

## Spin Networks in Nonperturbative Quantum Gravity

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### Abstract

A spin network is a generalization of a knot or link: a graph embedded in space, with edges labelled by representations of a Lie group, and vertices labelled by intertwining operators. Such objects play an important role in 3-dimensional topological quantum field theory, functional integration on the space  $\mathcal{A}/\mathcal{G}$  of connections modulo gauge transformations, and the loop representation of quantum gravity. Here, after an introduction to the basic ideas of nonperturbative canonical quantum gravity, we review a rigorous approach to functional integration on  $\mathcal{A}/\mathcal{G}$  in which  $L^2(\mathcal{A}/\mathcal{G})$  is spanned by states labelled by spin networks. Then we explain the ‘new variables’ for general relativity in 4-dimensional spacetime and describe how canonical quantization of gravity in this formalism leads to interesting applications of these spin network states.

## 1 Introduction

Spin networks are a generalization of knots and links, and in what follows we would like to describe some of their recent applications to quantum gravity, and also a little bit of their role in knot theory. Knot theory is a well-established branch of mathematics, full of interesting theorems, and it is easy as a mathematician to acquaint oneself with it by reading any of a number of texts. Quantum gravity, on the other hand, is more difficult for the

mathematician, consisting as it does mainly of unsolved and usually imprecisely posed questions. Thus it seems worthwhile to start with a thumbnail sketch of quantum gravity, focusing on an approach known as ‘nonperturbative canonical quantization’.

After this heuristic introduction, we give a mathematically rigorous account of spin networks in Section 2. For us, spin networks will be graphs embedded in a manifold  $S$ , with edges labelled by representations of a Lie group  $G$  and with vertices labelled by intertwining operators. Spin networks define gauge-invariant functions on the space  $\mathcal{A}$  of connections on any  $G$ -bundle over  $S$ , and we shall show that these functions span the gauge-invariant subspace  $L^2(\mathcal{A}/\mathcal{G})$  of a certain Hilbert space  $L^2(\mathcal{A})$ . In Sections 3 and 4 we go back to the physics and give a more detailed description of the role spin networks play in quantum gravity.

In the latter part of the twentieth century, much energy has been spent in a struggle to reconcile two brilliant accomplishments of the early part of the century: general relativity and quantum theory. So far there is remarkably little to show for all this work when it comes to verified predictions of experimental results. Indeed, it is quite possible that the new predictions of a theory of quantum gravity can only be tested at extremely small length scales, far below those that can be probed by current experimental techniques. The reason for this is simple dimensional analysis: from Planck’s constant  $\hbar$ , Newton’s gravitational constant  $\kappa$ , and the speed of light  $c$  one can form a quantity with units of length, the Planck length

$$\ell_P = (\hbar\kappa/c^3)^{1/2},$$

in a unique way (up to dimensionless constant factors). This works out to be about  $10^{-35}$  meters. If we are hoping to get experimental evidence for any theory of quantum gravity in the foreseeable future, we have to hope that somehow this simple argument is wrong. There are certainly many phenomena already observed, but so far unexplained, that a theory of quantum gravity might hope to ‘retrodict’: phenomena from particle physics, phenomena from cosmology, even phenomena such as the 4-dimensionality of spacetime that are taken for granted in all current theories of physics. Physicists have suggested many ideas along these lines, but none command widespread acceptance at this time; they are all somewhere between controversial theories and sheer lunacy.

In short, if there were several competing well-established theories of quantum gravity, the lack of empirical evidence to decide between them could easily be a serious problem. Luckily nature has been kind to us, in that we have been unable to formulate *even one* theory combining general relativity and quantum theory in a manner that is *mathematically consistent, not in obvious contradiction with experiment, and elegant*. The importance of mathematical consistency and non-contradiction with experiment should be obvious, but the role of elegance deserves some comment. First, it is always possible to find infinitely many consistent theories that fit a given finite set of experimental results, just as there are infinitely many curves through a finite set of points, and without the freedom to reject most of them as appallingly ugly, we would never get anywhere. Second, there is the fact that taken separately, general relativity and quantum theory are strikingly esthetic as pure mathematics. This could be a mere coincidence, but in the absence of evidence to the contrary it is probably best to take it as a hint, and search for a theory combining the most beautiful aspects of both in an integral manner. Indeed, though the ‘unreasonable effectiveness’ of elegant mathematics is still a great mystery, it is also an empirical fact, so even as hard-nosed empiricists we should heed it.

Many physicists, particularly in string theory, have argued that it is impossible to reconcile general relativity and quantum theory without taking all the other forces and particles into account. Here however we concentrate on a diametrically opposite approach, which tries to construct a quantum version of the *vacuum* Einstein equations — the theory of gravity with *no* matter around. We shall not attempt to discuss the relative merits of the two approaches, but simply note the curious fact that some of the same mathematics shows up in both, possibly for some deep reasons that would be good to understand [11].

In what follows we need to assume a nodding acquaintance with some differential geometry. Let us take as our spacetime a 4-manifold  $M$  diffeomorphic to  $\mathbf{R} \times S$ , where  $\mathbf{R}$  represents time and the 3-manifold  $S$  (compact and orientable to simplify the discussion) represents space. The vacuum Einstein equation concerns a Lorentzian metric  $g$  on  $M$ . Given such a metric there is a unique metric-preserving, torsion-free connection  $\Gamma$ , the Levi-Civita connection. The curvature of this connection is described by the Riemann tensor  $R$ , which in local coordinates is written  $R^\alpha{}_{\beta\gamma\delta}$ . A certain amount of information about the curvature of spacetime is thus contained in the Ricci

tensor  $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$ , and the vacuum Einstein equation says simply that

$$R_{\mu\nu} = 0.$$

To understand why precisely *these* are the equations describing the curvature of empty spacetime, though, it is necessary to step back and consider the case where there is matter present. In the presence of matter, the density of energy and momentum and their rate of flow in different spatial directions are summarized by a symmetric tensor  $T_{\mu\nu}$ . The sense in which energy and momentum are conserved in general relativity is quite subtle, but the simplest way of stating this conservation principle is that  $T_{\mu\nu}$  is ‘divergence-free’:

$$\nabla^\mu T_{\mu\nu} = 0,$$

where  $\nabla$  is the operator of covariant differentiation corresponding to the Levi-Civita connection. If we wish to say that spacetime is curved by energy and momentum, it is natural to try some equation like

$$C_{\mu\nu} = T_{\mu\nu}$$

where  $C$  is built up from the Riemann tensor in some simple way. However, are constrained by the fact that  $C$  must be symmetric and divergence-free. The simplest thing to try is some multiple of

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R^\alpha{}_\alpha g_{\mu\nu},$$

because this is symmetric and, by the Bianchi identity, also divergence-free. In fact, Einstein’s equation in the presence of matter says that

$$G_{\mu\nu} = 8\pi\kappa T_{\mu\nu}.$$

But in a vacuum  $T_{\mu\nu} = 0$ , so we obtain

$$G_{\mu\nu} = 0,$$

which turns out to be equivalent to the vanishing of the Ricci tensor.

To see how the vacuum Einstein equation describes the *dynamics* of the metric, it is useful to split spacetime into space and time. The manifold  $M$  is diffeomorphic to  $\mathbf{R} \times S$ , but not in any canonical way. Nonetheless, let us

go ahead and pick a diffeomorphism between them. This lets us transfer the usual coordinate function on  $\mathbf{R}$  to a ‘time’ function  $x^0$  on  $M$ , so we can talk about the metric on space at time  $t$ , that is, the metric  $g$  restricted to the slice

$$S_t = \{x^0 = t\} \subseteq M.$$

Assume we can choose the slices  $S_t$  to be spacelike, meaning that the restriction of  $g$  to each one is a Riemannian metric. We can only do this if  $g$  is ‘globally hyperbolic’, but physically this is a reasonable condition. Then, near any point  $p \in M$ , we can find local coordinates  $x^\mu$  such that  $x^0$  is the above time function, the vector field  $\partial_0$  is normal to the slices, and the vector fields  $\partial_i$  corresponding to the remaining ‘space’ coordinates  $x^i$  ( $i = 1, 2, 3$ ) are tangent to the slices. One does not really need to use coordinates with these special properties, but it simplifies the discussion below.

Since  $G_{\mu\nu}$  is a symmetric  $4 \times 4$  matrix, the vacuum Einstein equation really has 10 components. Splitting spacetime into space and time lets us interpret these components as follows. The equation  $G_{00} = 0$  and the 3 equations  $G_{0i} = 0$  are *constraints* on the metric on space,  $g_{ij}$ , and its first time derivative  $\dot{g}_{ij}$ . In other words,  $G_{00}$  and  $G_{0i}$  can be computed on the slice  $S_t$  in terms of the ‘initial data’  $g_{ij}$  and  $\dot{g}_{ij}$  on that slice, and for there to be a solution of Einstein’s equation having given initial data, the initial data must satisfy the constraints  $G_{00} = G_{0i} = 0$ . The remaining 6 equations  $G_{ij} = 0$  are *evolutionary equations* involving the second time derivative  $\ddot{g}_{ij}$ . They allow us to compute how the metric evolves in time, given that the constraints hold initially.

In fact, the constraints have a simple meaning, which turns out to be very important. To understand this meaning, however, we need a brief detour into classical mechanics. It is worth recalling the simplest classical mechanics problem of all, the motion of a particle on the line in a potential  $V$ . The particle’s position is a point  $q$  in the space  $\mathbf{R}$ , and one calls this space of possible positions the ‘configuration space’. The motion of the particle is determined by its position  $q$  and velocity  $\dot{q}$  at time zero, but it is often handier to work not with its velocity but with its momentum  $p = m\dot{q}$ , where  $m$  is its mass. The position and momentum determine a point  $(q, p)$  in the ‘phase space’  $T^*\mathbf{R}$ , since the momentum is most naturally regarded as a cotangent vector to the configuration space. This phase space has a closed

nondegenerate 2-form, or ‘symplectic structure’, on it, given by

$$\omega = dp \wedge dq.$$

The energy, or Hamiltonian, of the particle is a function on phase space, given by the sum of kinetic and potential energy:

$$H = \frac{p^2}{2m} + V(q).$$

The Hamiltonian gives rise to a vector field  $v_H$  on phase space by

$$\omega(\cdot, v_H) = dH.$$

This vector field generates a one-parameter group of diffeomorphisms of phase space, or ‘flow’, which describes the time evolution of the particle. For short, one says that the Hamiltonian generates time evolution. Indeed, quite often in classical mechanics the phase space is a symplectic manifold and time evolution is generated by the Hamiltonian in this manner. And quite often the phase space is the cotangent bundle of some configuration space; cotangent bundles are equipped with a canonical symplectic structure.

In general relativity, the quantity analogous to the ‘position’ is the metric  $g_{ij}$  on space, and we shall henceforth write this as  $q_{ij}$  to emphasize the analogy. The configuration space of general relativity is thus the space  $\mathcal{M}$  of all Riemannian metrics on  $S$ . The quantity analogous to the ‘velocity’ is the time derivative  $\dot{q}_{ij}$ . Following certain standard recipes, the analog of the momentum works out to be

$$p^{ij} = \frac{1}{2}(\det q_{ij})^{1/2}(\dot{q}^{ij} - \dot{q}_k^i q^{kj}).$$

The phase space of general relativity is the cotangent bundle  $T^*\mathcal{M}$ , and the pair  $(q_{ij}, p^{ij})$  determines a point in this phase space.

