \mathcal{H}_{aux} . The induced scalar product on \mathcal{V}_{diff} is unique up to an overall multiplicative constant on each diff-superselected sector. It automatically incorporates the physical reality conditions. (The ambiguity of multiplicative constants would be reduced if there exist weak observables which mix these sectors which are superselected by strong observables.)

How does this situation compare to the one in the general algebraic quantization program of Ref. [17,18]? In the final picture, the inner product is determined by the reality conditions. However, the group averaging strategy enables one to find this inner product *without* having to find the physical observables explicitly; the inner product on \mathcal{H}_{aux} which incorporates the kinematical reality conditions on the full phase space descends to *vp*. This is an enormous technical simplification. On the

conceptual side, there are now four inputs into the program: choice of a set of elementary functions (labeled by loops and strips in our case), of a representation of the corresponding algebra (on $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$ in our case), of expressions of the regularized constraint operators (which, in our case, implement the natural action of the diffeomorphism group on \mathcal{H}_{aux}), and of the subspace Φ ($\text{Cyl}^\infty(\overline{\mathcal{A}/\mathcal{G}})$ in our case). We have shown that the choices we made are viable and quantization can be completed. There may of course be other, inequivalent quantum theories, which correspond to different choices. Indeed, even in Minkowski space, a classical *field* theory can be quantized in inequivalent ways. We expect, however, that there exists an appropriate uniqueness theorem which singles out our solution, analogous to the theorem that singles out the Fock representation for free field theories.

What are the implications of these results to the specific models discussed in Sec. III? For the Husain-Kuchař model, we have complete solutions. For Riemannian general relativity, on the other hand, we have only a partial result since the Hamiltonian constraint is yet to be incorporated. However, our analysis does provide a natural strategy to complete the quantization. For, we already have indications that the projective methods can be used also to regulate the Hamiltonian constraint operator *on diffeomorphism invariant states*. If this step can be completed, one would check for anomalies. If there are none, one would again apply the group averaging procedure to find solutions. This task may even be simpler now because, given the structure of the classical constraint algebra, one would expect the Hamiltonian constraints to *commute* on diffeomorphism invariant states. The procedure outlined in Appendix A would then lead to the physical Hilbert space for the full theory. As indicated in Sec. VI E, however, subtleties will arise because of the observables which commute with the constraints only weakly and the final Hilbert space is likely to contain elements from *different* diff-superselected sectors. Furthermore, to extract “physical” predictions, one would almost certainly have to develop suitable approximation schemes. However, this task would be simplified considerably if we already know that a consistent quantum theory exists. Indeed, in this respect, the situation would be comparable to the one currently encountered in atomic and molecular physics where approximations schemes are essential in practice but the knowledge that the exact Hamiltonian exists as a well-defined self-adjoint operator goes a long way

in providing both confidence in and guidelines for these approximations.

For Lorentzian general relativity, one can begin with the formulation in which the spin connection is the configuration variable. For this case, the results of this paper again lead to a complete solution to the Gauss and the diffeomorphism constraints. Unfortunately, as mentioned in the Introduction, the Hamiltonian constraint is unmanageable in these variables and the best strategy is to perform a transformation and work with *self-dual* connections [2]. Classically, the required canonical transformation is well-understood. Its quantum analog is an appropriate “coherent state transform” which would map complex-valued functions of spin connections to holomorphic functions of the self-dual connections. Such a transform is already available [3] and it seems fairly straightforward to carry over our treatment of the diffeomorphism constraint to the holomorphic representation. However, it is far from being obvious that the Hamiltonian constraint can be treated so easily in the holomorphic representation. Another strategy is to begin with the Riemannian model, obtain physical states and *then* pass to the holomorphic representation via an appropriate generalization of the Wick rotation procedure. Thus, whereas in the Riemannian case, results of this paper provide a clear avenue, in the Lorentzian case, new inputs are needed. Work is in progress along the two lines indicated above.

