An Introduction to Geometric Algebra

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History

Geometric algebra is the Clifford algebra of a finite dimensional vector space over real scalars cast in a form most appropriate for physics and engineering. This was done by David Hestenes (Arizona State University) in the 1960's. From this start he developed the geometric calculus whose fundamental theorem includes the generalized Stokes theorem, the residue theorem, and new integral theorems not realized before. Hestenes likes to say he was motivated by the fact that physicists and engineers did not know how to multiply vectors.

Researchers at Arizona State and Cambridge have applied these developments to classical mechanics,quantum mechanics, general relativity (gauge theory of gravity), projective geometry, conformal geometry, etc.

Axioms of Geometric Algebra

Let $\mathcal{V}(p,q)$ be a finite dimensional vector space of signature (p,q)[1] over $\Re$. Then $\forall a,b,c \in \mathcal{V}$ there exists a geometric product with the properties -

$$(ab)c = a(bc)$$
$$a(b+c) = ab + ac$$
$$(a+b)c = ac + bc$$
$$aa \in \Re$$

If $a^2 \neq 0$ then $a^{-1} = \frac{1}{a^2}a$.

[1] To be completely general we would have to consider $\mathcal{V}(p,q,r)$ where the dimension of the vector space is $n = p+q+r$ and p, q, and r are the number of basis vectors respectively with positive, negative and zero squares.

Why Learn This Stuff?

The geometric product of two (or more) vectors produces something "new" like the $\sqrt{-1}$ with respect to real numbers or vectors with respect to scalars. It must be studied in terms of its effect on vectors and in terms of its symmetries. It is worth the effort. Anything that makes understanding rotations in a N dimensional space simple is worth the effort! Also, if one proceeds on to geometric calculus many diverse areas in mathematics are unified and many areas of physics and engineering are greatly simplified.

Inner, $\cdot$, and outer, $\wedge$, product of two vectors and their basic properties

$$a \cdot b \equiv \frac{1}{2}(ab + ba) \tag{1}$$

$$a \wedge b \equiv \frac{1}{2}(ab - ba) \tag{2}$$

$$ab = a \cdot b + a \wedge b \tag{3}$$

$$a \wedge b = -b \wedge a \tag{4}$$

$$\begin{aligned} c &= a + b \\ c^2 &= (a+b)^2 \\ c^2 &= a^2 + ab + ba + b^2 \\ 2a \cdot b &= c^2 - a^2 - b^2 \\ a \cdot b &\in \Re \end{aligned} \tag{5}$$

$$a \cdot b = |a|\,|b| \cos(\theta) \text{ if } a^2, b^2 > 0 \tag{6}$$

Orthogonal vectors are defined by $a \cdot b = 0$.
For orthogonal vectors $a \wedge b = ab$.

Now compute $(a \wedge b)^2$

$$
\begin{aligned}
(a \wedge b)^2 &= -(a \wedge b)(b \wedge a) && (7)\\
&= -(ab - a \cdot b)(ba - a \cdot b) && (8)\\
&= -\left(abba - (a \cdot b)(ab + ba) + (a \cdot b)^2\right) && (9)\\
&= -\left(a^2 b^2 - (a \cdot b)^2\right) && (10)\\
&= -a^2 b^2 \left(1 - \cos^2(\theta)\right) && (11)\\
&= -a^2 b^2 \sin^2(\theta) && (12)
\end{aligned}
$$

Thus in a Euclidean space, $a^2, b^2 > 0$, $(a \wedge b)^2 \leq 0$ and $a \wedge b$ is proportional to $\sin(\theta)$. If $e_{\|}$ and $e_{\perp}$ are any two orthonormal unit vectors in a Euclidean space then $\left(e_{\|} e_{\perp}\right)^2 = -1$. Who needs the $\sqrt{-1}$?

Outer, $\wedge$, product for r Vectors in terms of the geometric product

We define the outer product of r vectors to be

$$a_1 \wedge \ldots \wedge a_r \equiv \frac{1}{r!} \sum_{i_1,\ldots,i_r} \varepsilon^{i_1 \ldots i_r}_{1 \ldots r} a_{i_1} \ldots a_{i_r} \tag{13}$$

Thus

$$\begin{aligned} a_1 \wedge \ldots \wedge (a_j + b_j) \wedge \ldots \wedge a_r &= \\ & a_1 \wedge \ldots \wedge a_j \wedge \ldots \wedge a_r + a_1 \wedge \ldots \wedge b_j \wedge \ldots \wedge a_r \end{aligned} \tag{14}$$

and

$$\begin{aligned} a_1 \wedge \ldots \wedge a_j \wedge a_{j+1} \wedge \ldots \wedge a_r &= \\ & -a_1 \wedge \ldots \wedge a_{j+1} \wedge a_j \wedge \ldots \wedge a_r \end{aligned} \tag{15}$$

The outer product of r vectors is called a blade of grade r.