Since one can write the constraints  $G_{00}$  and  $G_{0i}$  in terms of  $q_{ij}$  and  $p^{ij}$ , one can think of these constraints as functions on  $T^*\mathcal{M}$ . From this point of view, they play a dual role [27]. First, as already noted, for the pair  $(q_{ij}, p^{ij})$  to determine a solution of Einstein’s equation, it must lie on the subspace of  $T^*\mathcal{M}$  where the constraints vanish. Second, and more subtly, the constraints generate physically important flows on phase space.

For example, if we integrate  $G_{00}$  over  $S$ , we obtain a function on  $T^*\mathcal{M}$  which generates time evolution with respect to the time coordinate  $x^0$ . For this reason physicists call this constraint the ‘Hamiltonian constraint’, and use the notation

$$\mathcal{H} = G_{00}$$

in this context. On the other hand, if we take a vector field  $N$  on  $S$  and integrate  $N^i G_{0i}$  over  $S$ , we obtain a function on  $T^*\mathcal{M}$  generating a flow which is the same as that induced by the one-parameter group of diffeomorphisms of  $S$  generated by  $N$ . Physicists call this constraint the ‘diffeomorphism constraint’, and write it as

$$\mathcal{H}_i = G_{0i}.$$

It is important to note, however, that both  $\mathcal{H}$  and  $\mathcal{H}_i$  correspond to one-parameter groups of diffeomorphisms of spacetime:  $\mathcal{H}$  corresponds to diffeomorphisms that ‘push  $S$  forwards in time’, while  $\mathcal{H}_i$  corresponds to diffeomorphisms that map  $S$  to itself.

Now let us turn the problem of quantizing general relativity, that is, guessing a quantum theory that reduces to Einstein’s equation in the limit  $\hbar \rightarrow 0$ . Again, it is good to recall the example of a particle on the line. In the quantum theory of a particle on a line, the particle’s state is given by a ‘wavefunction’, that is, an  $L^2$  function  $\psi$  on the configuration space  $\mathbf{R}$ . If we assume  $\psi$  is normalized so that  $\|\psi\| = 1$ , the probability of finding the particle in any open subset  $U \subseteq \mathbf{R}$  is then given by

$$\int_U |\psi(x)|^2 dx.$$

Now, classically the Hamiltonian is

$$H(p, q) = \frac{p^2}{2m} + V(q),$$

and to quantize the particle one simply replaces  $p$  and  $q$  in this formula with certain operators  $\hat{p}, \hat{q}$  on  $L^2(\mathbf{R})$ :

$$(\hat{p}\psi)(x) = \frac{\hbar}{i} \frac{d}{dx} \psi(x), \quad (\hat{q}\psi)(x) = x\psi(x),$$

obtaining an operator  $\hat{H}$ , the quantum Hamiltonian:

$$(\hat{H}\psi)(x) = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) + V(x)\psi(x).$$

The Hamiltonian  $\hat{H}$  then describes the evolution of the wavefunction in time via Schrödinger's equation:

$$i\hbar \frac{d}{dt}\psi = \hat{H}\psi.$$

This recipe for replacing  $p$  and  $q$  by operators  $\hat{p}$  and  $\hat{q}$  is known as ‘canonical quantization’. If we try to copy this recipe in the case of general relativity, we quickly notice some serious problems. In the case of the particle on the line, the ‘configuration space’ in which  $q$  lies is just the real line, and we can define  $L^2(\mathbf{R})$  using Lebesgue measure. In the case of general relativity, the configuration space is the space  $\mathcal{M}$  of all Riemannian metrics on  $S$ . This is an *infinite-dimensional* manifold, and there is no good notion of ‘Lebesgue measure’ on such spaces, so defining the Hilbert space  $L^2(\mathcal{M})$  is not so easy.

Of course, this problem will arise whenever we try to canonically quantize a theory with infinite-dimensional configuration space. Thus it occurs, not only in quantum gravity, but in other quantum field theories. When quantizing linear equations, where the configuration space is an infinite-dimensional vector space  $\mathcal{V}$ , one can use an infinite-dimensional analog of a Gaussian measure to define  $L^2(\mathcal{V})$ , and the resulting theories make fine mathematical sense. This strategy is not sufficient by itself to deal with nonlinear equations such as Einstein's equation, however. Indeed,  $\mathcal{M}$  is not even a vector space.

Another problem is that unlike the particle on the line, whose position and velocity at a given time are arbitrary, general relativity involves *constraints*. One should take these constraints into account when quantizing the theory, but the correct way to do so is not at all clear! Indeed, much ink has been spilled concerning the quantization of constrained systems, and we cannot go into all the nuances of this issue here. Instead, let us simply give an oversimplified account of Dirac's approach to constraints [24], as applied to gravity by DeWitt [23] in the late 1960s.

Suppose we could somehow get ahold of a Hilbert space  $L^2(\mathcal{M})$ . The idea is that not all vectors in this so-called ‘kinematical’ state space represent physical states of quantum gravity, but only those satisfying certain quantum versions of the constraints. To describe these quantized constraints, first we try to define operators  $\hat{q}_{ij}$  and  $\hat{p}^{ij}$  on  $L^2(\mathcal{M})$  by the formulas

$$(\hat{p}^{ij}\psi)(q) = \frac{\hbar}{i} \frac{\partial}{\partial q_{ij}}\psi(q), \quad (\hat{q}_{ij}\psi)(q) = q_{ij}\psi(q),$$

where  $q \in \mathcal{M}$ . These formulas are merely heuristic, and part of the problem is making good enough sense of them for the task at hand. For example, in analogy with  $p^{ij}$  and  $q_{ij}$  in the first place,  $\hat{p}^{ij}$  and  $\hat{q}_{ij}$  are really something like operator-valued functions on the subset of the spacelike slice contained in our coordinate chart. When one tries to be rigorous one discovers that they are operator-valued distributions. As noted above, one can work out formulas for the constraints  $\mathcal{H}$  and  $\mathcal{H}_i$  in terms of  $p^{ij}$  and  $q_{ij}$ . The idea is then to take these formulas, and replace all appearances of  $p^{ij}$  and  $q_{ij}$  in them by  $\hat{p}^{ij}$  and  $\hat{q}_{ij}$ , obtaining operator-valued distributions  $\hat{\mathcal{H}}$  and  $\hat{\mathcal{H}}_i$ . For a wavefunction  $\psi \in L^2(\mathcal{M})$  to represent a physical state of quantum gravity, the following quantum versions of the constraint equations should hold:

$$\hat{\mathcal{H}}\psi = 0, \quad \hat{\mathcal{H}}_i\psi = 0.$$

There are some deep conceptual problems associated with this approach to quantum gravity. For example, instead of Schrödinger's equation, in which the Hamiltonian describes the time evolution of the wavefunction, all we have is the so-called 'Wheeler-DeWitt equation'  $\hat{\mathcal{H}}\psi = 0$ . What does this mean? Where has the *dynamics* of quantum gravity gone, if physical states are annihilated by the operator whose classical counterpart generates time evolution? Briefly, the answer appears to be this: the constraint equations say that  $\psi$  only describes *diffeomorphism-invariant* information about the world. In other words, we cannot use  $\psi$  to answer questions about what is happening at a certain point whose location is specified using a coordinate system, such as 'what is the metric at the point  $x^\mu = a^\mu$ ?'. Unfortunately, we are not used to doing physics without asking such questions! This is known as the 'problem of time' in quantum gravity [33].

Just as bad as the conceptual problems are the sheer technical problems involved in making mathematical sense out of DeWitt's strategy for quantizing gravity. In fact, nobody has yet succeeded. It is worth noting that there are other constrained nonlinear equations with infinite-dimensional configuration spaces where many of the same problems show up: for example, the Yang-Mills equation, various versions of which appear to describe all the forces *except* gravity. Quantizing these in a rigorous way is *also* extremely difficult, but physicists have had much more success with them at the practical level. Let us briefly summarize the difference between the two cases.

Physicists often quantize nonlinear field theories by treating the nonlinearities as 'small perturbations' of some linear equation. There are also

perturbative methods for dealing with constraints. These methods can be problematic, mathematically speaking, but unless afflicted by an unusual desire for rigor, physicists are often happy to do perturbation theory using formal power series in some coupling constant that measures the nonlinearity. Due to the nonrigorous nature of these computations, the coefficients of these power series usually come out to be ill-defined unless one performs some clever maneuvers known as renormalization. One says the theory is renormalizable if these maneuvers work, but renormalizability by itself does not mean that the power series are convergent, or even asymptotic. Thus, strictly speaking, renormalizability is not a sufficient condition for a nonlinear quantum field theory to make rigorous sense. Nor, on the other hand, is renormalizability a *necessary* condition to be able to quantize a nonlinear wave equation, for there are also other ‘nonperturbative’ approaches. In practice, however, particle physicists often restrict themselves to renormalizable theories, make physical predictions using the first few terms in the power series, and compare these predictions with experiment to see if the theory is on the right track.

For the Yang-Mills equation, and indeed for the whole ‘Standard Model’ of particles and forces other than gravity, this strategy has been quite successful. But the strategy fails miserably with quantum gravity, for the theory resists all attempts at renormalization. As it turns out, this is closely related to the fact that the constraints are not polynomials in the basic ‘position’ and ‘momentum’ variables,  $q_{ij}$  and  $p^{ij}$ , and their derivatives. Instead, they contain nasty factors of  $(\det q_{ij})^{-1/2}$ , essentially because the formula for  $p^{ij}$  contains a factor of  $(\det q_{ij})^{1/2}$ . Standard methods for replacing  $q_{ij}$  with the operator  $\hat{q}_{ij}$  lead to all sorts of problems when applied to such non-polynomial expressions. (Similar problems arise in the path-integral approach which is more commonly used in renormalization.) To overcome the nonrenormalizability of quantum gravity, particle physicists constructed ever more complicated models containing gravity and other forces, in order to cancel out unruly infinities. This led to the study of supergravity and eventually superstring theory, which attempts to model *all* known forces and particles, and unfortunately many unknown ones as well.

Meanwhile, many general relativists were suspicious of the whole idea of perturbatively quantizing gravity. This amounts to treating all metrics as small perturbations of a fixed ‘background metric’, usually taken to be the flat Minkowski metric on  $\mathbf{R}^4$ . Dimensional analysis, however, suggests that

one should expect more and more extreme quantum fluctuations of the metric at smaller and smaller length scales, becoming very significant at about the Planck length. So perhaps one should really adopt some *nonperturbative* approach to canonical quantum gravity.

Saying this is easy: the problem is actually doing it. In principle, the most direct way would be to make DeWitt's approach into rigorous mathematical physics without reference to a fixed background metric. But attempting to do this led immediately into a quagmire which for two decades seemed impassable. It was not until the late 1980s, when Ashtekar [1] found a clever change of variables, that a way around began to seem possible.