The canonical approach to quantum gravity is quite old; foundations of the geometrodynamical framework were laid by Dirac and Bergmann already in the late fifties. The precise mathematical structure of the classical configuration and phase spaces became clear in the seventies. However, these analyses dealt only with smooth fields while, as is well-known, in quantum field theory one has to go beyond such configurations. The required extensions are non-trivial and are, in fact, yet to be carried out in the metric representation. Consequently, in the traditional geometrodynamical approach, the formulation and imposition of quantum constraints have remained at a formal level *even for the diffeomorphism constraint*. We have seen that the situation changes dramatically if one shifts the emphasis and works with connections. (Note that these can be $SU(2)$ spin connections; they don't have to be self-dual. Since the spin connection is completely determined by the triads, the corresponding representation provides an alternative framework to solve the quantum Gauss and diffeomorphism constraints of the triad geometrodynamics.) Now, problems of quantum field theory

can be faced directly and the general level of mathematical precision is comparable to that encountered in rigorous quantum field theory. Finally, note that this became possible *only* because of the availability of a calculus on the quantum configuration space which does not refer to a background field such as a metric. Thus, the projective techniques summarized in Sec. IV are not a luxury; they are essential if one wants to ensure that inner products and operators are well-defined in the quantum theory.

Most of theoretical physics, however, does not require such a high degree of precision. Why, then, is so much care necessary here? The main reason is that we have very little experience with non-perturbative techniques. We have already seen that the perturbative strategy, which is so successful in theories of other forces of Nature, fails in the case of gravity. Hence, if one wishes to pursue a new approach, it is important to have an assurance that the quantum theory we are dealing with is internally consistent and that the problems that arise in perturbative treatments are not just swept under a rug. An obvious way to achieve certainty is to work at a high level of mathematical precision.

The mathematical framework could, however, be improved in two directions. First, the functional calculus we used is based, in an essential way, on the assumption that all edges of our graphs are analytic. If we weaken this assumption and allow edges which are only C^∞ , a number of technical problems can arise since, for example, two C^∞ curves can have an infinite number of intersections in a finite interval. On physical grounds, on the other hand, smoothness seems more appropriate than analyticity and it would be desirable to extend this framework accordingly. Furthermore, if we could work with smooth loops, the discussion of the “independent sectors” in Sec. VI would simplify considerably; it would not be necessary to divide the spin networks into types. The second improvement would be more substantial. The present mathematical framework is based on the assumption that traces of holonomies should become well-defined operators on the auxiliary Hilbert space. Once this assumption is made, one is naturally led to regard $\overline{\mathcal{A}/\mathcal{G}}$ as the quantum configuration space and use on it the calculus that is induced by the projective techniques. The assumption is not unreasonable for a diffeomorphism invariant theory and has led to a rich structure which, as we saw, is directly useful in a number of models. (The framework has also been used to find new results in 2-dimensional Yang-Mills theory [37] which happens to be invariant under all volume preserving diffeo-

morphisms.) However, it is quite possible that, ultimately, the assumption will have to be weakened. To do so, we may need to feed more information about the underlying manifold into the quantum configuration space. Our present construction does capture a part of the manifold structure through its use of analytic graphs and also has some topological information, e.g., of the first homotopy group of the manifold. However, it does not use the notion of convergence of a sequence of graphs which knows much more about the topology of the underlying manifold. In the language of projective techniques (see Appendix B), it would be desirable to use the underlying manifold to introduce a topology on the label set and see how it influences the rest of the construction. These issues are currently being investigated.

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APPENDIX A: SUPER-SELECTION RULES FOR ABELIAN CONSTRAINTS

To illustrate the quantization program, we discussed a number of simple examples in section II B. To bring out some subtleties associated with the group averaging procedure, in this Appendix, we will consider a somewhat more general situation which, however, is simpler than the one considered in Sec. VI.

In section II B, the group generated by the quantum constraints was Abelian and was represented by unitary operators $U(g)$ in a Hilbert space \mathcal{H}_{aux} . The definition of the physical inner product involved a map η from a space Φ of test functions to its topological dual Φ' which was defined by integrating over the volume of the gauge group, $\eta|f\rangle = (\int dg U(g)|f\rangle)^\dagger$. As such, it is clearly important that no infinite subgroup should leave $|f\rangle$ invariant so that the integral does not diverge. Thus, it is natural to ask if this method can be suitably modified to incorporate the case when some $U(g)$

have eigenstates in \mathcal{H}_{aux} with eigenvalue 1. In this Appendix, we will analyze this issue in the general setting of Abelian constraints and show that the answer is always ‘yes,’ though the procedure is somewhat more subtle.

Recall that our intent is to construct an irreducible representation of a \star -algebra of *physical* operators and that we suppose this algebra to be represented on \mathcal{H}_{aux} . At least when this algebra is generated by bounded operators, we will see that the representation on \mathcal{H}_{aux} is reducible whenever 1 is a part of the discrete as well as the continuous spectrum of some $U(g)$.