Alternate Definition of Outer, $\wedge$, product for r Vectors

Let $e_1, e_2, \ldots, e_r$ be an orthogonal basis for the set of linearly independent vectors $a_1, a_2, \ldots, a_r$ so that we can write

$$a_i = \sum_j \alpha_{ij} e_j \tag{16}$$

Then

$$\begin{aligned} a_1 a_2 \ldots a_r &= \left(\sum_{j_1} \alpha_{1j_1} e_{j_1}\right)\left(\sum_{j_2} \alpha_{2j_2} e_{j_2}\right) \ldots \left(\sum_{j_r} \alpha_{rj_r} e_{j_r}\right) \\ &= \sum_{j_1,\ldots,j_r} \alpha_{1j_1}\alpha_{2j_2}\ldots\alpha_{rj_r} e_{j_1} e_{j_2} \ldots e_{j_r} \end{aligned} \tag{17}$$

Now define a blade of grade n as the geometric product of n orthogonal vectors. Thus the product $e_{j_1}e_{j_2}\dots e_{j_r}$ in equation 17 could be a blade of grade r, $r-2$, $r-4$, etc. depending upon the number of repeated factors.

If there are no repeated factors in the product we have that

$$e_{j_1}\dots e_{j_r} = \varepsilon^{j_1\dots j_r}_{1\dots r} e_1 \dots e_r \tag{18}$$

Due to the fact that interchanging two adjacent orthogonal vectors in the geometric product will reverse the sign of the product and we can define the outer product of r vectors as

$$\begin{aligned} a_1 \wedge \dots \wedge a_r &= \sum_{j_1,\dots,j_r} \varepsilon^{j_1\dots j_r}_{1\dots r} \alpha_{1j_1}\dots\alpha_{rj_r} e_1\dots e_r \quad (19) \\ &= \det(\alpha)\, e_1 \dots e_r \quad (20) \end{aligned}$$

Thus the outer product of r independent vectors is the part of the geometric product of the r vectors that is of grade r. Equation 19 is equivalent to equation 13.

This can be proved by substituting equation 17 into equation 13 to get

$$a_1 \wedge \ldots \wedge a_r = \frac{1}{r!} \sum_{i_1,\ldots,i_r} \sum_{j_1,\ldots,j_r} \varepsilon^{i_1 \ldots i_r}_{1 \ldots r} \alpha_{i_1 j_1} \ldots \alpha_{i_r j_r} e_{j_1} \ldots e_{j_r} \tag{21}$$

$$= \frac{1}{r!} \sum_{i_1,\ldots,i_r} \sum_{j_1,\ldots,j_r} \varepsilon^{i_1 \ldots i_r}_{1 \ldots r} \varepsilon^{j_1 \ldots j_r}_{1 \ldots r} \alpha_{i_1 j_1} \ldots \alpha_{i_r j_r} e_1 \ldots e_r \tag{22}$$

$$= \frac{1}{r!} \sum_{j_1,\ldots,j_r} \varepsilon^{j_1 \ldots j_r}_{1 \ldots r} \varepsilon^{j_1 \ldots j_r}_{1 \ldots r} \det(\alpha)\, e_1 \ldots e_r \tag{23}$$

$$= \det(\alpha)\, e_1 \ldots e_r \tag{24}$$

We go from equation 22 to equation 23 by noting that $\sum_{i_1,\ldots,i_r} \varepsilon^{i_1 \ldots i_r}_{1 \ldots r} \alpha_{i_1 j_1} \ldots \alpha_{i_r j_r}$ is just $\det(\alpha)$ with the columns permuted.

Multiplying $\det(\alpha)$ by $\varepsilon^{j_1\ldots j_r}_{1\ldots r}$ gives the correct sign for the determinant with the columns permuted.

If $e_1, \ldots, e_n$ is an orthonormal basis for vector space the unit psuedoscalar is defined as

$$I = e_1 \ldots e_n \tag{25}$$

In equation 24 let $r = n$ and the $a_1, \ldots, a_n$ be another orthonormal basis for the vector space. Then we may write

$$a_1 \ldots a_n = \det(\alpha)\, e_1 \ldots e_n \tag{26}$$

Since both the a's and the e's form orthonormal bases the matrix α is orthogonal and $\det(\alpha) = \pm 1$. All psuedoscalars for the vector space are identical to within a scale factor of ± 1.[2]

Likewise $a_1 \wedge \ldots \wedge a_n$ is equal to I times a scale factor.

[2]It depends only upon the ordering of the basis vectors.

Useful Relation's

1. For a set of r orthogonal vectors, $e_1, \ldots, e_r$

$$e_1 \wedge \ldots \wedge e_r = e_1 \ldots e_r \tag{27}$$

2. For a set of r linearly independent vectors, $a_1, \ldots, a_r$, there exists a set of r orthogonal vectors, $e_1, \ldots, e_r$, such that

$$a_1 \wedge \ldots \wedge a_r = e_1 \ldots e_r \tag{28}$$

   If the vectors, $a_1, \ldots, a_r$, are not linearly independent then

$$a_1 \wedge \ldots \wedge a_r = 0 \tag{29}$$

The product $a_1 \wedge \ldots \wedge a_r$ is call a "blade" of grade r. The dimension of the vector space is the highest grade any blade can have.