We postpone a detailed discussion of these 'new variables' to Section 3. For now, let us simply note that in terms of them, the configuration space of general relativity space is not the space of *metrics* on  $S$ , but the space of *connections* on some  $SL(2, \mathbf{C})$  bundle over  $S$ . Since  $SL(2, \mathbf{C})$  is a complex Lie group, this configuration space is a complex manifold. It turns out that in the quantum theory, kinematical states should be holomorphic functions on this configuration space. However, using the fact that  $SL(2, \mathbf{C})$  has  $SU(2)$  as a real form, one expects the kinematical state space to be isomorphic to  $L^2(\mathcal{A})$ , where  $\mathcal{A}$  is the space of connections on a certain  $SU(2)$  bundle over  $S$ . Thus, at least naively, one expects the physical states of quantum gravity in the new variables formalism to be wavefunctions  $\psi \in L^2(\mathcal{A})$  annihilated by certain constraints. These constraints include Hamiltonian and diffeomorphism constraints as before, which express the invariance of  $\psi$  under diffeomorphisms of spacetime, but also a 'Gauss law' constraint which expresses the invariance of  $\psi$  under gauge transformations.

Now the configuration space of the Yang-Mills equation is also a space of connections, and the only constraint in Yang-Mills theory is the Gauss law. Thus in terms of the new variables, canonical quantum gravity is very similar to  $SU(2)$  Yang-Mills theory with some extra constraints! This is one of the main advantages of the new variables: they allow techniques from Yang-Mills theory to be imported to quantum gravity. Another more technical advantage is that in the new variables, the constraints are polynomial functions of the analogs of position and momentum (and their derivatives). Because the nonrenormalizability of quantum gravity in the traditional metric formulation was closely related to the non-polynomial nature of the constraints, this created a lot of excitement. Unfortunately, at the present time the Hamiltonian constraint still presents thorny problems.

Shortly after the discovery of the new variables, Rovelli and Smolin [45] used them to develop a ‘loop representation’ of quantum gravity, and this is when the relationship to knot theory became very apparent. The idea of the loop representation of a gauge theory had been developed by Gambini and Trias [31], and basically it consists of writing all the equations one can in terms of ‘Wilson loops’. Wilson loops are certain functions on the space of connections: at a point  $A \in \mathcal{A}$ , the value of the Wilson loop is just the trace of the holonomy of  $A$  around some loop  $\alpha$  in  $S$ , taken in some (finite-dimensional) representation  $\rho$  of  $G$ , written:

$$\text{tr}(\rho(T \exp \int_{\alpha} A)).$$

One reason why physicists like them is that they are invariant under gauge transformations, and one expects the physically observable aspects of a gauge theory to be gauge-invariant.

Rovelli and Smolin argued that a state  $\psi$  of quantum gravity should give rise to link invariant  $\hat{\psi}$ , the ‘loop transform’ of  $\psi$ , whose value on the link with components given by the loops  $\alpha_1, \dots, \alpha_n$  is

$$\hat{\psi}(\alpha_1, \dots, \alpha_n) = \int_{\mathcal{A}} \prod_{i=1}^n \text{tr}(\rho((T \exp \int_{\alpha_i} A)) \psi(A) \mathcal{D}A,$$

where  $\rho$  is the fundamental representation and  $\mathcal{D}A$  is the purely formal ‘Lebesgue measure’ on  $\mathcal{A}$ . The reason  $\hat{\psi}$  should be a link invariant is that the diffeomorphism constraint says  $\psi$  is invariant under diffeomorphisms of  $S$ ! Of course, this argument is merely heuristic, owing to the mysterious nature of  $\mathcal{D}A$ , but it is no worse than much of the reasoning in quantum gravity. Indeed, a similar sort of argument led Witten [48] to discover the relation between knot theory and another quantum field theory, Chern-Simons theory. In fact, the link invariant coming from  $SU(2)$  Chern-Simons theory, namely the Kauffman bracket, appears to be the loop transform of a state of quantum gravity ‘with cosmological constant’, meaning that the Einstein equation has been modified to give the vacuum a nonzero stress-energy tensor. We will not go into this here, since there are a number of expository treatments already [11, 12, 14, 18, 41], but certainly it is one of the main reasons for interest in the interface between knots and quantum gravity.

There is much here that needs to be made more precise in the following sections. However, we are at least in a position now to describe what needs

to be done. First, we need to develop integration theory on  $\mathcal{A}$ , in order to escape the use of purely formal entities like the ‘Lebesgue measure’  $\mathcal{D}\mathcal{A}$ . In Section 2 we do this when  $\mathcal{A}$  is the space of smooth connections on any principal  $G$ -bundle  $P$  over a manifold  $S$ , where  $G$  is a compact connected Lie group and  $S$  is real-analytic. There is an especially nice ‘generalized measure’ on  $\mathcal{A}$  which is a substitute for the nonexistent Lebesgue measure. Using it we can define  $L^2(\mathcal{A})$ , and it turns out that the gauge-invariant subspace of  $L^2(\mathcal{A})$  is spanned by ‘spin network states’. These are described by graphs analytically embedded in  $S$ , with oriented edges labelled by representations of  $G$ , and with vertices labelled by intertwining operators from the tensor product of representations labelling ‘incoming’ edges to the tensor product of representations labelling ‘outgoing’ edges. Equipped with this mathematical technology, we then return to quantum gravity.

## 2 Spin Networks

In quantum field theory computations, physicists often try to do integrals over a space  $\mathcal{A}$  of connections using a strange thing they call ‘Lebesgue measure’ on  $\mathcal{A}$ , usually written  $\mathcal{D}\mathcal{A}$ . This exploits the fact that  $\mathcal{A}$  is an affine space, or, arbitrarily choosing one point as the origin, a vector space. The idea is that one should be able to pick a basis for  $\mathcal{A}$ , thus setting up an isomorphism

$$\mathcal{A} \cong \mathbf{R}^\infty,$$

and then, working in the coordinates defined by this basis, let

$$\mathcal{D}\mathcal{A} = \prod_{i=1}^{\infty} dx_i.$$

There are many problems with this idea, however! First, while  $\mathcal{A}$  can indeed be identified with an infinite-dimensional vector space, a basis of  $\mathcal{A}$  does not really give an isomorphism between  $\mathcal{A}$  and an infinite product of copies of  $\mathbf{R}$ . Second, there is no good theory of infinite products of arbitrary measure spaces. Of course one can be more sophisticated about these issues, but the fact remains that there is no such thing as ‘Lebesgue measure’ on an infinite-dimensional vector space. Thus it is not surprising that when physicists actually do integrals over  $\mathcal{A}$  using ‘Lebesgue measure’ they often get infinite or ill-defined answers until they perform various sneaky tricks.

On the other hand, there is a different way of doing these integrals which is used in lattice gauge theory. Lattice gauge theory is an approximation to gauge theory on  $\mathbf{R}^n$  in which one replaces the continuum by an infinite graph having the points of a lattice as vertices and the line segments between neighboring vertices as edges. A ‘connection’ on the lattice is simply an assignment of an element  $g_e$  of the gauge group  $G$  to each edge  $e$  of the graph, representing the effect of parallel transport along  $e$ . The space of connections in this context is thus very different: it is

$$\mathcal{A} \cong G^\infty,$$

not an infinite product of copies of  $\mathbf{R}$ , but an infinite product of copies of  $G$ ! Now, while an infinite product of arbitrary measure spaces is ill-defined, an infinite product of *probability* measure spaces *is* well-defined. (Recall that a probability measure space is a space  $X$  with positive measure  $\mu$  such that  $\int_X \mu = 1$ .) If  $G$  is compact, it is equipped with a very natural probability measure, namely Haar measure, the unique left- and right-invariant Borel measure  $dg$  with  $\int_G dg = 1$ . So in lattice gauge theory with compact gauge group, one can work with the measure

$$\mathcal{D}A = \prod_e dg_e$$

on  $\mathcal{A}$ , instead of the nonexistent ‘Lebesgue measure’. This is precisely what is done.

Now, while lattice gauge theory is easier to make rigorous, it has the disadvantage of being only an approximation to the continuum theories we are really interested in. Indeed, in practice much work in lattice gauge theory is computational in nature. Here one uses not a lattice but a finite graph, so that  $\mathcal{A}$  becomes a *finite* product of copies of  $G$ , and one numerically calculates integrals over  $\mathcal{A}$  by Monte Carlo methods. To apply these results to gauge theory on  $\mathbf{R}^n$  one must then investigate not only the ‘continuum limit’ in which the lattice spacing goes to zero, but also the ‘large-volume limit’.

Approximating the continuum by a lattice or a fixed finite graph is particularly distressing in the case of general relativity, which is so intimately connected to the differential geometry of manifolds. It would be nice if one could have ones cake and eat it too, working with connections in the continuum context and preserving diffeomorphism-invariance, but still thinking of

the space of connections as something like a product of copies of  $G$ . This is precisely what the theory of ‘generalized measures’ on the space of connections seeks to achieve.

The traditional approach to measure theory is to pick a  $\sigma$ -algebra of ‘measurable’ subsets of some space  $X$ , assign measures to them in a manner satisfying some axioms, and then define a vector space of ‘integrable’ complex-valued functions on  $X$ . Then one figures out how to integrate functions in this space, shows that integration is linear, and shows how to pass limits through integrals under certain conditions. Then, typically, one forgets the proofs of these results and simply uses them. A more modern approach is to shortcut this process and simply choose a space  $\text{Fun}(X)$  of functions on  $X$ , equip it with a topology, and define a ‘generalized measure’  $\mu$  on  $X$  to be a continuous linear functional

$$\mu: C \rightarrow \mathbf{C},$$

writing  $\mu(f)$  as  $\int_X f d\mu$  solely out of deference to tradition. Actually, of course, the modern approach complements the old approach rather than replaces it; they are related by many useful theorems, some of which we have summarized elsewhere [9].

Now suppose that  $P$  is a principal  $G$ -bundle over  $S$ , where  $G$  is compact and connected, and let  $\mathcal{A}$  be the space of smooth connections on  $P$ . To define generalized measures on  $\mathcal{A}$  we need to choose the space  $\text{Fun}(\mathcal{A})$  of functions we want to integrate. Following the idea of the loop representation, for example, we could choose  $\text{Fun}(\mathcal{A})$  to be the algebra of functions on  $\mathcal{A}$  generated by all Wilson loops, or a completion of this algebra in some topology. In fact, this is how Ashtekar and Isham [5] proceeded in their original attempt to use ideas from the loop representation to set up a rigorous integration theory on  $\mathcal{A}$ . Subsequent work by Ashtekar, Lewandowski, and the author [6, 7, 9, 10, 37] improved things in a number of ways. For one, it turns out to be simpler to let  $\text{Fun}(\mathcal{A})$  contain arbitrary continuous functions of the holonomy of  $A$  along *paths* in  $S$ . Also, working with smooth paths turns out to be a nuisance, because they can intersect each other in horribly complicated ways. For this reason we shall follow Ashtekar and Lewandowski and require  $S$  to be a real-analytic manifold, and work with piecewise real-analytic paths. In a sense this is not so drastic, because every compact smooth manifold can be given a real-analytic structure. One pays a price, however, since one is

no longer doing differential geometry in the category of smooth manifolds. Eventually it would be nice to understand the smooth case too.