Suppose that the representation of the gauge group is generated by some set U_i of unitary operators for i in some label set I . Denote by $\mathcal{S}_i^{(d)}$ the subspace of \mathcal{H}_{aux} which is left invariant by U_i , i.e., the space of eigenvectors with discrete eigenvalue 1. Since $\{1\}$ is a set of zero measure in \mathbb{R} , any state in \mathcal{H}_{aux} which is orthogonal to \mathcal{S}_i^1 can be built from spectral subspaces of U_i with eigenvalue $\neq 1$. Now, solutions to the constraints in Φ' are of two types. First, each element of \mathcal{S}_i^d , regarded as an element of Φ' , is a solution. Second, there is a subspace \mathcal{S}_i^c obtained by group-averaging elements of Φ which are orthogonal to \mathcal{S}_i^d . These two subspaces of physical states are orthogonal to each other. Consider now a bounded operator A which commutes with each U_i . It is straightforward to check that the action of A preserves each of the two orthogonal subspaces; the action of A on Φ' does not mix the discrete and continuous eigenvalue 1 distributions of U_i in Φ' .

We now refine our group-averaging procedure as follows. First, decompose \mathcal{H}_{aux} as a direct sum of the subspaces $\mathcal{H}_{aux}^\lambda$, where λ is a map $\lambda : I \rightarrow \{d, c\}$. Thus, $\mathcal{H}_{aux}^\lambda$ is the subspace on which U_i has continuous spectrum for $\lambda(i) = c$ but has discrete spectrum for $\lambda(i) = d$. Since these subspaces are superselected, it is only meaningful to define a physical Hilbert space $\mathcal{H}_{phys}^\lambda$ for each $\mathcal{H}_{aux}^\lambda$ separately. This is done by projecting $\mathcal{H}_{aux}^\lambda$ to the zero spectrum of each U_i with $\lambda(i) = d$ and averaging as in section II B over the Abelian group generated by the U_i with $\lambda(i) = c$. It then follows that operators induced by physical operators on \mathcal{H}_{aux} have the required \star -relations on each $\mathcal{H}_{phys}^\lambda$.

We would like to emphasize that, when the U_i 's generate the entire gauge group, these superselection rules are not just an artifact of the mathematics but are important for a physical understanding of the system. They imply that the representation of the physical algebra on \mathcal{H}_{aux} is reducible, so that each $\mathcal{H}_{phys}^\lambda$ contains a separate representation of the algebra of physical operators.

Which $\mathcal{H}_{phys}^\lambda$ is realized in a given situation must be determined experimentally.

Furthermore, the super-selection rules described above have a close classical analogue due to Liouville's theorem. Consider a classical constraint function C_i and a strong observable A that is a smooth function on the phase space. (The use of strong observables is not essential but simplifies the discussion.) The Hamiltonian vector field h_A of any such A has the property that it maps any orbit of C_i in the unconstrained phase space onto another orbit of C_i . Heuristically, regions of the phase space that contain compact orbits correspond to the discrete spectrum of a corresponding U_i and regions that contain non-compact orbits correspond to the continuous spectrum. Now, consider any set of compact orbits with non-zero but finite phase space volume. By Liouville's theorem, the exponentiated action of any Hamiltonian vector field preserves the finite volume of this set. As a result, h_A cannot map this set of compact orbits to a bundle of non-compact orbits. Note that this is a direct analogy with the super-selection laws described above.

Of course, if these orbits are not the full gauge orbits, but only those of a gauge subgroup, then such arguments are inconclusive when applied to weakly physical operators. This is because, under the action of the full gauge group, many of the above compact orbits may combine to form a single non-compact orbit, which could then be mapped onto non-compact orbits in a volume preserving way.

APPENDIX B: PROJECTIVE LIMITS

A general setting for functional integration over an infinite dimensional, locally convex, topological space V is provided by the notion of "projective families" [38,39]. This framework can be naturally extended to theories of connections where the relevant space $\overline{\mathcal{A}/\mathcal{G}}$ is *non-linear* [6,8–12]. In the present appendix we will summarize the basic ideas which are implicitly used in the main text.