Projection Operator

A multivector, the basic element of the geometric algebra, is made of of a sum of scalars, vectors, blades. A multivector is homogeneous (pure) if all the blades in it are of the same grade. The grade of a scalar is 0 and the grade of a vector is 1. The general multivector A is decomposed with the grade projection operator $\langle A \rangle_r$ as (N is dimension of the vector space):

$$A = \sum_{r=0}^{N} \langle A \rangle_r \tag{30}$$

As an example consider ab, the product of two vectors. Then

$$ab = \langle ab \rangle_0 + \langle ab \rangle_2 \tag{31}$$

We define $\langle A \rangle \equiv \langle A \rangle_0$ for any multivector A

Basis Blades

The geometric algebra of a vector space, $\mathcal{V}(p,q)$, is denoted $\mathcal{G}(p,q)$ or $\mathcal{G}(\mathcal{V})$ where (p,q) is the signature of the vector space (first p unit vectors square to $+1$ and next q unit vectors square to -1, dimension of the space is $p+q$). Examples are:

p	q	Type of Space
3	0	3D Euclidean
1	3	Relativistic Space Time
4	1	3D Conformal Geometry

If the orthonormal basis set of the vector space is $e_1, \ldots, e_N$, the basis of the geometric algebra (multivector space) is formed from the geometric products (since we have chosen an orthonormal basis) of the basis vectors. For grade r multivectors the basis blades are all the combinations of basis vectors products taken r at a time from the set of N vectors. Thus the number basis blades of r rank are $\binom{N}{r}$, the binomial expansion coefficient and the total dimension of the multivector space is the sum of $\binom{N}{r}$ over r which is 2^N. Thus the basis blades for $\mathcal{G}(3,0)$ are:

Grade			
0	1	2	3
1	e_1	e_1e_2	$e_1e_2e_3$
	e_2	e_1e_3	
	e_3	e_2e_3	

The multiplication table for the $\mathcal{G}(3,0)$ basis blades is

	1	e_1	e_2	e_3	e_1e_2	e_1e_3	e_2e_3	$e_1e_2e_3$
1	1	e_1	e_2	e_3	e_1e_2	e_1e_3	e_2e_3	$e_1e_2e_3$
e_1	e_1	1	e_1e_2	e_1e_3	e_2	e_3	$e_1e_2e_3$	e_2e_3
e_2	e_2	$-e_1e_2$	1	e_2e_3	$-e_1$	$-e_1e_2e_3$	e_3	$-e_1e_3$
e_3	e_3	$-e_1e_3$	$-e_2e_3$	1	$e_1e_2e_3$	$-e_1$	$-e_2$	e_1e_2
e_1e_2	e_1e_2	$-e_2$	e_1	$e_1e_2e_3$	-1	$-e_2e_3$	e_1e_3	$-e_3$
e_1e_3	e_1e_3	$-e_3$	$-e_1e_2e_3$	e_1	e_2e_3	-1	$-e_1e_2$	e_2
e_2e_3	e_2e_3	$e_1e_2e_3$	$-e_3$	e_2	$-e_1e_3$	e_1e_2	-1	$-e_1$
$e_1e_2e_3$	$e_1e_2e_3$	e_2e_3	$-e_1e_3$	e_1e_2	$-e_3$	e_2	$-e_1$	-1

Note that the squares of all the grade 2 and 3 basis blades are -1. The highest rank basis blade (in this case $e_1e_2e_3$) is ususally denoted by I and is called the pseudoscalar.

The multiplication table for the $\mathcal{G}(1,3)$ basis blades is (Part I)

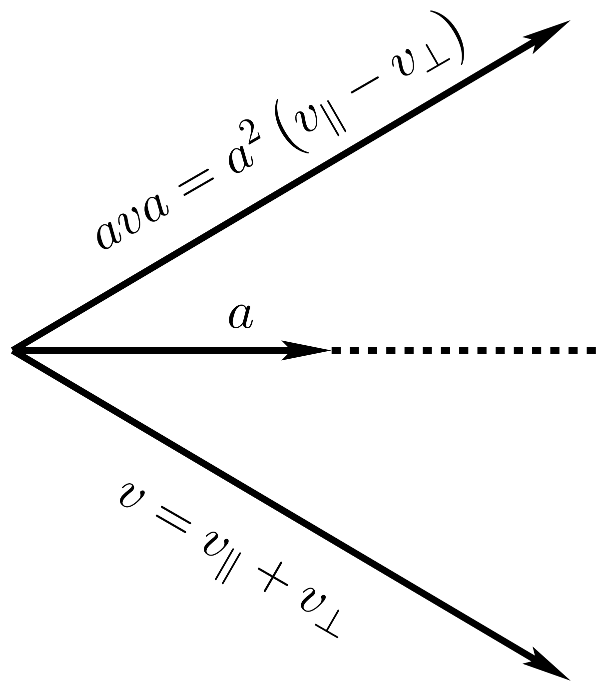
	1	γ_0	γ_1	γ_2	γ_3	$\gamma_0\gamma_1$	$\gamma_0\gamma_2$	$\gamma_1\gamma_2$
1	1	γ_0	γ_1	γ_2	γ_3	$\gamma_0\gamma_1$	$\gamma_0\gamma_2$	$\gamma_1\gamma_2$
γ_0	γ_0	1	$\gamma_0\gamma_1$	$\gamma_0\gamma_2$	$\gamma_0\gamma_3$	γ_1	γ_2	$\gamma_0\gamma_1\gamma_2$
γ_1	γ_1	$-\gamma_0\gamma_1$	-1	$\gamma_1\gamma_2$	$\gamma_1\gamma_3$	γ_0	$-\gamma_0\gamma_1\gamma_2$	$-\gamma_2$
γ_2	γ_2	$-\gamma_0\gamma_2$	$-\gamma_1\gamma_2$	-1	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2$	γ_0	γ_1
γ_3	γ_3	$-\gamma_0\gamma_3$	$-\gamma_1\gamma_3$	$-\gamma_2\gamma_3$	-1	$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$
$\gamma_0\gamma_1$	$\gamma_0\gamma_1$	$-\gamma_1$	$-\gamma_0$	$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_3$	1	$-\gamma_1\gamma_2$	$-\gamma_0\gamma_2$
$\gamma_0\gamma_2$	$\gamma_0\gamma_2$	$-\gamma_2$	$-\gamma_0\gamma_1\gamma_2$	$-\gamma_0$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2$	1	$\gamma_0\gamma_1$
$\gamma_1\gamma_2$	$\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2$	γ_2	$-\gamma_1$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_2$	$-\gamma_0\gamma_1$	-1
$\gamma_0\gamma_3$	$\gamma_0\gamma_3$	$-\gamma_3$	$-\gamma_0\gamma_1\gamma_3$	$-\gamma_0\gamma_2\gamma_3$	$-\gamma_0$	$\gamma_1\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$
$\gamma_1\gamma_3$	$\gamma_1\gamma_3$	$\gamma_0\gamma_1\gamma_3$	γ_3	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_1$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_2\gamma_3$
$\gamma_2\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$	γ_3	$-\gamma_2$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_3$	$\gamma_1\gamma_3$
$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2$	$\gamma_1\gamma_2$	$\gamma_0\gamma_2$	$-\gamma_0\gamma_1$	$\gamma_0\gamma_1\gamma_2\gamma_3$	γ_2	$-\gamma_1$	$-\gamma_0$
$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_1\gamma_3$	$\gamma_1\gamma_3$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_1$	γ_3	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_2\gamma_3$
$\gamma_0\gamma_2\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_2$	$\gamma_1\gamma_2\gamma_3$	γ_3	$\gamma_0\gamma_1\gamma_3$
$\gamma_1\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_2\gamma_3$	$\gamma_1\gamma_3$	$-\gamma_1\gamma_2$	$\gamma_0\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_3$	$-\gamma_3$
$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_3$	$-\gamma_0\gamma_1\gamma_2$	$\gamma_2\gamma_3$	$-\gamma_1\gamma_3$	$-\gamma_0\gamma_3$