In any event, let us define  $\text{Fun}(A)$  as follows. If  $\gamma: [0, 1] \rightarrow S$  is a piecewise analytic path, let  $\mathcal{A}_\gamma$  denote the space of smooth maps  $F: P_{\gamma(0)} \rightarrow P_{\gamma(1)}$  that are compatible with the right action of  $G$  on  $P$ :

$$F(xg) = F(x)g.$$

Note that for any connection  $A \in \mathcal{A}$ , the parallel transport map

$$T \exp \int_\gamma A : P_{\gamma(0)} \rightarrow P_{\gamma(1)}$$

lies in  $\mathcal{A}_\gamma$ . If we fix a trivialization of  $P$  at the endpoints of  $\gamma$ , we can identify  $\mathcal{A}_\gamma$  with the group  $G$ . This makes  $\mathcal{A}_\gamma$  into a compact manifold in a manner which one can check is independent of the trivialization. Let  $\text{Fun}_0(\mathcal{A})$  be the algebra of functions on  $\mathcal{A}$  generated by those of the form

$$\psi(A) = f\left(T \exp \int_{\gamma_1} A, \dots, T \exp \int_{\gamma_n} A\right) \quad (1)$$

where  $\gamma_1, \dots, \gamma_n$  are piecewise analytic paths in  $S$ , and  $f$  is a continuous function on  $\mathcal{A}_{\gamma_1} \times \dots \times \mathcal{A}_{\gamma_n}$ . Then let  $\text{Fun}(\mathcal{A})$  be the completion of  $\text{Fun}_0(\mathcal{A})$  in the sup norm:

$$\|\psi\|_\infty = \sup_{A \in \mathcal{A}} |\psi(A)|.$$

It is easy to check that the Wilson loops lie in this algebra; in fact, they lie in  $\text{Fun}_0(\mathcal{A})$ .

A generalized measure on  $\mathcal{A}$  is then defined to be a continuous linear functional on  $\text{Fun}(\mathcal{A})$ . This may seem rather abstract, so let us see how to get our hands on one. There exists a nice general recipe for constructing *any* generalized measure on  $\mathcal{A}$ , which we have described elsewhere [10, 12], but for now let us concentrate on a simple and important example: the uniform generalized measure  $\mu_u$ . This is a kind of replacement for the nonexistent ‘Lebesgue measure’ on  $\mathcal{A}$ , and as we shall see, it is closely modelled after the measure  $\mathcal{D}A$  in lattice gauge theory.

To define  $\mu_u$ , first we define it as a linear functional on  $\text{Fun}_0(\mathcal{A})$ . Because we are working with real-analytic paths, it turns out that any  $\psi \in \text{Fun}_0(\mathcal{A})$  can be written in the form given by equation (1) with the paths  $\{\gamma_i\}$  forming

an *embedded graph* in  $S$ . By this we mean that each path  $\gamma_i: [0, 1] \rightarrow S$  is one-to-one and restricts to an embedding of  $(0, 1)$ , and the images  $\gamma_i[0, 1]$  and  $\gamma_j[0, 1]$  intersect, if at all, only at their endpoints when  $i \neq j$ . For functions  $\psi$  written in this special form we define

$$\int_{\mathcal{A}} \psi d\mu_u = \int_{G^n} f(g_1, \dots, g_n) dg_1 \cdots dg_n.$$

Here we are using trivializations of  $P$  at the endpoints of the paths  $\gamma_i$  to identify  $\mathcal{A}_{\gamma_1} \times \cdots \times \mathcal{A}_{\gamma_n}$  with  $G^n$ , but the right-hand side is independent of the choice of trivializations, because Haar measure is right- and left-invariant. All the real work goes into checking that the right-hand side does not depend on *how* we wrote  $\psi$  in this special form involving an embedded graph. Given that, it is easy to check that  $\mu_u$  is a linear functional on  $\text{Fun}_0(\mathcal{A})$ . The bound

$$\left| \int_{\mathcal{A}} \psi d\mu_u \right| \leq \|\psi\|_{\infty}$$

then holds because Haar measure is a probability measure. Since  $\text{Fun}_0(\mathcal{A})$  is dense in  $\text{Fun}(\mathcal{A})$ , this bound implies that  $\mu_u$  extends uniquely to a continuous linear functional on all of  $\text{Fun}(\mathcal{A})$ , which we again call  $\mu_u$ . This is the uniform generalized measure on the space of connections!

If we examine what we have just done, the relationship to lattice gauge theory should become clear. Instead of working with a fixed graph embedded in  $S$  as one does in lattice gauge theory, we have considered *all possible* embedded graphs  $\gamma = \{\gamma_i\}$  in  $S$ . Each one of these indeed has the topology of a graph with the paths  $\gamma_i$  as edges and the endpoints  $\gamma_i(0), \gamma_i(1)$  as vertices. Following the spirit of lattice gauge theory, we can define a finite-dimensional space of ‘connections’ on  $\gamma$ ,

$$\mathcal{A}_{\gamma} = \mathcal{A}_{\gamma_1} \times \cdots \times \mathcal{A}_{\gamma_n},$$

on which there is a natural measure, namely a product of copies of Haar measure, one copy for each edge of  $\gamma$ . To do the integral of a function that only depends on the holonomies along the edges of  $\gamma$ , we simply use this natural measure. The key fact is that although a function  $\psi \in \text{Fun}_0(\mathcal{A})$  can be expressed in terms of many different embedded graphs in this way — after all, if it can be expressed in terms of a graph  $\gamma$ , it can also be expressed in terms of any graph  $\gamma'$  containing  $\gamma$  — the answer we get for the integral of

$\psi$  is independent of this choice. This is why we obtain a well-defined linear functional  $\mu_u$  on  $\text{Fun}_0(\mathcal{A})$ , which then extends to all of  $\text{Fun}(\mathcal{A})$ . In short, we are getting a generalized measure on  $\mathcal{A}$  as a kind of ‘limit’ of measures on the spaces of connections on all graphs embedded in  $S$ ! This can be made perfectly precise using the language of projective limits [7].

We can now define the Hilbert space  $L^2(\mathcal{A})$  as the completion of  $\text{Fun}(\mathcal{A})$  with respect to the norm

$$\|\psi\|_2 = \left( \int_{\mathcal{A}} |\psi|^2 d\mu_u \right)^{1/2}.$$

In the special case where  $G = \text{SU}(2)$  and  $S$  is a compact oriented 3-manifold, this Hilbert space will be our space of ‘kinematical states’ for quantum gravity in the new variables formalism. The Gauss law constraint then amounts to requiring that the state  $\psi \in L^2(\mathcal{A})$  be invariant under gauge transformations, so it is important to find gauge-invariant vectors in  $L^2(\mathcal{A})$ . Finding these vectors is actually of interest no matter what  $G$  and  $S$  happen to be. It turns out that these vectors can also be thought of as wavefunctions on the quotient of  $\mathcal{A}$  by the group  $\mathcal{G}$  of smooth gauge transformations [12], so we will denote the space of gauge-invariant vectors in  $L^2(\mathcal{A})$  by  $L^2(\mathcal{A}/\mathcal{G})$ .

The most obvious examples of vectors in  $L^2(\mathcal{A}/\mathcal{G})$  are the Wilson loops. If we have an analytic loop  $\alpha$  in  $S$  and a representation  $\rho$  of  $G$ , the function

$$\psi(A) = \text{tr}(\rho(T \exp \int_{\alpha} A))$$

is gauge-invariant, and it lies in  $L^2(\mathcal{A})$ . More generally, any product of Wilson loops is a gauge-invariant element of  $L^2(\mathcal{A})$ .

More generally still, we can get vectors in  $L^2(\mathcal{A}/\mathcal{G})$  from ‘spin networks’ [13]. Take a graph  $\gamma$  embedded in  $S$  and label each of its edges  $e$  with a representation  $\rho_e$  of  $G$ . Let

$$H_e(A) = \rho_e(T \exp \int_e A).$$

If we trivialize  $P$  at the vertices of  $\gamma$ , and pick a basis for  $\rho_e$ , we can think of  $H_e(A)$  as a matrix  $H_e(A)_j^i$ . Now form the tensor product of all these matrices, one for each edge of  $\gamma$ . We get a big tensor  $H(A)$  having one superscript and one subscript for each edge; it is too ugly to bother writing down, but we hope the reader gets the idea.

Next, for each vertex  $v$  of  $\gamma$  let  $S(v)$  be the set of edges having  $v$  as ‘source’ — where the source of an edge  $\gamma_i$  is defined to be  $\gamma_i(0)$  — and let  $T(v)$  be the set of edges having  $v$  as ‘target’ — where the target of  $\gamma_i$  is  $\gamma_i(1)$ . For each vertex  $v$ , pick an intertwining operator

$$I_v: \bigotimes_{e \in T(v)} \rho_e \rightarrow \bigotimes_{e \in S(v)} \rho_e.$$

We can think of  $I_v$  as a tensor with one superscript for each edge  $e \in T(v)$  and one subscript for each edge  $e \in S(v)$ . Then form the tensor product of all these tensors  $I_v$ , one for each vertex. We get a big tensor  $I$ . Then form the tensor product of  $H(A) \otimes I$ . Note that each superscript of  $H(A)$  corresponds to a particular subscript of  $I$  and vice versa, because each edge of  $\gamma$  lies in  $S(v)$  for one vertex  $v$  and lies in  $T(w)$  for one vertex  $w$ . So we can contract the tensor  $H(A) \otimes I$  to get a number, which of course depends on  $A$ . This is our ‘spin network state’  $\psi(A)$ . Note that a Wilson loop is just a special case of a spin network with only one edge and one vertex, with the intertwining operator taken to be the identity operator.

One can show directly from this explicit definition that the spin network states are gauge-invariant and lie in  $L^2(\mathcal{A})$ , in fact in  $\text{Fun}_0(\mathcal{A})$ . But we want to show more: we want to show they span the whole space of gauge-invariant vectors in  $L^2(\mathcal{A})$ . For this a more abstract approach is better.

The group  $\mathcal{G}$  acts on  $\mathcal{A}$ , and in fact there is a unitary representation of  $\mathcal{G}$  on  $L^2(\mathcal{A})$  given by

$$g\psi(A) = \psi(g^{-1}A).$$

To see this it is useful to introduce work ‘one embedded graph at a time’. One can show that for any embedded graph  $\gamma$ ,  $L^2(\mathcal{A}_\gamma)$  can be identified with the smallest closed subspace of  $L^2(\mathcal{A})$  containing all functions of the form given in equation (1). If we do a gauge transformation  $g \in \mathcal{G}$  on the connection  $A$ , the holonomy along any edge  $\gamma_i$  transforms to

$$T \exp \int_{\gamma_i} gA = g(\gamma_i(1)) (T \exp \int_{\gamma_i} A) g(\gamma_i(0))^{-1}, \quad (2)$$

so if  $\psi$  lies in  $L^2(\mathcal{A}_\gamma)$ , so does  $g\psi$ . Thus we get a representation of  $\mathcal{G}$  on each subspace  $L^2(\mathcal{A}_\gamma)$ , and these representations are unitary because Haar measure is left- and right-invariant. Since the union of the subspaces  $L^2(\mathcal{A}_\gamma)$  is dense in  $L^2(\mathcal{A})$ , we obtain a unitary representation of  $\mathcal{G}$  on all of  $L^2(\mathcal{A})$ .