Let L be a partially ordered directed set; i.e. a set equipped with a relation ' \geq ' such that, for all S, S' and S'' in L we have:

$$S \geq S ; \quad S \geq S' \text{ and } S' \geq S \Rightarrow S = S' ; \quad S \geq S' \quad (\text{B.1})$$

and

$$S' \geq S'' \Rightarrow S \geq S'' ; \quad (\text{B.2})$$

and, given any $S', S'' \in L$, there exists $S \in L$ such that

$$S \geq S' \quad \text{and} \quad S \geq S'' . \quad (\text{B.3})$$

L will serve as the label set. A *projective family* $(\mathcal{X}_S, p_{SS'})_{S, S' \in L}$ consists of sets \mathcal{X}_S indexed by elements of L , together with a family of surjective *projections*,

$$p_{SS'} : \mathcal{X}_{S'} \rightarrow \mathcal{X}_S, \quad (\text{B.4})$$

assigned uniquely to pairs (S', S) whenever $S' \geq S$ such that

$$p_{SS'} \circ p_{S'S''} = p_{SS''} . \quad (\text{B.5})$$

We will assume that \mathcal{X}_S are all topological, compact, Hausdorff spaces and that the projections $p_{SS'}$ are continuous.

In the application of this framework to theories of connections, carried out in Sec. IV, the labels S can be thought of as general lattices (which are not necessarily rectangular) and the members \mathcal{X}_S of the projective family, as the spaces of configurations associated with these lattices. The continuum theory will be recovered in the limit as one considers lattices with increasing number of loops of arbitrary complexity.

Note that, in the projective family there will, in general, be no set $\overline{\mathcal{X}}$ which can be regarded as the largest, from which we can project to any of the \mathcal{X}_S . However, such a set does emerge in an appropriate limit, which we now define. The *projective limit* $\overline{\mathcal{X}}$ of a projective family $(\mathcal{X}_S, p_{SS'})_{S, S' \in L}$ is the subset of the Cartesian product $\times_{S \in L} \mathcal{X}_S$ that satisfies certain consistency conditions:

$$\overline{\mathcal{X}} := \{(x_S)_{S \in L} \in \times_{S \in L} \mathcal{X}_S : \quad S' \geq S \Rightarrow p_{SS'} x_{S'} = x_S\} . \quad (\text{B.6})$$

(This is the limit that gave us in Sec. IV the quantum configuration $\overline{\mathcal{A}/\mathcal{G}}$ for theories of connections.) We provide $\overline{\mathcal{X}}$ with the product topology that descends from $\times_{S \in L} \mathcal{X}_S$. This is the *Tychonov topology*. In the Tychonov topology the product space is known to be compact and Hausdorff. Furthermore, as noted in [9], $\overline{\mathcal{X}}$ is closed in $\times_{S \in L} \mathcal{X}_S$, whence $\overline{\mathcal{X}}$ is also compact (and Hausdorff). Note that the limit $\overline{\mathcal{X}}$ is naturally equipped with a family of projections:

$$p_S : \overline{\mathcal{X}} \rightarrow \mathcal{X}_S, \quad p_S((x_{S'})_{S' \in L}) := x_S . \quad (\text{B.7})$$

Next, we introduce certain function spaces. For each S consider the space $C^0(\mathcal{X}_S)$ of the complex valued, continuous functions on \mathcal{X}_S . In the union

$$\bigcup_{S \in L} C^0(\mathcal{X}_S) \quad (\text{B.8})$$

we define the following equivalence relation. Given $f_{S_i} \in C^0(\mathcal{X}_{S_i})$, $i = 1, 2$, let us say:

$$f_{S_1} \sim f_{S_2} \quad \text{if} \quad p_{S_1 S_3}^* f_{S_1} = p_{S_2 S_3}^* f_{S_2} \quad (\text{B.9})$$

for every $S_3 \geq S_1, S_2$, where p_{S_1, S_3}^* denotes the pull-back map from the space of functions on \mathcal{X}_{S_1} to the space of functions on \mathcal{X}_{S_3} . Using the equivalence relation we can now introduce the set of *cylindrical functions* associated with the projective family $(\mathcal{X}_S, p_{SS'})_{S, S' \in L}$,

$$\text{Cyl}(\overline{\mathcal{X}}) := \left(\bigcup_{S \in L} C^0(\mathcal{X}_S) \right) / \sim . \quad (\text{B.10})$$

The quotient just gets rid of a redundancy: pull-backs of functions from a smaller set to a larger set are now identified with the functions on the smaller set. Note that in spite of the notation, as defined, an element of $\text{Cyl}(\overline{\mathcal{X}})$ is *not* a function on $\overline{\mathcal{X}}$; it is simply an equivalence class of continuous functions on some of the members \mathcal{X}_S of the projective family. The notation is, however, justified because, one *can* identify elements of $\text{Cyl}(\overline{\mathcal{X}})$ with continuous functions on $\overline{\mathcal{X}}$. This identification was implicitly used in (IV.22). If the \mathcal{X}_S are differentiable manifolds then one can define spaces $\text{Cyl}^n(\overline{\mathcal{X}})$ of differentiable cylindrical functions in a completely analogous way. These spaces play a crucial role in defining measures and regulated operators.