The multiplication table for the $\mathcal{G}(1,3)$ basis blades is (Part II)

	$\gamma_0\gamma_3$	$\gamma_1\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$
1	$\gamma_0\gamma_3$	$\gamma_1\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$
γ_0	γ_3	$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2$	$\gamma_1\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$
γ_1	$-\gamma_0\gamma_1\gamma_3$	$-\gamma_3$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_2$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_2\gamma_3$	$\gamma_0\gamma_2\gamma_3$
γ_2	$-\gamma_0\gamma_2\gamma_3$	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_3$	$-\gamma_0\gamma_1$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_3$	$\gamma_1\gamma_3$	$-\gamma_0\gamma_1\gamma_3$
γ_3	γ_0	γ_1	γ_2	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_1$	$-\gamma_0\gamma_2$	$-\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2$
$\gamma_0\gamma_1$	$-\gamma_1\gamma_3$	$-\gamma_0\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$	γ_2	γ_3	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_2\gamma_3$	$\gamma_2\gamma_3$
$\gamma_0\gamma_2$	$-\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_3$	$-\gamma_1$	$\gamma_1\gamma_2\gamma_3$	γ_3	$\gamma_0\gamma_1\gamma_3$	$-\gamma_1\gamma_3$
$\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_2\gamma_3$	$-\gamma_1\gamma_3$	$-\gamma_0$	$\gamma_0\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_3$	$-\gamma_3$	$-\gamma_0\gamma_3$
$\gamma_0\gamma_3$	1	$\gamma_0\gamma_1$	$\gamma_0\gamma_2$	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_1$	$-\gamma_2$	$-\gamma_0\gamma_1\gamma_2$	$\gamma_1\gamma_2$
$\gamma_1\gamma_3$	$-\gamma_0\gamma_1$	-1	$\gamma_1\gamma_2$	$-\gamma_0\gamma_2\gamma_3$	$-\gamma_0$	$\gamma_0\gamma_1\gamma_2$	γ_2	$\gamma_0\gamma_2$
$\gamma_2\gamma_3$	$-\gamma_0\gamma_2$	$-\gamma_1\gamma_2$	-1	$\gamma_0\gamma_1\gamma_3$	$-\gamma_0\gamma_1\gamma_2$	$-\gamma_0$	$-\gamma_1$	$-\gamma_0\gamma_1$
$\gamma_0\gamma_1\gamma_2$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_3$	-1	$\gamma_2\gamma_3$	$-\gamma_1\gamma_3$	$-\gamma_0\gamma_3$	$-\gamma_3$
$\gamma_0\gamma_1\gamma_3$	$-\gamma_1$	$-\gamma_0$	$\gamma_0\gamma_1\gamma_2$	$-\gamma_2\gamma_3$	-1	$\gamma_1\gamma_2$	$\gamma_0\gamma_2$	γ_2
$\gamma_0\gamma_2\gamma_3$	$-\gamma_2$	$-\gamma_0\gamma_1\gamma_2$	$-\gamma_0$	$\gamma_1\gamma_3$	$-\gamma_1\gamma_2$	-1	$-\gamma_0\gamma_1$	$-\gamma_1$
$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2$	γ_2	$-\gamma_1$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_2$	$\gamma_0\gamma_1$	1	$-\gamma_0$
$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_1\gamma_2$	$\gamma_0\gamma_2$	$-\gamma_0\gamma_1$	γ_3	$-\gamma_2$	γ_1	γ_0	-1

Reflections

We wish to show that $a, v \in \mathcal{V} \rightarrow ava \in \mathcal{V}$ and v is reflected about a if $a^2 = 1$.