Let  $L^2(\mathcal{A}_\gamma/\mathcal{G}_\gamma)$  denote the gauge-invariant subspace of  $L^2(\mathcal{A}_\gamma)$ . Since the action of  $\mathcal{G}$  on  $L^2(\mathcal{A})$  preserves each subspace  $L^2(\mathcal{A}_\gamma)$ , the union of the subspaces  $L^2(\mathcal{A}_\gamma/\mathcal{G}_\gamma)$  must span  $L^2(\mathcal{A}/\mathcal{G})$ . So, what does  $L^2(\mathcal{A}_\gamma/\mathcal{G}_\gamma)$  look like? For this we need a very precise picture of the action of  $\mathcal{G}$  on  $L^2(\mathcal{A}_\gamma)$ .

Write  $E$  for the set of edges of  $\gamma$ , and  $V$  for the set of vertices. Then picking a trivialization of  $P$  over the vertices of  $\gamma$ , we obtain an isomorphism

$$\begin{aligned} L^2(\mathcal{A}_\gamma) &\cong L^2(G^E) \\ &\cong \bigotimes_{e \in E} L^2(G). \end{aligned}$$

To see how  $\mathcal{G}$  acts on the right-hand side, we use the Peter-Weyl theorem. Note that  $G \times G$  acts on  $G$  by

$$(g_1, g_2)(h) = g_1 h g_2^{-1},$$

giving a unitary representation of  $G \times G$  on  $L^2(G)$ . The Peter-Weyl theorem describes how  $L^2(G)$  decomposes into irreducible unitary representations of  $G \times G$ . These are all of the form  $\rho_1 \otimes \rho_2$ , where  $\rho_1$  and  $\rho_2$  are irreducible unitary representations of  $G$ . Let  $R$  be a set containing one irreducible unitary representation of  $G$  from each equivalence class. Then the Peter-Weyl theorem says that

$$L^2(G) \cong \bigoplus_{\rho \in R} \rho \otimes \rho^*.$$

It follows that

$$L^2(\mathcal{A}_\gamma) \cong \bigotimes_{e \in E} \bigoplus_{\rho \in R} \rho \otimes \rho^*,$$

and in terms of this description, any  $g \in \mathcal{G}$  acts on  $L^2(\mathcal{A}_\gamma)$  as the operator

$$\bigotimes_{e \in E} \bigoplus_{\rho \in R} \rho(g(s(e)) \otimes \rho^*(g(t(e))).$$

The reason is that the holonomy of  $A$  along each edge  $e$  transforms in precisely the manner suited to applying the Peter-Weyl theorem, as we saw in equation (2).

Next, using the associativity of tensor product over direct sum, we obtain

$$L^2(\mathcal{A}_\gamma) \cong \bigoplus_{\rho \in R^E} \bigotimes_{e \in E} \rho_e \otimes \rho_e^*,$$

where  $R^E$  is the set of all labellings of edges  $e \in E$  by representations  $\rho_e \in R$ . Grouping the edges by their source and target, this gives

$$L^2(\mathcal{A}_\gamma) \cong \bigoplus_{\rho \in R^E} \bigotimes_{v \in V} \left( \bigotimes_{e \in S(v)} \rho_e \otimes \bigotimes_{e \in T(v)} \rho_e^* \right).$$

In these terms any  $g \in \mathcal{G}$  acts on  $L^2(\mathcal{A}_\gamma)$  as the operator

$$\bigoplus_{\rho \in R^E} \bigotimes_{v \in V} \left( \bigotimes_{e \in S(v)} \rho_e(g_v) \otimes \bigotimes_{e \in T(v)} \rho_e^*(g_v) \right).$$

Thus we have

$$L^2(\mathcal{A}_\gamma/\mathcal{G}_\gamma) \cong \bigoplus_{\rho \in R^E} \bigotimes_{v \in V} \text{Inv} \left( \bigotimes_{e \in S(v)} \rho_e \otimes \bigotimes_{e \in T(v)} \rho_e^* \right),$$

where for any representation  $\lambda$  of  $G$ ,  $\text{Inv}(\lambda)$  is the subspace of  $G$ -invariant vectors. But  $\text{Inv}(\lambda_2 \otimes \lambda_1^*)$  is isomorphic to the space  $\text{Hom}(\lambda_1, \lambda_2)$  of intertwining operators from  $\lambda_1$  to  $\lambda_2$ , so finally we have

$$L^2(\mathcal{A}_\gamma/\mathcal{G}_\gamma) \cong \bigoplus_{\rho \in R^E} \bigotimes_{v \in V} \text{Hom} \left( \bigotimes_{e \in T(v)} \rho_e, \bigotimes_{e \in S(v)} \rho_e \right).$$

In other words, a complete set of gauge-invariant vectors in  $L^2(\mathcal{A}_\gamma)$  is obtained by labelling each edge of  $\gamma$  with an irreducible unitary representation of  $G$  and labelling each vertex of  $\gamma$  with an intertwining operator from the tensor product of representations labelling incoming edges to the tensor product of representations labelling outgoing edges. If one unravels the logic of the proof, one sees that these are special cases of the spin network states described more explicitly above!

Our work has shown that spin network states span  $L^2(\mathcal{A}/\mathcal{G})$ . But in fact we have shown more. For each embedded graph  $\gamma$  we obtain an orthonormal basis of  $L^2(\mathcal{A}_\gamma/\mathcal{G}_\gamma)$  by letting  $\rho$  range over all labellings of edges by irreducible unitary representations, and, for each  $\rho$ , picking an orthonormal basis of

$$\text{Hom} \left( \bigotimes_{e \in T(v)} \rho_e, \bigotimes_{e \in S(v)} \rho_e \right)$$

for each vertex  $v$ . While it is a nuisance to assemble all these orthonormal bases into a single orthonormal basis for all of  $L^2(\mathcal{A}/\mathcal{G})$ , for practical computations having an orthonormal basis for each graph is usually sufficient.

Of course, for *really* practical computations one might need a recipe for picking an orthonormal basis of the space of intertwining operators

$$\text{Hom} \left( \bigotimes_{e \in T(v)} \rho_e, \bigotimes_{e \in S(v)} \rho_e \right).$$

For example, Brüggemann [19] has tried some computer simulations of quantum gravity using spin networks, and computers are notorious for wanting everything to be very explicit. To pick such an orthonormal basis, one needs to understand the representation theory of  $G$ . Luckily, the representation theory of  $\text{SU}(2)$  is quite simple, so let us consider that case. A more detailed treatment can be found in the work of Rovelli and Smolin [46].

The irreducible unitary representations of  $\text{SU}(2)$  can be labelled by their dimension  $d = 1, 2, 3, \dots$ , or equivalently by their ‘spin’  $j = 0, \frac{1}{2}, 1, \dots$ , where  $d = 2j + 1$ . The tensor product of two such representations decomposes as follows:

$$j_1 \otimes j_2 = |j_1 - j_2| \oplus |j_1 - j_2| + 1 \oplus \dots \oplus j_1 + j_2.$$

This means that for a *trivalent* graph  $\gamma$  we do not need to worry much about labelling the vertices with intertwining operators. Suppose, for example, that a vertex has two incoming edges labelled with spins  $j_1$  and  $j_2$ , and one outgoing edge labelled  $j_3$ . Unless the Clebsch-Gordon condition

$$|j_1 - j_2| \leq j_3 \leq j_1 + j_2, \quad j_1 + j_2 + j_3 \in \mathbf{Z}$$

holds, the representation  $j_3$  will not appear as a summand of the tensor product  $j_1 \otimes j_2$ , and there is no way to get a nonzero spin network state from this labelling. If the Clebsch-Gordon condition does hold,  $j_3$  appears with multiplicity 1 in  $j_1 \otimes j_2$ , so up to a constant factor which one must fix in some standard way, there is no choice about which intertwining operator to pick.

Graphs that are not trivalent can be reduced to the trivalent case as follows. Take each  $n$ -valent vertex and replace it in ones mind by a  $n$ -leaved tree each of whose vertices is trivalent. If the original vertex was  $n$ -valent, this tree will have  $n - 3$  new ‘internal edges’. A basis of intertwining operators

for the original vertex is then given by all labellings of these internal edges by spins satisfying the Clebsch-Gordon condition. Of course, there are many different trees with  $n$  leaves, so there is a great deal of arbitrariness in this choice of basis. However, to change from one basis to another simply requires repeated use of some matrices known as the  $6j$  symbols, which are familiar in  $SU(2)$  representation theory and whose generalizations play an important role in topological quantum field theory [26, 36].

Before returning to a more thorough treatment of the new variables and the use of spin networks of quantum gravity, let us say a little about the history of spin networks. Because of the immense difficulties they have had in quantizing gravity, physicists have often considered the possibility that our picture of spacetime as a manifold breaks down at the Planck scale. This has led to various attempts to reformulate physics in terms of more discrete, or combinatorial, ideas. In fact, spin networks were invented in the early 1970s by Penrose [39] in one such attempt. However, while our spin networks involve graphs *embedded* in a pre-existing manifold that represents space, his spin networks were *abstract* graphs labelled by representations and intertwining operators. In other words, rather than serving as a tool for describing the geometry of a spacetime manifold, his spin networks were intended as a purely combinatorial substitute for a spacetime manifold. The problem with this sort of radical idea has always been bridging the gap between it and existing theories of physics. Thus the more recent discovery that spin networks also arise naturally in attempts to quantize Einstein's equation is quite intriguing.

If we succeeded in constructing quantum gravity as a field theory on a manifold, we might simply decide to forget about more combinatorial approaches to quantum gravity. However, as we shall see, the Hamiltonian constraint poses serious problems even in the new variables formalism, so-called 'ultraviolet problems', which might be due to falsely extrapolating our picture of spacetime as a manifold to arbitrarily small length scales. Thus it is still worth keeping in mind the possibility that some combinatorial approach is the fundamental one. For this reason, any clues about the relation between 'abstract' and 'embedded' spin networks are likely to be interesting.

Luckily, there are quite a few clues along these lines! While Penrose's ideas yield a purely combinatorial recipe for defining an invariant of abstract  $SU(2)$  spin networks, this invariant can also be described in terms of spin networks embedded in  $\mathbf{R}^3$ . One can obtain a generalized measure  $\mu$  on the

space  $\mathcal{A}$  of  $SU(2)$  connections on  $\mathbf{R}^3$  by taking a finite Borel measure  $\mu_0$  on the subspace  $\mathcal{A}_0$  of flat connections, and defining

$$\int_{\mathcal{A}} \psi(A) d\mu = \int_{\mathcal{A}_0} \psi(A) d\mu_0. \quad (3)$$

Then if  $\psi$  is a spin network state associated to some graph  $\gamma$ , the integral  $\int_{\mathcal{A}} \psi(A) d\mu$  does not depend on the choice of embedding of  $\gamma$ , essentially because the holonomy of a *flat* connection along a path does not change when the path is deformed. Thus we obtain an invariant of abstract spin networks. If  $\mu_0$  is a probability measure, this is the same invariant that one can define combinatorially following the ideas of Penrose (while carefully adjusting his sign conventions [46]).