APPENDIX C: UNEXPECTED CONSEQUENCES OF DIFFEOMORPHISM INVARIANCE

In section VI, we used a group averaging procedure to solve the quantum diffeomorphism constraint. It was therefore natural to use the finite –rather than the infinitesimal– form of constraints. It turns out, however, that there is really no choice: it is not possible to define the infinitesimal form of the diffeomorphism constraints on *any* $\mathcal{H}_{aux} = L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu)$ which carries a faithful representation of the holonomy algebra when $d\mu$ is diffeomorphism invariant. In this appendix, we will discuss this somewhat surprising technical point.

Let N^a denote a complete analytic vector field on Σ and φ_t the corresponding flow of analytic diffeomorphisms. Then, from (III.2), we see that the smeared version of the diffeomorphism constraint is given by:

$$\begin{aligned} V_N &:= \int_{\Sigma} N^a(x) V_a(x) d^d x \\ &= \int_{\Sigma} N^a(x) \text{tr} [F_{ab}(x) \tilde{E}^b(x)] d^d x . \end{aligned} \quad (\text{C.1})$$

Let us equip \mathcal{A}/\mathcal{G} with one of the standard Sobolev topologies [40] and denote by \tilde{N} and $\tilde{\varphi}_t$ the vector field and the flow on \mathcal{A}/\mathcal{G} induced by N^a . Given a smooth function ψ on \mathcal{A}/\mathcal{G} , it is then easy to write out the action of the desired operator \hat{V}_N on ψ :

$$\hat{V}_N \circ \psi = \int_{\Sigma} d^d x N^a \text{tr} F_{ab} \frac{\delta \psi}{\delta A_b} = \mathcal{L}_{\tilde{N}} \psi . \quad (\text{C.2})$$

Hence, the exponentiated version of the constraint is given simply by: $U_N(t) \circ \psi = (\tilde{\varphi}_t)_* \cdot \psi$. Since $\tilde{\varphi}$ extends naturally to $\overline{\mathcal{A}/\mathcal{G}}$, it is straightforward to extend the action of U_N to our \mathcal{H}_{aux} , which we will denote again by U_N . If the measure on $\overline{\mathcal{A}/\mathcal{G}}$ is diffeomorphism invariant, U_N are unitary operators, hence defined on all of \mathcal{H}_{aux} . It is now obvious that the algebra of these operators is closed; *there are no anomalies*. The result is, however, non-trivial because our constraint operators $U_N(t)$ are rigorously defined on the auxiliary Hilbert space. It is known, for example, that if one uses a lattice regularization to give meaning to the formally defined constraint operators, anomalies do result.

What would happen if we try to extend to \mathcal{H}_{aux} the action of the infinitesimal constraints \hat{V}_N instead? Since Wilson loop variables are smooth functions on \mathcal{A}/\mathcal{G} , let us begin by setting $\psi(A) = T_{\alpha}(A)$ on \mathcal{A}/\mathcal{G} . Then, we have:

$$\left(\hat{V}_N \circ T_{\alpha} \right) (A) = \lim_{t \rightarrow 0} \frac{T_{\alpha_t} - T_{\alpha}}{t} (A) , \quad (\text{C.3})$$

where $\alpha_t = \phi_t \alpha$ and the point A indicates that the limit is taken pointwise in \mathcal{A}/\mathcal{G} . The limit is of course a well-behaved smooth function on \mathcal{A}/\mathcal{G} . However, it fails to be a cylindrical function. (Note that $U_N(t) \circ T_{\alpha} = T_{\phi(t) \cdot \alpha}$, on the other hand, *is* cylindrical.) Hence, one might suspect that there may be a difficulty in extending the operator \hat{V}_N to \mathcal{H}_{aux} . We will see that this is the case.

More precisely, we now show that for a diffeomorphism invariant measure μ on $\overline{\mathcal{A}/\mathcal{G}}$ to be compatible with a well defined infinitesimal generator of the diffeomorphism constraint, μ must have a very special support. The resulting representation of the \hat{T}_{α} algebra would then be so unfaithful as to be physically irrelevant.