1. Decompose $v = v_{\parallel} + v_{\perp}$ where $v_{\parallel}$ is the part of v parallel to a and $v_{\perp}$ is the part perpendicular to a.

2. $av = av_{\|} + av_{\perp} = v_{\|}a - v_{\perp}a$ since a and $v_{\perp}$ are orthogonal.

3. $ava = a^2 \left(v_{\|} - v_{\perp}\right)$ is a vector since a^2 is a scalar.

4. ava is the reflection of v about the direction of a if $a^2 = 1$.

5. Thus $a_1 \ldots a_r v a_r \ldots a_1 \in \mathcal{V}$ and produces a composition of reflections of v if $a_1^2 = \ldots = a_r^2 = 1$.

Rotations, Part 1

First define the reverse of a product of vectors. If $R = a_1 \dots a_s$ then the reverse is $R^\dagger = (a_1 \dots a_s)^\dagger = a_r \dots a_1$, the order of multiplication is reversed. Then let $R = ab$ so that

$$RR^\dagger = (ab)(ba) = ab^2a = a^2b^2 = R^\dagger R \tag{32}$$

Let $RR^\dagger = 1$ and calculate $\left(RvR^\dagger\right)^2$, where v is an arbitrary vector.

$$\left(RvR^\dagger\right)^2 = RvR^\dagger RvR^\dagger = Rv^2R^\dagger = v^2RR^\dagger = v^2 \tag{33}$$

Thus $RvR^\dagger$ leaves the length of v unchanged.

Now we must also prove $Rv_1R^\dagger \cdot Rv_2R^\dagger = v_1 \cdot v_2$. Since $Rv_1R^\dagger$ and $Rv_2R^\dagger$ are both vectors we can use the definition of the dot product for two vectors

$$
\begin{aligned}
Rv_1R^\dagger \cdot Rv_2R^\dagger &= \frac{1}{2}\left(Rv_1R^\dagger Rv_2R^\dagger + Rv_2R^\dagger Rv_1R^\dagger\right) \\
&= \frac{1}{2}\left(Rv_1v_2R^\dagger + Rv_2v_1R^\dagger\right) \\
&= \frac{1}{2}R\left(v_1v_2 + v_2v_1\right)R^\dagger \\
&= R\left(v_1 \cdot v_2\right)R^\dagger \\
&= v_1 \cdot v_2 RR^\dagger \\
&= v_1 \cdot v_2
\end{aligned}
$$

Thus the transformation $RvR^\dagger$ preserves both length and angle and must be a rotation. The normal designation for R is a rotor.

If we have a series of successive rotations $R_1, R_2, \ldots, R_k$ to be applied to a vector v then the result of the k rotations will be

$$R_k R_{k-1} \ldots R_1 v R_1^\dagger R_2^\dagger \ldots R_k^\dagger$$

Since each individual rotation can be written as the geometric product of two vectors, the composition of k rotations can be written as the geometric product of $2k$ vectors. The multivector that results from the geometric product of r vectors is called a **versor** of order r. A composition of rotations is always a versor of even order.

Rotations, Part 2

The general rotation can be represented by $R = e^{\frac{\theta}{2}u}$ where u is a unit bivector in the plane of the rotation and θ is the rotation angle in the plane.[3] The two possible non-degenerate cases are $u^2 = \pm 1$

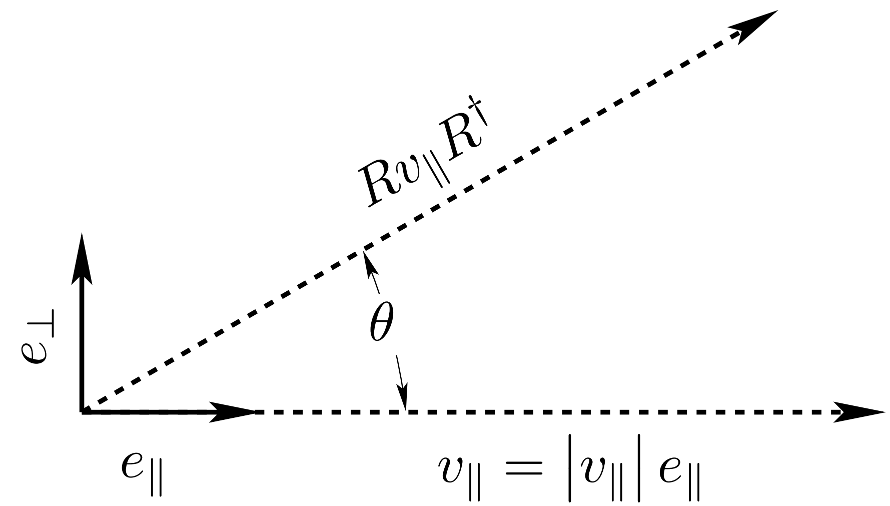
$$e^{\frac{\theta}{2}u} = \left\{ \begin{array}{lrl} \text{(Euclidean plane)} & u^2 = -1: & \cos\left(\frac{\theta}{2}\right) + u\sin\left(\frac{\theta}{2}\right) \\ \text{(Minkowski plane)} & u^2 = 1: & \cosh\left(\frac{\theta}{2}\right) + u\sinh\left(\frac{\theta}{2}\right) \end{array} \right\} \tag{34}$$