A striking generalization of this idea arises from the work of Witten [48] on Chern-Simons theory and the Jones polynomial, and its subsequent reinterpretation using quantum groups by Reshtekhin and Turaev [42]. Heuristically, Chern-Simons theory defines an isotopy invariant of spin networks embedded in  $S^3$  by means of a measure on the space of connections:

$$\int_{\mathcal{A}} \psi(A) e^{\frac{ik}{4\pi} \int_{S^3} \text{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)} \mathcal{D}A \quad (4)$$

Indeed, Witten was able by formal manipulations to compute these invariants in certain cases. Unfortunately, unlike equation (3), this formula is difficult to make rigorous. It involves the purely formal ‘Lebesgue measure’  $\mathcal{D}A$ , and replacing  $\mathcal{D}A$  by the uniform generalized measure does not help much, since the quantity integrated against it does not lie in  $\text{Fun}(\mathcal{A})$ . In fact, one cannot really compute these spin network invariants using a generalized measure in our precise sense of the term [12]. One reason, though not the only one, is that the invariants depend on a choice of extra structure known as a ‘framing’. Nonetheless, Reshetikhin and Turaev were able to rigorously construct these isotopy invariants of framed embedded spin networks by a combinatorial procedure that generalizes Penrose’s recipe to quantum groups. In the case of links labelled by the spin- $\frac{1}{2}$  representation of  $SU(2)$ , this invariant is just the Kauffman bracket.

These ideas interact with the subject of quantum gravity in many ways. In Section 1 we cited some expository accounts of the relation between Chern-Simons theory and quantum gravity in 4-dimensional spacetime. However, spin networks and Chern-Simons theory are also closely related to quantum

gravity with cosmological constant in 3-dimensional spacetime [28, 36, 40, 43, 44, 47, 49]. In this case, quantum gravity can be rigorously described *either* by starting with Einstein’s equation on a smooth 3-manifold, phrased in terms of an appropriate version of the new variables, *or* purely combinatorially in terms of a simplicial 3-manifold with edges labelled by spins. Better yet, although there is still a bit of work left in proving this rigorously, it appears that the two approaches are equivalent! In short, in this toy model the question as to whether spacetime is continuous or discrete has no simple yes-or-no answer; in some sense, the answer is ‘both’.

It is natural to wonder if the same might be true in 4 dimensions. There are a few hints here. On the one hand, Einstein’s equation in the new variables formulation has a simpler relative called  $BF$  theory [12], which is much easier to canonically quantize. On the other hand, Crane and Yetter have combinatorially constructed a relatively simple topological quantum field theory in 4 dimensions using spin networks [22]. It is not known if these are equivalent, though there are some clues suggesting it, such as the relation of both to Chern-Simons theory. This would be worth understanding better, even though we expect full-fledged quantum gravity in 4 dimensions to be more complicated.

### 3 The New Variables

Now let us return to general relativity and describe more precisely how the new variables make it look more like other gauge theories. In its original formulation, general relativity is all about a *metric* on spacetime, while gauge theories are all about a *connection* on some bundle over spacetime. Of course there is a connection involved in general relativity too, the Levi-Civita connection, but this is traditionally regarded as a subsidiary entity, since it can be computed starting from the metric. To emphasize the gauge-theoretic aspects of general relativity, one needs to rewrite it so a connection has the starring role, and the metric appears as more of a minor character. There have been many different attempts to do this, and for a more thorough exploration of them the reader will have to turn elsewhere [38]. Here we only describe two: the Palatini formalism, and the Plebanski formalism. The latter is the one directly related to the ‘new variables’, but the former serves as a good warmup exercise. This section makes somewhat greater demands

on the reader’s acquaintance with differential geometry, and also uses some variational calculus. Luckily, there happens to be a textbook that explains most of what we need [14]. We will skip over all sorts of important subtleties, which are discussed in Ashtekar’s books [2, 3].

In the Palatini formalism there will be two basic fields, a connection and a ‘soldering form’. The clever idea (in the modern version of this approach) is to fix an oriented bundle  $\mathcal{T}$  over  $M$  that is isomorphic to the tangent bundle  $TM$ , but not canonically so. We can think of  $\mathcal{T}$  as a kind of ‘imitation tangent bundle’. Physicists usually call it, or any of its fibers, the ‘internal space’. We assume  $\mathcal{T}$  is equipped with a Lorentzian metric  $\eta$  — the ‘internal metric’ — and assume that the spacetime metric  $g$  is obtained from  $\eta$  via an isomorphism  $e: TM \rightarrow \mathcal{T}$ . In other words,

$$g(v, w) = \eta(e(v), e(w)) \tag{5}$$

for any vector fields  $v$  and  $w$ . We may regard  $e$  as a  $\mathcal{T}$ -valued 1-form, and then it is called the soldering form. We should add, in the interests of cultural literacy, that  $e$  is also called the ‘coframe field’, while  $e^{-1}$  is known as the ‘frame field’, ‘tetrad field’, or ‘vierbein’.

In Palatini formalism the two basic fields are this soldering form  $e$  and a connection  $A$  on  $\mathcal{T}$  preserving the metric  $\eta$ , usually called a ‘Lorentz connection’. One virtue of this formalism is that it makes sense even when  $e: TM \rightarrow \mathcal{T}$  is *not* an isomorphism. Thus the Palatini formalism extends general relativity to certain cases where the metric  $g$  is degenerate.

In computations it is handy to use differential forms on  $M$  taking values in the exterior algebra bundle  $\Lambda\mathcal{T}$ . These form an algebra, where the product (written as  $\wedge$ ) is built from the exterior product in  $\Lambda\mathcal{T}$  together with the usual wedge product of differential forms. In complete analogy with the volume form on an oriented Lorentzian manifold, the orientation and internal metric on  $\mathcal{T}$  give rise to an ‘internal volume form’, i.e. a section  $\nu$  of  $\Lambda^4\mathcal{T}$ . This in turn yields a map from  $\Lambda^4\mathcal{T}$ -valued forms to ordinary differential forms, denoted by ‘tr’, and given by

$$\text{tr}(\nu \otimes \omega) = \omega$$

for any differential form  $\omega$  on  $M$ . Now the curvature of the connection  $A$  can, as usual, be regarded as an  $\text{End}(\mathcal{T})$ -valued 2-form, but the internal metric provides an isomorphism  $\mathcal{T} \cong \mathcal{T}^*$ , so we may think of it as  $\mathcal{T} \otimes \mathcal{T}$ -valued,

and then the fact that  $A$  is metric-preserving means the curvature actually takes values in  $\Lambda^2\mathcal{T}$ . We call this  $\Lambda^2\mathcal{T}$ -valued 2-form  $F$ .

One can then obtain the vacuum Einstein equation from a variational principle, starting with the ‘Palatini action’ given by

$$S_{Pal}(A, e) = \int_M \text{tr}(e \wedge e \wedge F).$$

Let us sketch how this goes. The idea is to compute the variation  $\delta S_{Pal}$ , demand that it vanish for all compactly supported variations  $\delta A$  and  $\delta e$ , and see what this implies. We will need to use the wonderful formula

$$\delta F = d_A \delta A, \tag{6}$$

where  $d_A$  denotes the exterior covariant derivative of  $\Lambda\mathcal{T}$ -valued forms with respect to the connection  $A$ , and  $\delta A$  is treated as a  $\Lambda^2\mathcal{T}$ -valued 1-form. We begin by computing the variation, or differential, of  $S_{Pal}$  as a function of  $A$  and  $e$ :

$$\begin{aligned} \delta S_{Pal} &= \int \delta \text{tr}(e \wedge e \wedge F) \\ &= \int \text{tr}(2\delta e \wedge e \wedge F + e \wedge e \wedge \delta F), \end{aligned}$$

using the standard rules for differentiation. Using equation (6) we obtain

$$\delta S_{Pal} = \int \text{tr}(2\delta e \wedge e \wedge F + e \wedge e \wedge d_A \delta A).$$

Finally, integrating by parts as one always does in these variational computations,

$$\delta S_{Pal} = 2 \int \text{tr}(\delta e \wedge e \wedge F - e \wedge d_A e \wedge \delta A).$$

This only vanishes for *all* compactly supported  $\delta A$ ,  $\delta e$  if

$$e \wedge F = 0, \quad e \wedge d_A e = 0.$$

These are our ‘classical equations of motion’.

When  $e$  is an isomorphism, these equations are really just the vacuum Einstein equation in disguise. First, one can show that in this case  $e \wedge d_A e = 0$  implies  $d_A e = 0$ . Then, using the isomorphism  $e: TM \rightarrow \mathcal{T}$  to transfer  $A$  to

a metric-preserving connection on  $TM$ , say  $\Gamma$ , the equation  $d_A e = 0$  implies that  $\Gamma$  is torsion-free, hence equal to the Levi-Civita connection of  $g$ . This lets us translate the equation  $e \wedge F = 0$  into an equation about the curvature of  $\Gamma$ , i.e., the Riemann tensor. When we do so, we obtain the vacuum Einstein equation!

The Plebanski formalism works in much the same way. Indeed, at first it may seem like just an unnecessarily complicated version of the Palatini formalism. As noted by Ashtekar, however, the *constraints* work out much more nicely when one tries to canonically quantize the theory. The Plebanski formalism is also called the ‘self-dual’ formalism, because it takes advantage of self-duality, a very special feature of 4 dimensions. Recall that given a 4-dimensional real inner product space  $V$ , the Hodge star operator maps the second exterior power of  $V$  to itself:

$$\star: \Lambda^2 V \rightarrow \Lambda^2 V.$$

Since  $\star^2 = 1$ , we can split any element  $\omega \in \Lambda^2 V$  into a self-dual and an anti-self-dual part:

$$\omega = \omega_+ + \omega_-, \quad \star \omega_{\pm} = \pm \omega_{\pm}.$$

Another way to think of this is as follows. The inner product gives an isomorphism between  $\text{End}(V) = V \otimes V^*$  and  $V \otimes V$ , which restricts to an isomorphism between  $\mathfrak{so}(V)$  and  $\Lambda^2 V$ . The splitting of  $\Lambda^2 V$  into self-dual and anti-self-dual parts then turns out to correspond to a splitting of the 6-dimensional Lie algebra  $\mathfrak{so}(V)$  into two 3-dimensional ones. Taking  $V = \mathbf{R}^4$  this gives

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3). \tag{7}$$

These simple facts have all sorts of ramifications for 4-dimensional physics and topology, especially in gauge theory, where a connection on a Riemannian manifold having self-dual curvature 2-form is automatically a solution of the Yang-Mills equation, called an ‘instanton’ [29]. When  $M$  is compact, the space of instantons modulo gauge transformations is finite-dimensional, and one can extract powerful invariants of the smooth structure of  $M$  from this ‘moduli space’ using techniques developed by Donaldson and others [25]. For example, one can use these to show that  $\mathbf{R}^4$  admits uncountably many different smooth structures, unlike any other  $\mathbf{R}^n$ ! Seiberg and Witten have

recently simplified some aspects of Donaldson theory by further recourse to ideas from physics, but self-duality still plays a key role.

The sort of self-duality relevant to the Plebanski formalism is rather different, though with tantalizing relationships to the above [21]. First, given a Lorentzian rather than Riemannian 4-manifold, one has  $\star^2 = -1$  on 2-forms. This means that the eigenvalues of the Hodge star operator are  $\pm i$ , so one needs to work with *complex-valued* 2-forms to take advantage of self-duality. When we tensor both sides of equation (7) with  $\mathbf{C}$ , we obtain

$$\mathfrak{so}(4, \mathbf{C}) \cong \mathfrak{sl}(2, \mathbf{C}) \oplus \mathfrak{sl}(2, \mathbf{C}).$$

For these reasons, the Plebanski formalism applies most naturally to *complexified* general relativity, and some extra work is needed to restrict to real-valued metrics.