Indeed, let μ denote a diffeomorphism invariant measure and $\alpha_t = \varphi_t \alpha$ as above. For the diffeomorphism constraint to be well defined we must have (at least for “most” of the loops α in Σ)

$$\lim_{t \rightarrow 0} \| T_{\alpha_t} - T_\alpha \|_{L^2}^2 = \lim_{t \rightarrow 0} \int_{\mathcal{A}/\mathcal{G}} (T_{\alpha_t} - T_\alpha)^2 d\mu = 0 . \quad (\text{C.4})$$

From diffeomorphism invariance of the measure it is clear that

$$\int_{\mathcal{A}/\mathcal{G}} T_\alpha^2 d\mu = \int_{\mathcal{A}/\mathcal{G}} T_{\alpha_t}^2 d\mu , \quad \forall t$$

and that there exists $t_0 > 0$ such that

$$\int_{\mathcal{A}/\mathcal{G}} T_\alpha T_{\alpha_t} d\mu = k = \text{const} , \quad \text{for } t : 0 < t < t_0 . \quad (\text{C.5})$$

(To see this we can consider a flow φ'_s of analytic diffeomorphisms that leave α invariant and such that $\varphi'_s \alpha_t = \alpha_{\nu'(t,s)}$.) For the limit in (C.4) to be equal to zero we must have $k = \int_{\mathcal{A}/\mathcal{G}} T_\alpha^2 d\mu$, which from (C.4) implies that in $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu)$

$$T_{\alpha_t} = T_\alpha , \quad \forall t : 0 < t < t_0 . \quad (\text{C.6})$$

Now, (C.6) implies that the representation ρ (see (IV.4)) of the holonomy algebra on $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu)$ is not faithful since $T_{\alpha_t} - T_\alpha \neq 0$ as elements of \mathcal{HA} , while $\rho(T_{\alpha_t} - T_\alpha) = 0$ as operators on $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu)$. Thus, the support of the measure μ is so special that it is not suitable as a kinematical measure in quantum theory. Put differently, in any interesting representation of the holonomy algebra,

$$\int_{\mathcal{A}/\mathcal{G}} T_{\alpha_t} T_\alpha d\mu \neq \int_{\mathcal{A}/\mathcal{G}} T_\alpha^2 d\mu , \quad \forall t : 0 < t < t_0 \quad (\text{C.7})$$

and therefore the infinitesimal generators of the diffeomorphism constraints can not be well defined.

APPENDIX D: GEOMETRICAL OPERATORS

On the phase space of Riemannian general relativity, the momentum variable \tilde{E}_i^a has the interpretation of a density weighted triad. Hence, one can use it to construct functions on the phase space that carry geometrical information. For example, the volume of a region R within Σ is given by:

$$V_R := \int_R d^3x |\eta_{abc} \epsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b \tilde{E}_k^c|^{\frac{1}{2}} , \quad (\text{D.1})$$

where η_{abc} is the Levi-Civita tensor density on Σ . Similarly, the area of a 2-surface S within Σ defined by, say, $x_3 = \text{const}$ is given by:

$$A_S := \int_S d^2x |\tilde{E}_i^a \tilde{E}^{bi} \nabla_a x_3 \nabla_b x_3|^{\frac{1}{2}} \quad (\text{D.2})$$

The question then arises: are there well-defined geometric operators \hat{V}_R and \hat{A}_S on \mathcal{H}_{aux} ? In absence of matter fields, V_R and A_S fail to be observables since they are not diffeomorphism invariant. Hence, the corresponding operators will not represent physical observables. However, if we bring in matter sources and *define* the regions R and surfaces S using these fields, *then* \hat{V}_R and \hat{A}_S would be observables with respect to the diffeomorphism constraints [41]. Therefore, it is of considerable interest to try to construct these operators in the kinematical setting of Sec.IV and explore their properties.

At first sight, it seems difficult to make sense out of these operators. To begin with, \tilde{E}_i^a itself is not a well-defined operator on \mathcal{H}_{aux} . Second, the desired operators would require *products* of \tilde{E}_i^a evaluated at the same point, and, furthermore, a square-root! Nonetheless, it turns out that these formal expressions *can be* regulated satisfactorily to yield well-defined operators on \mathcal{H}_{aux} . The regularization procedure involves point-splitting and it is necessary to fix a gauge and a background metric (or coordinate system) in the intermediate stage. However, when the regulator is removed, the final expression is not only well-defined but independent of the background structures used in the procedure. The overall procedure is similar to the one used in rigorous quantum field theories. Furthermore, somewhat surprisingly, for suitable operators such as \hat{V}_R and \hat{A}_S , the situation is better than what one might have expected: there is no need to renormalize, whence the final answers have no free parameters. Finally, the operators are essentially self-adjoint on \mathcal{H}_{aux} and their spectra are often discrete. Thus, the “quantum geometry” that emerges from our framework has certain essentially discrete elements which suggest that the use of a continuum picture at the Planck scale is flawed. These results are analogous to the ones obtained by Rovelli and Smolin [21] in the loop representation. However, the precise relation is not known.