Decompose $v = v_{\|} + \left(v - v_{\|}\right)$ where $v_{\|}$ is the projection of v into the plane defined by u. Note the $v - v_{\|}$ is orthogonal to all vectors in the u plane. Now let $u = e_{\perp}e_{\|}$ where $e_{\|}$ is parallel to $v_{\|}$ and of course $e_{\perp}$ is in the plane u and orthogonal to $e_{\|}$. $v - v_{\|}$ anticommutes with $e_{\|}$ and $e_{\perp}$ and $v_{\|}$ anticommutes with $e_{\perp}$ (it is left to the viewer to show $RR^{\dagger} = 1$).

[3] e^A is defined as the Taylor series expansion $e^A = \sum_{j=0}^{\infty} \frac{A^j}{j!}$ where A is any multivector.

Euclidean Case

For the case of $u^2 = -1$



$$RvR^{\dagger} = \left(\cos\left(\frac{\theta}{2}\right) + e_{\perp}e_{\|}\sin\left(\frac{\theta}{2}\right)\right)\left(v_{\|} + (v - v_{\|})\right)\left(\cos\left(\frac{\theta}{2}\right) + e_{\|}e_{\perp}\sin\left(\frac{\theta}{2}\right)\right)$$

Since $v - v_{\|}$ anticommutes with $e_{\|}$ and $e_{\perp}$ it commutes with R and

$$RvR^{\dagger} = Rv_{\|}R^{\dagger} + \left(v - v_{\|}\right) \tag{35}$$

So that we only have to evaluate

$$Rv_{\|}R^{\dagger} = \left(\cos\left(\frac{\theta}{2}\right) + e_{\perp}e_{\|}\sin\left(\frac{\theta}{2}\right)\right) v_{\|} \left(\cos\left(\frac{\theta}{2}\right) + e_{\|}e_{\perp}\sin\left(\frac{\theta}{2}\right)\right) \tag{36}$$

Since $v_{\|} = \left|v_{\|}\right| e_{\|}$

$$Rv_{\|}R^{\dagger} = \left|v_{\|}\right| \left(\cos\left(\theta\right) e_{\|} + \sin\left(\theta\right) e_{\perp}\right) \tag{37}$$

and the component of v in the u plane is rotated correctly.

Minkowski Case

For the case of $u^2 = 1$ there are two possibilities, $v_{\|}^2 > 0$ or $v_{\|}^2 < 0$. In the first case $e_{\|}^2 = 1$ and $e_{\perp}^2 = -1$. In the second case $e_{\|}^2 = -1$ and $e_{\perp}^2 = 1$. Again $v - v_{\|}$ is not affected by the rotation so that we need only evaluate

$$Rv_{\|}R^{\dagger} = \left(\cosh\left(\frac{\theta}{2}\right) + e_{\perp}e_{\|}\sinh\left(\frac{\theta}{2}\right)\right) v_{\|} \left(\cosh\left(\frac{\theta}{2}\right) + e_{\|}e_{\perp}\sinh\left(\frac{\theta}{2}\right)\right)$$

Note that in this case $\left|v_{\|}\right| = \sqrt{\left|v_{\|}^2\right|}$ and

$$Rv_{\|}R^{\dagger} = \left\{ \begin{array}{l} v_{\|}^2 > 0 : \left|v_{\|}\right| \left(\cosh\left(\theta\right) e_{\|} + \sinh\left(\theta\right) e_{\perp}\right) \\ v_{\|}^2 < 0 : \left|v_{\|}\right| \left(\cosh\left(\theta\right) e_{\|} - \sinh\left(\theta\right) e_{\perp}\right) \end{array} \right\} \tag{38}$$

Expansion of geometric product and generalization of $\cdot$ and $\wedge$

If A_r and B_s are respectively grade r and s pure grade multivectors then

$$A_rB_s = \langle A_rB_s\rangle_{|r-s|} + \langle A_rB_s\rangle_{|r-s|+2} + \cdots + \langle A_rB_s\rangle_{\min(r+s,2N-(r+s))} \quad (39)$$

$$A_r \cdot B_s \equiv \langle A_rB_s\rangle_{|r-s|} \quad (40)$$

$$A_r \wedge B_s \equiv \langle A_rB_s\rangle_{r+s} \quad (41)$$

Thus if $r+s>N$ then $A_r \wedge B_s = 0$, also note that these formulas are the most efficient way of calculating $A_r \cdot B_s$ and $A_r \wedge B_s$.