Second, in the Plebanski formalism self-duality first shows up not with respect to ‘honest’ 2-forms, but with respect to sections of  $\Lambda^2 \mathbf{CT}$ , the second exterior power of the complexified internal space. The internal metric  $\eta$  extends by complexification to a bilinear pairing on  $\mathbf{CT}$ , which we again call  $\eta$ , and this together with the orientation give rise to an ‘internal’ Hodge star operator  $*$  acting on sections of  $\Lambda^2 \mathbf{CT}$ . We can thus split sections of this bundle into self-dual and anti-self-dual parts:

$$\omega = \omega_+ + \omega_-, \quad *\omega_{\pm} = \pm i\omega_{\pm}.$$

Now, the orthonormal frame bundle of  $\mathbf{CT}$  is a principal bundle  $P$  with structure group  $SO(4, \mathbf{C})$ . Assume we have a spin structure for  $\mathbf{CT}$ , that is, a double cover  $\tilde{P}$  of  $P$  with structure group  $\widetilde{SO}(4, \mathbf{C}) = \mathrm{SL}(2, \mathbf{C}) \oplus \mathrm{SL}(2, \mathbf{C})$ . Then  $\tilde{P}$  is the sum  $P_+ \oplus P_-$  of two principal bundles with structure group  $\mathrm{SL}(2, \mathbf{C})$ . A metric-preserving connection on  $\mathbf{CT}$  is usually called a ‘complex Lorentz connection’. By the above, a complex Lorentz connection  $A$  is equivalent to a pair of connections  $A_{\pm}$  on  $P_{\pm}$ , which we call the self-dual and anti-self-dual parts of  $A$ . The curvature of  $A_{\pm}$  is a 2-form  $F_{\pm}$  taking values in the associated bundle  $\mathrm{Ad}P_{\pm}$ , but sections of these bundles can be identified with self-dual (resp. anti-self-dual) sections of  $\Lambda^2 \mathbf{CT}$ . The curvature of  $A$  is a  $\Lambda^2 \mathbf{CT}$ -valued 2-form  $F$  with  $F = F_+ + F_-$ .

The two basic fields in the Plebanski formalism are a *complex-valued soldering form*, that is, 1-form on  $M$  with values in  $\mathbf{CT}$ , and a *self-dual*

Lorentz connection  $A_+$ , that is, a connection on  $P_+$ . The Plebanski action is given by:

$$S_{Pl}(A_+, e) = \int_M \text{tr}(e \wedge e \wedge F_+)$$

This is very much like the Palatini action, and, much as before, the classical equations of motion are

$$e \wedge F_+ = 0, \quad e \wedge d_{A_+} e = 0.$$

These give Einstein's equation for complex-valued metrics on  $M$  when  $e$  defines an isomorphism between the complexified tangent bundle of  $M$  and  $\mathbf{CT}$ . To see this, first note that  $e$  gives rise to a complex-valued metric  $g$  on  $M$  by equation (5). Moreover, we can use  $e$  to transfer  $A_+$  to a connection  $\Gamma_+$  on the complexified tangent bundle of  $M$ . The equation  $e \wedge d_{A_+} e = 0$  then implies  $\Gamma_+$  is torsion-free, hence equal to the self-dual part of the Levi-Civita connection  $\Gamma$  of  $g$ . We can transfer  $\Gamma$  back to  $\mathbf{CT}$  and get a connection  $A$  having  $A_+$  as its self-dual part. Then it turns out that  $e \wedge F_+ = e \wedge F$ , so as in the Palatini formalism we obtain Einstein's equation for  $g$ .

In terms of these 'new variables', as they are often called, the configuration space of general relativity is no longer the space of all metrics on  $S$ . Instead, it is the space  $\mathcal{A}_+$  of all connections on  $P_+$  restricted to  $S$ , or more precisely, to some fixed spacelike slice  $S_t \subset M$ . As in the metric formulation, one can separate the equations of motion into evolutionary equations and constraints. We shall not go through the calculations here, but simply state some of the results [2, 3]. Recall that in the metric formulation of general relativity, the analogs of the position and momentum are the fields  $q_{ij}$  and  $p^{ij}$  on  $S$ . In the new variables, the analogs of position and momentum are instead the connection  $A_+$  and the field  $(e \wedge e)_+$  restricted to  $S$ . Since  $P_+$  is trivial when restricted to  $S$ ,  $A_+$  can be identified with an  $\mathfrak{sl}(2, \mathbf{C})$ -valued 1-form on  $S$ , usually denoted simply by  $A$ . Similarly,  $(e \wedge e)_+$  can be identified with an  $\mathfrak{sl}(2, \mathbf{C})$ -valued 2-form on  $S$ , but physicists often use the isomorphism

$$\Lambda^2 T^* S \cong TS \otimes \Lambda^3 T^* S,$$

given by the interior product, to treat it as a 'densitized vector field' on  $S$  with values in  $\mathfrak{sl}(2, \mathbf{C})$ , usually denoted by  $\tilde{E}$ . Actually, using coordinates, they often write  $A$  and  $\tilde{E}$  as  $A_i^a$  and  $\tilde{E}^{ia}$ , where  $a = 1, 2, 3$  indexes a basis of  $\mathfrak{sl}(2, \mathbf{C})$  that is orthonormal with respect to the Killing form.

In these terms, the constraints in the new variables approach are

$$\mathcal{G}^a = \partial_i \tilde{E}^{ia} + [A_i, \tilde{E}^i]^a, \quad (8)$$

$$\mathcal{H}_j = F_{ij}^a \tilde{E}_a^i, \quad (9)$$

and

$$\mathcal{H} = \epsilon_{abc} F_{ij}^c \tilde{E}^{ia} \tilde{E}^{jb}, \quad (10)$$

where  $F$  is the curvature of  $A$  and  $\epsilon_{abc}$  is the completely antisymmetric tensor with  $\epsilon_{123} = 1$ .

It is probably not obvious from our description where the advantage of the self-dual formalism over the Palatini formalism lies. The key point is the construction of the field  $\tilde{E}$  from  $(e \wedge e)_+$ . One can similarly construct  $\tilde{E}$  from  $e \wedge e$  in the Palatini formalism, but then  $\tilde{E}$  winds up being subject to extra constraints [3] which negate the advantages of this approach. In the self-dual formalism, no conditions on  $\tilde{E}$  need hold for it to come from a complex soldering form  $e$ .

## 4 Canonical Quantization

Now let us try to canonically quantize gravity in terms of the new variables, with an eye to the importance of spin networks. The basic idea of DeWitt's approach goes over to this context with only a few small modifications. In its naive form, the idea would be to define a Hilbert space  $L^2(\mathcal{A}_+)$  and define operators on this space by

$$(\hat{A}_i^a \psi)(A) = A_i^a \psi(A), \quad (\hat{E}_a^i \psi)(A) = \hbar \frac{\delta \psi}{\delta A_i^a}(A).$$

(Again, we have suppressed the dependence of these operators on the point in our spacelike slice.) We would then substitute these operators for  $A$  and  $\tilde{E}$  in formulas (8-10), obtaining quantized constraints  $\hat{\mathcal{H}}$ ,  $\hat{\mathcal{H}}_j$ , and  $\hat{\mathcal{G}}^a$ . Physical states should then be annihilated by these constraints.

There are, however, a number of subtleties that we need to address in order to do things right. First there is the fact that the Plebanski formalism most naturally describes complex general relativity, and needs some adjustment to become a theory of honest real-valued Lorentzian metrics. This

issue has been a murky one for some time, and only now is it beginning to become clear. At the classical level one can simply impose extra constraints saying that the metric is Lorentzian. This would not be very nice in the quantum theory, though, because the whole point of the new variables was to simplify the constraints! Luckily it appears that at the quantum level we can deal with the issue in quite a different way, namely by restricting our attention to a space  $HL^2(\mathcal{A}_+)$  of holomorphic wavefunctions on  $\mathcal{A}_+$ . Rather than really explaining this here, let us merely note that it is closely related to the Bargmann-Segal formulation of the quantum theory of a particle on a line [3, 4]. The Bargmann-Segal formulation makes use of an isomorphism between  $L^2(\mathbf{R})$  and the Hilbert space  $HL^2(\mathbf{C})$  consisting of holomorphic functions that are square-integrable with respect to a Gaussian measure. A similar theory for complex Lie groups and their compact real forms, due to Hall [32], gives an isomorphism between a Hilbert space  $HL^2(\mathrm{SL}(2, \mathbf{C}))$  and the Hilbert space  $L^2(\mathrm{SU}(2))$  defined using Haar measure. A generalization of this [8] gives an isomorphism, the ‘coherent state transform’, between a Hilbert space  $HL^2(\mathcal{A}_+)$  and the space  $L^2(\mathcal{A})$ , where  $\mathcal{A}$  is the space of connections on an  $\mathrm{SU}(2)$  bundle over  $S$ . Thus the theory described in Section 2 is relevant to quantum gravity even though  $\mathrm{SL}(2, \mathbf{C})$  is not compact.

Second, there is the issue of rigorously interpreting the constraints and finding solutions to them. When we try to replace the fields  $A$  and  $\tilde{E}$  by  $\hat{A}$  and  $\hat{\tilde{E}}$  in formulas (8-10), we run into difficulties. One problem is that operators do not commute, so different orderings of the same polynomial in the classical fields can have different meanings at the quantum level. The orderings given here seem best when one wants operators on some space of functions on  $\mathcal{A}_+$ ; elsewhere one may see the opposite orderings [12], but that is the natural consequence of working dually on some space of generalized measures. A more profound problem is that  $\hat{A}$  and  $\hat{\tilde{E}}$  are not really operator-valued functions, but only operator-valued distributions — one must integrate them against ‘test functions’ (really bundle sections) to obtain operators — and like ordinary distributions, the product of operator-valued distributions is only defined under special conditions, or using special tricks.

Luckily, the Gauss law constraint and diffeomorphism constraint have simple geometrical interpretations which relieve us of the need for making sense of the quantum analogs of formulas (8) and (9). At the classical level, functions of the form  $\int_S X_a \mathcal{G}^a$  generate all the flows on the phase space  $T^*\mathcal{A}$

that correspond to the action of one-parameter groups of gauge transformations on  $T^*\mathcal{A}$ . So at the quantum level we can interpret the Gauss law constraint as saying that a state is invariant under ‘small’ gauge transformations, that is, those lying in the component of  $\mathcal{G}$  containing the identity. One can show that any state  $\psi \in L^2(\mathcal{A})$  that is invariant under small gauge transformations is invariant under all gauge transformations. As we have seen, there is an ample supply of such states, and the space of these, which we write as  $L^2(\mathcal{A}/\mathcal{G})$ , is spanned by spin network states.

Similarly, any function on phase space of the form  $\int_S N^i \mathcal{H}_i$  generates a flow corresponding to the action of a one-parameter group of diffeomorphisms. Thus at the quantum level we can interpret the diffeomorphism constraint as saying that a state is invariant under the group  $\text{Diff}_0(S)$  of small analytic diffeomorphisms, that is, those lying in the connected component containing the identity. Here we run into a problem. It is easy to see that the only element of  $\text{Fun}_0(\mathcal{A})$  invariant under  $\text{Diff}_0(S)$  is the function 1; any other can be written as in equation (1) only for some nonempty graph  $\gamma$ , and then there is a small diffeomorphism taking it to some other function, in fact one orthogonal to it in  $L^2(\mathcal{A})$ . A simple approximation argument then shows that no  $\psi \in L^2(\mathcal{A})$  except  $\psi = 1$  is invariant under  $\text{Diff}_0(S)$ !