Here, we will illustrate these results with the area operator. For simplicity, let us suppose that we can choose coordinates on Σ in a neighborhood of S such that S is given by $x_3 = \text{const}$ and x_1, x_2 coordinatize S . Then, we can write A_S as: $A_S = \int_S d^2x \sqrt{|O(x)|}$, where $O(x) = \tilde{E}_i^3(x) \tilde{E}^{3i}(x)$. To define \hat{A}_S , let us use a point splitting procedure and consider the regulated operator

$$\widehat{O}_\epsilon(x) := \int d^3 y f_\epsilon(x, y) \int d^3 z f_\epsilon(x, z) \frac{\delta}{\delta A_3^i(y)} \frac{\delta}{\delta A_3^i(z)}, \quad (\text{D.3})$$

where $f_\epsilon(x, y)$ (is a density of weight 1 in x and function in y and that) tends to $\delta^3(x, y)$ in the limit. For concreteness, we will construct it from Θ density/functions:

$$f_\epsilon(x, y) = \frac{1}{\epsilon^3} [\Theta(\frac{\epsilon}{2} - |x_1 - y_1|) \Theta(\frac{\epsilon}{2} - |x_2 - y_2|) \times \Theta(\frac{\epsilon}{2} - |x_3 - y_3|)]. \quad (\text{D.4})$$

(There is thus an implicit background density of weight one in x in the expression of Θ .)

Now, let us begin by considering a cylindrical function F_γ on the space \mathcal{A} of smooth connections. By using a group-valued chart on \mathcal{A}_γ , F_γ can be expressed as $F_\gamma(A) = f(g_1, \dots, g_N)$ where N is the number of edges in γ and $g_I = \mathcal{P} \exp \int_{e_I} A$. A simple calculation yields:

$$\begin{aligned} & \widehat{O}_\epsilon(x) \circ F_\gamma \\ &= \int d^3 y f_\epsilon(x, y) \frac{\delta}{\delta A_3^i(y)} \sum_{e_J} \int_{e_J} ds f_\epsilon(x, e_J(s)) \\ & \quad \dot{e}_J^3 \text{Tr}(h_J(1, s) \tau_i h_J(s, 0) \frac{\partial}{\partial h_J}) f_\gamma \\ &= \sum_{I, J} \int_{e_I} dt \dot{e}_I^3(t) \int_{e_J} ds \dot{e}_J^3(s) f_\epsilon(x, e_I(t)) \times \\ & \quad f_\epsilon(x, e_J(s)) \text{Tr}(h_I(1, t) \tau_i h_I(t, 0) \frac{\partial}{\partial h_I}) \times \\ & \quad \text{Tr}(h_J(1, s) \tau_i h_J(s, 0) \frac{\partial}{\partial h_J}) f_\gamma \\ & \quad + 2 \sum_I \int_{e_I \times e_I, t \geq s} dt ds \dot{e}_I^3(t) \dot{e}_I^3(s) \\ & \quad f_\epsilon(x, e_I(t)) f_\epsilon(x, e_I(s)) \\ & \quad [\text{Tr}(h_I(1, t) \tau_i h_I(t, s) \tau_j h_I(s, 0) \frac{\partial}{\partial h_I}) + \\ & \quad \text{Tr}(h_I(1, t) \tau_i h_I(t, s) \tau_j h_I(s, 0) \frac{\partial}{\partial h_I})] f_\gamma \\ & =: (\widehat{O}_\epsilon^I(x) + \widehat{O}_\epsilon^{II}(x)) \circ f \end{aligned} \quad (\text{D.5})$$

The right side is a well-defined function of smooth connections A . However, it is no more a cylindrical function because of the form of the terms involving integrals over edges. We thus have two problems: the action of $O_\epsilon(x)$ is not well-defined on functions of *generalized* connections, and, even while operating on functions of smooth connections, the operator sends cylindrical functions to more general ones. We will see that the two problems go away once the regulator is removed.

Let us consider the first term in detail; an analogous treatment of the second term shows that it does not contribute to the final result.