Using equations 28 and 39 we can prove that for a vector a and a grade r multivector B_r

$$a \cdot B_r = \frac{1}{2}\left(aB_r - (-1)^r B_r a\right) \tag{42}$$

$$a \wedge B_r = \frac{1}{2}\left(aB_r + (-1)^r B_r a\right) \tag{43}$$

If equations 42 and 43 are true for a grade r blade they are also true for a grade r multivector (superposition of grade r blades). By equation 28 let $B_r = e_1 \ldots e_r$ where the $e's$ are orthogonal and expand a

$$a = a_\perp + \sum_{j=1}^{r} \alpha_j e_j \tag{44}$$

where $a_\perp$ is orthogonal to all the $e's$. Then[4]

$$\begin{aligned} aB_r &= \sum_{j=1}^{r} (-1)^{j-1} \alpha_j e_j^2 e_1 \cdots \breve{e}_j \cdots e_r + a_\perp e_1 \ldots e_r \\ &= a \cdot B_r + a \wedge B_r \end{aligned} \tag{45}$$

Now calculate

$$\begin{aligned} B_r a &= \sum_{j=1}^{r} (-1)^{r-j} \alpha_j e_j^2 e_1 \cdots \breve{e}_j \cdots e_r - (-1)^{r-1} a_\perp e_1 \ldots e_r \\ &= (-1)^{r-1} \left(\sum_{j=1}^{r} (-1)^{j-1} \alpha_j e_j^2 e_1 \cdots \breve{e}_j \cdots e_r - a_\perp e_1 \ldots e_r \right) \\ &= (-1)^{r-1} (a \cdot B_r - a \wedge B_r) \end{aligned} \tag{46}$$

Adding and subtracting equations 45 and 46 gives equations 42 and 43.

[4] $e_1 \ldots e_{j-1} \breve{e}_j e_{j+1} \ldots e_r = e_1 \ldots e_{j-1} e_{j+1} \ldots e_r$

Duality and the Pseudoscalar

If $e_1, \ldots, e_n$ is an orthonormal basis for the vector space the the pseudoscalar I is defined by

$$I = e_1 \ldots e_n \tag{47}$$

Since one can transform one orthonormal basis to another by an orthogonal transformation the I's for all orthonormal bases are equal to within a ± 1 scale factor with depends on the ordering of the basis vectors.

If A_r is a pure r grade multivector $(A_r = \langle A_r \rangle_r)$ then

$$A_r I = \langle A_r I \rangle_{n-r} \tag{48}$$

or $A_r I$ is a pure $n - r$ grade multivector. Further by the symmetry properties of I we have

$$IA_r = (-1)^{(n-1)r} A_r I \tag{49}$$

I can also be used to exchange the $\cdot$ and $\wedge$ products as follows using equations 42 and 43

$$\begin{aligned} a \cdot (A_r I) &= \frac{1}{2}\left(aA_r I - (-1)^{n-r} A_r I a\right) & (50) \\ &= \frac{1}{2}\left(aA_r I - (-1)^{n-r} (-1)^{n-1} A_r a I\right) & (51) \\ &= \frac{1}{2}\left(aA_r + (-1)^{r} A_r a\right) I & (52) \\ &= (a \wedge A_r) I & (53) \end{aligned}$$

More generally if A_r and B_s are pure grade multivectors with $r+s \leq n$ we have using equation 40 and 48

$$
\begin{aligned}
A_r \cdot (B_s I) &= \langle A_r B_s I \rangle_{|r-(n-s)|} && (54) \\
&= \langle A_r B_s I \rangle_{n-(r+s)} && (55) \\
&= \langle A_r B_s \rangle_{r+s} I && (56) \\
&= (A_r \wedge B_s) I && (57)
\end{aligned}
$$

Finally we can relate I to $I^\dagger$ by

$$
I^\dagger = (-1)^{\frac{n(n-1)}{2}} I \qquad (58)
$$

Reciprocal Frames

Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be a set of linearly independent vectors that span the vector space that are not necessarily orthogonal. These vectors define the frame (frame vectors are shown in bold face since they are almost always associated with a particular coordinate system) with volume element

$$E_n \equiv \mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_n \tag{59}$$

So that $E_n \propto I$. The reciprocal frame is the set of vectors $\mathbf{e}^1, \ldots, \mathbf{e}^n$ that satisfy the relation

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j, \quad \forall i, j = 1, \ldots, n \tag{60}$$

The $\mathbf{e}^i$ are constructed as follows

$$\mathbf{e}^j = (-1)^{j-1} \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \ldots \wedge \breve{\mathbf{e}}_j \wedge \ldots \wedge \mathbf{e}_n E_n^{-1} \tag{61}$$

So that the dot product is (using equation 53 since $E_n^{-1} \propto I$)

$$\begin{aligned} \mathbf{e}_i \cdot \mathbf{e}^j &= (-1)^{j-1}\, \mathbf{e}_i \cdot \left(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \ldots \wedge \breve{\mathbf{e}}_j \wedge \ldots \wedge \mathbf{e}_n E_n^{-1}\right) && (62) \\ &= (-1)^{j-1} \left(\mathbf{e}_i \wedge \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \ldots \wedge \breve{\mathbf{e}}_j \wedge \ldots \wedge \mathbf{e}_n\right) E_n^{-1} && (63) \\ &= 0, \quad \forall i \neq j && (64) \end{aligned}$$

and

$$\begin{aligned} \mathbf{e}_1 \cdot \mathbf{e}^1 &= \mathbf{e}_1 \cdot \left(\mathbf{e}_2 \wedge \ldots \wedge \mathbf{e}_n E_n^{-1}\right) && (65) \\ &= \left(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \ldots \wedge \mathbf{e}_n\right) E_n^{-1} && (66) \\ &= 1 && (67) \end{aligned}$$

Coordinates

The reciprocal frame can be used to develop a coordinate representation for multivectors in an arbitrary frame $\mathbf{e}_1, \ldots, \mathbf{e}_n$ with reciprocal frame $\mathbf{e}^1, \ldots, \mathbf{e}^n$.