In fact this is not as bad as it may seem. Quite often in the quantization of systems with constraints, the physical states are not really vectors in the kinematical state space, but only vectors in some larger topological vector space having the kinematical state space as a dense subspace. Consider a simple example: the particle on the line, regarded as a particle on the plane whose position  $(q_1, q_2)$  is subject to the constraint  $q_2 = 0$ . Naively, one would start with the kinematical state space  $L^2(\mathbf{R}^2)$  and look for states  $\psi$  satisfying the constraint  $\hat{q}_2\psi = 0$ , i.e.,

$$q_2\psi(q_1, q_2) = 0.$$

The only  $L^2$  function on the plane satisfying this equation is zero! However, there are *distributions* on the plane satisfying this equation. The space of such solutions is not itself a Hilbert space. However, it has a subspace isomorphic to  $L^2(\mathbf{R})$ , given by the distributions of the form  $\psi(q_1)\delta(q_2)$ . This subspace is correct Hilbert space for the particle on the line.

Similarly, while there are practically no solutions of the diffeomorphism constraint living in  $L^2(\mathcal{A})$ , there are plenty of ‘distributional’ ones. In fact,

there are plenty of generalized measures on  $\mathcal{A}$  invariant under small diffeomorphisms of  $S$ . The most interesting of these are the gauge-invariant ones, since we would like to solve the Gauss law constraint as well. Note that the inner product on  $L^2(\mathcal{A}/\mathcal{G})$  sets up a chain of inclusions

$$\text{Fun}(\mathcal{A}/\mathcal{G}) \subset L^2(\mathcal{A}/\mathcal{G}) \subset \text{Fun}(\mathcal{A}/\mathcal{G})^*,$$

where  $\text{Fun}(\mathcal{A}/\mathcal{G})$  is the gauge-invariant subspace of  $\text{Fun}(\mathcal{A})$ , and its Banach space dual  $\text{Fun}(\mathcal{A}/\mathcal{G})^*$  may be identified with the space of gauge-invariant generalized measures on  $\mathcal{A}$ . The group  $\text{Diff}_0(S)$  acts in a consistent way on all these spaces. A natural candidate for a space of simultaneous solutions of the Gauss law and diffeomorphism constraints is the space

$$\text{Fun}(\mathcal{A}/\mathcal{G})_{inv}^* = \{\mu \in \text{Fun}(\mathcal{A}/\mathcal{G})^* : \forall g \in \text{Diff}_0(S) \ g\mu = \mu\}.$$

This space is not itself a Hilbert space, but it may have some subspace deserving to be called the ‘Hilbert space of diffeomorphism-invariant states’. On the other hand, perhaps there are physically relevant diffeomorphism-invariant states that are not contained in  $\text{Fun}(\mathcal{A}/\mathcal{G})^*$ , but only in some still larger space containing  $L^2(\mathcal{A}/\mathcal{G})$ . An example would be the ‘Chern-Simons state’. To understand these would require further study of generalized functions on  $\mathcal{A}/\mathcal{G}$ .

Anyway, at least a general recipe for finding elements of  $\text{Fun}(\mathcal{A}/\mathcal{G})_{inv}^*$  is known, as are many interesting examples [7, 9]. Among the most interesting are the ‘knot states’, which appeared already in a nonrigorous way in the pioneering work of Rovelli and Smolin [45]. These are most easily described using spin networks, although they were not originally constructed that way [4]. Fix an isotopy class  $C$  of analytic knots and an irreducible unitary representation  $\rho$  of  $\text{SU}(2)$ . This determines a set  $S$  of spin network states, namely the Wilson loops  $\text{tr}(\rho(T \exp \int_\alpha A))$  with  $\alpha \in C$ . Since linear combinations of spin network states are dense in  $\text{Fun}(\mathcal{A}/\mathcal{G})$ , any generalized measure  $\mu \in \text{Fun}(\mathcal{A}/\mathcal{G})^*$  is determined by its values on spin network states. We define the knot state  $\mu$  by setting  $\mu(\psi) = 1$  if  $\psi \in S$  and  $\mu(\psi) = 0$  if  $\psi$  is a spin network state not in  $S$ . One must check, of course, that  $\mu$  is a well-defined generalized measure, which involves proving a certain bound. By construction  $\mu$  is invariant under small diffeomorphisms, so we have  $\mu \in \text{Fun}(\mathcal{A}/\mathcal{G})_{inv}^*$ .

Since a Wilson loop is just a special sort of spin network, it is natural to ask if this procedure generalizes to yield ‘diffeomorphism-invariant spin

network states'. The idea would be to use an arbitrary isotopy class of spin networks to obtain a set  $S$  of spin network states, and to define  $\mu \in \text{Fun}(\mathcal{A}/\mathcal{G})_{inv}^*$  by setting  $\mu(\psi) = 1$  for  $\psi \in S$  and  $\mu(\psi) = 0$  if  $\psi$  is a spin network state not in  $S$ . The problem is simply to prove that  $\mu$  is a well-defined generalized measure. This seems to be true when we start with an ambient isotopy class of *links* in  $S$  labelled with representations, but the general spin network case is more subtle and still under investigation.

Now let us say a bit about the Hamiltonian constraint. This is perhaps the most controversial aspect of the whole subject, and certainly one of the most important ones: if we find a way to rigorously treat the Hamiltonian constraint, we will be quite close to a rigorous quantization of Einstein's equation, but if not, the whole approach described here may be fundamentally flawed, or at least in need of very new ideas. A lot of work has been done on the Hamiltonian constraint, which we cannot really do justice to here [15, 16, 17, 20, 30, 34]. Naively, the problem is to make sense of

$$\hat{\mathcal{H}}(x) = \hbar^2 \epsilon_{abc} F_{ij}^c(x) \frac{\partial}{\partial A^{ia}(x)} \frac{\partial}{\partial A^{jb}(x)}$$

as a distribution on  $S$  taking values in operators on some space of holomorphic functions on  $\mathcal{A}_+$ . Eventually, however, we want to find states that are annihilated by this constraint together with the Gauss law and diffeomorphism constraints. Thus we would be perfectly happy if we could first use the coherent state transform to transfer the constraint to  $L^2(\mathcal{A})$ , and then find some subspace of  $\text{Fun}(\mathcal{A}/\mathcal{G})_{inv}^*$  — or some related space of solutions of the Gauss law and diffeomorphism constraints — that we could argue was annihilated by the Hamiltonian constraint. The problem is that, in contrast to the Gauss law and diffeomorphism constraints, there is no easy geometrical interpretation of the Hamiltonian constraint in terms of connections on  $S$  to fall back upon.

At first it might seem foolish to even *hope* for a simple geometrical 3-dimensional interpretation of the Hamiltonian constraint. After all, the Hamiltonian constraint expresses the *4-dimensional* diffeomorphism invariance of general relativity; or in other words, it encodes the dynamics of the theory. It is all the more tantalizing, therefore, that there are some hints of such an interpretation. Rovelli and Smolin's discovery of these was one of the early successes of the loop representation [45], but making them precise has proved to be very difficult.

Stripped of all nuance, the observation of Rovelli and Smolin reduces to the following. Given a knot  $\alpha$  in  $S$ , the Wilson loop

$$\psi(A) = \text{tr}(\rho(T \exp \int_{\alpha} A))$$

is a holomorphic function on  $\mathcal{A}_+$ . Heuristically speaking, when one applies the functional derivative  $\partial/\partial A_i^a(x)$  to  $\psi$  one brings down a factor of the tangent vector  $\dot{\alpha}^i(t)$  if  $\alpha(t) = x$ . So the double functional derivative in  $\hat{\mathcal{H}}(x)$  brings down a factor of  $\dot{\alpha}^i(t)\dot{\alpha}^j(t)$ , which is symmetric in  $i$  and  $j$ . Since  $F_{ij}^c$  is antisymmetric in  $i$  and  $j$ , one obtains

$$\hat{\mathcal{H}}(x)\psi = 0.$$

In fact, when one goes through the argument more carefully one discovers that the double functional derivative of a Wilson loop is very singular, so the result  $\hat{\mathcal{H}}(x)\psi = 0$  is a purely formal one unless one can devise some regularization procedure to make the argument rigorous. This has not yet been achieved. Still, upon examination, the argument seems to suggest a 3-dimensional geometrical interpretation of the Hamiltonian constraint as some kind of ‘shift operator’ generating the motion of a Wilson loop, or more generally the edges of a spin network, along its tangent vectors [45]. For this reason, much effort has been expended to understand the argument and find a context in which it can be made rigorous. One would be very happy, for example, to find by some heuristic argument a general formula for the action of the Hamiltonian constraint on spin network states, or perhaps ‘diffeomorphism-invariant spin network states’, which one could then subsequently justify by means of its good properties.

In short, quantum gravity continues to provide mathematical physics with challenging — indeed quite frustrating — problems. However, let us conclude on a more upbeat note! Even at the level of the kinematical state space  $L^2(\mathcal{A}/\mathcal{G})$ , there are some very intriguing applications of spin networks to physics. Classically, in the metric representation a state of gravity is described by the metric on  $S$  and its first time derivative. We can rewrite interesting functions of the metric in terms of the new variables, and then attempt to quantize them by replacing  $A$  and  $\tilde{E}$  by their quantum versions in these expressions, obtaining operators on  $L^2(\mathcal{A})$ . These operators should commute with the action of  $\mathcal{G}$ , hence give rise to operators on  $L^2(\mathcal{A}/\mathcal{G})$ . As

the example of the Hamiltonian constraint shows, carrying this out is by no means straightforward in all cases. However, Rovelli and Smolin [46] have considered the *area of a surface* and the *volume of a region* in  $S$ , which are technically simpler, and obtained explicit formulas for their quantum versions as operators on  $L^2(\mathcal{A}/\mathcal{G})$ , in terms of the spin network basis. These operators turn out to have discrete spectrum: certain multiples of Planck length squared for the area operator, and certain multiples of the Planck length cubed for the volume operator. In quantum theory, the spectrum of a self-adjoint operator corresponds to the values the corresponding observable can assume, so this is an indication that area and volume are ‘quantized’ in the very literal sense of assuming a discrete set of values! Moreover, there is a simple geometrical reason for this fact. To speak in rather oversimplified terms, the area operator applied to a given spin network state counts the number of points at which an *edge* of the spin network intersects the surface in question, weighted by a factor of  $\sqrt{j(j+1)}\ell_p^2$ , where  $j$  is the spin labelling that edge. Similarly, the volume operator applied to a spin network state counts the number of *vertices* of the spin network contained in the region in question, weighted by some function of their labellings by intertwiners and the geometry of the incident edges.

Until we have a fully working theory of quantum gravity, and understand how to take the other forces in account, it is dangerous to take too seriously any physical predictions a partial theory might make. Moreover, this prediction of discreteness of area and volume at the Planck scale is absurdly hard to test with present technology! Nonetheless, the idea that the marriage of Einstein’s equation and quantum theory could make such a fascinating prediction should serve as a kind of inspiration for mathematicians and physicists working on quantum gravity.

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