Ultimately, we want to integrate $\widehat{O}(x)$ over S . Hence, we want x to lie in S . Then, for sufficiently small ϵ , because of the f_ϵ terms, only the edges that intersect S contribute to the sum. (Furthermore, since only the third component of the tangent vectors count in O_ϵ^I , edges which lie within S do not contribute.) Without loss of generality, we can assume that intersections occur only at vertices of γ (since we can always add vertices in the beginning of the calculation to ensure this). Now, if we write out the functions f_ϵ explicitly and Taylor expand, around each vertex at which γ intersects S , the group elements that appear in the integrals we can express $O_\epsilon^I \circ f$ as a sum:

$$\begin{aligned} \widehat{O}_\epsilon^I(x) \circ f &= \sum_{v_\alpha} \sum_{I_\alpha} \sum_{J_\alpha} \frac{K(I_\alpha, J_\alpha)}{\epsilon^4} \\ & [\Theta(\frac{\epsilon}{2} - |x_1 - x_1^{(\alpha)}|) \Theta(\frac{\epsilon}{2} - |y_1 - y_1^{(\alpha)}|)]^2 \times \\ & \{X_{I_\alpha, i} X_{J_\alpha, i} + o(\epsilon)\} \circ f(g_1, \dots, g_n). \end{aligned} \quad (\text{D.6})$$

Here v_α are the vertices of γ that lie in S , $e_{I_\alpha}, e_{J_\alpha}$ are the edges passing through the vertex v_α , X_{I_α} is the right (left) invariant vector field on the copy of the group corresponding to the edge e_{I_α} which points at the identity of the group in the i -th direction, if the edge is oriented to be outgoing (incoming) at the vertex, and the constant $K(I_\alpha, J_\alpha)$ equals $+1$ if the two edges lie on the opposite side of S , -1 if they lie on the same side and vanishes if the tangent vector of either edge is tangential to S .

Let us try to take the limit of $\widehat{O}_\epsilon^I(x) \circ f$ as ϵ tends to zero. In this limit, each Θ_ϵ tends to a 1-dimensional Dirac δ -distribution, and the expression then diverges as $1/\epsilon^2$. As is usual in field theory, we can first renormalize the expression by ϵ^2 and *then* take the limit. Now, the limit clearly exists. However, it depends on the background density implicit in the expression of Θ and hence the resulting operator carries the memory of the background structure used in the regularization. That is, the ambiguity in the final answer is not of a multiplicative constant, but of a background density of weight one. (This is to be expected since the left hand side is a density of weight 2 (in x and y) while the 2-dimensional Dirac δ -distribution is only a density of weight 1.) Because of the background dependence, the resulting operator is not useful for our purposes.

However, if we take the square-root of the regulated operator and *then* take the limit, we obtain a well-defined result:

$$\lim_{\epsilon \rightarrow 0} |\hat{O}_\epsilon(x)|^{\frac{1}{2}} \circ f = \sum_{v_\alpha} \delta^2(x, v_\alpha) \times$$

$$\left| \sum_{I_\alpha} \sum_{J_\alpha} (K(I_\alpha, J_\alpha) \{X_{I_\alpha, i} X_{J_\alpha, i}\})^{\frac{1}{2}} \circ f(g_1, \dots, g_n) \right. \quad (D.7)$$

Note that, now, no renormalization is necessary. In the final result, both sides are densities of weight one and there is neither background dependence, nor any free parameters. With proper specification of domains, the operator under the square-root can be shown to be a non-negative self-adjoint operator on $L^2((SU(2))^n, d\mu_H)$. (For example, if there are just two edges at a vertex v_α , one on each side of S , then the operator is just the (negative of the) Laplacian.) Hence, the square-root is well-defined. We can therefore construct an area operator:

$$\hat{A}_S \circ F_\gamma = \int_S d^2x |\hat{O}(x)|^{\frac{1}{2}} \circ f. \quad (D.8)$$

Clearly, this operator maps cylindrical functions to cylindrical functions. It is straightforward to show it satisfies the compatibility conditions discussed in Sec. IV and thus leads to a well-defined operator on $\mathcal{H}_{aux} = L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_o)$. This operator is self-adjoint and has a discrete spectrum.

The volume operator can be treated in a similar manner.

To conclude, note that there is a striking qualitative resemblance between this analysis of properties of geometry and that of physical properties of polymers in condensed matter physics [42]. In both cases, the basic excitations are “loopy” rather than “wavy”; they reside along 1-dimensional graphs rather than on 3-dimensional volumes. However, under suitably complex conditions, they resemble genuinely 3-dimensional systems [42,43].

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