Since both the frame and it's reciprocal span the base vector space we can write any vector a in the vector space as

$$a = a^i \mathbf{e}_i = a_i \mathbf{e}^i \tag{68}$$

where if an index such as i is repeated it is assumes that the terms with the repeated index will be summed from 1 to n. Using that $\mathbf{e}_i \cdot \mathbf{e}^j = \delta_i^j$ we have

$$a_i = a \cdot \mathbf{e}_i \tag{69}$$

$$a^i = a \cdot \mathbf{e}^i \tag{70}$$

In tensor notation a_i would be the covarient representation and a^i the contravarient representation of the vector a.

Now consider the case of grade 2 and grade 3 blades:

$$\begin{aligned}
\mathbf{e}^i \cdot (a \wedge b) &= a \cdot \mathbf{e}^i b - b \cdot \mathbf{e}^i a \\
\mathbf{e}_i \left(a \cdot \mathbf{e}^i b - b \cdot \mathbf{e}^i a\right) &= ab - ba = 2a \wedge b \\
\mathbf{e}^i \cdot (a \wedge b \wedge c) &= \\
& a \cdot \mathbf{e}^i b \wedge c - b \cdot \mathbf{e}^i a \wedge c + c \cdot \mathbf{e}^i a \wedge b \\
\mathbf{e}_i \left(a \cdot \mathbf{e}^i b \wedge c - b \cdot \mathbf{e}^i a \wedge c + c \cdot \mathbf{e}^i a \wedge b\right) &= \\
& ab \wedge c - ba \wedge c + ca \wedge b = 3a \wedge b \wedge c
\end{aligned}$$

for an r-blade A_r we have (the proof is left to the student)

$$\mathbf{e}_i \mathbf{e}^i \cdot A_r = r A_r \tag{71}$$

Since $\mathbf{e}_i\mathbf{e}^i = n$ we have

$$\mathbf{e}_i\mathbf{e}^i \wedge A_r = \mathbf{e}_i\left(\mathbf{e}^i A_r - \mathbf{e}^i \cdot A_r\right) = (n-r)\,A_r \tag{72}$$

Flipping $\mathbf{e}^i$ and A_r in equations 71 and 72 and subtracting equation 71 from 72 gives

$$\mathbf{e}_i A_r \mathbf{e}^i = (-1)^r\,(n-2r)\,A_r \tag{73}$$

In Hestenes and Sobczyk (3.14) it is proved that

$$\left(\mathbf{e}^{k_r} \wedge \ldots \wedge \mathbf{e}^{k_1}\right) \cdot \left(\mathbf{e}_{j_1} \wedge \ldots \wedge \mathbf{e}_{j_r}\right) = \delta^{j_1}_{k_1}\delta^{j_2}_{k_2}\ldots\delta^{j_r}_{k_r} \tag{74}$$

so that the general multivector A can be expanded in terms of the blades of the frame and reciprocal frame as

$$A = \sum_{i<j<\cdots<k} A_{ij\cdots k}\mathbf{e}^i \wedge \mathbf{e}^j \wedge \cdots \wedge \mathbf{e}^k \tag{75}$$

where

$$A_{ij\cdots k} = (\mathbf{e}_k \wedge \cdots \wedge \mathbf{e}_j \wedge \mathbf{e}_i) \cdot A \tag{76}$$

The components $A_{ij\cdots k}$ are totally antisymmetric on all indices and are usually referred to as the components of an *antisymmetric tensor*.

Linear Transformations

Let f be a linear transformation $f : \mathcal{V} \rightarrow \mathcal{V}$ with $f(\alpha a + \beta b) = \alpha f(a) + \beta f(b)\ \forall a, b \in \mathcal{V}$ and $\alpha, \beta \in \Re$. Then define the action of f on a blade of the geometric algebra by

$$f(a_1 \wedge \ldots \wedge a_r) = f(a_1) \wedge \ldots \wedge f(a_1) \tag{77}$$

and the action of f on any two $A, B \in \mathcal{G}(\mathcal{V})$ by

$$f(\alpha A + \beta B) = \alpha f(A) + \beta f(B) \tag{78}$$

Since any multivector A can be expanded as a sum of blades $f(A)$ is defined. This has many consequences. Consider the following definition for the determinant of f, $\det(f)$.

$$f(I) = \det(f)\, I \tag{79}$$

First show that this definition is equivalent to the standard definition of the determinant (again $e_1, \ldots, e_N$ is an orthonormal basis for $\mathcal{V}$).

$$f\left(e_r\right) = \sum_{s=1}^{N} a_{rs} e_s \tag{80}$$

Then

$$\begin{aligned} f\left(I\right) &= \left(\sum_{s_1=1}^{N} a_{1s_1} e_s\right) \wedge \ldots \wedge \left(\sum_{s_N=1}^{N} a_{Ns_N} e_s\right) \\ &= \sum_{s_1,\ldots,s_N} a_{1s_1} \ldots a_{Ns_N} e_{s_1} \ldots e_{s_N} \end{aligned} \tag{81}$$

But

$$e_{s_1} \ldots e_{s_N} = \varepsilon_{1\ldots N}^{s_1\ldots s_N} e_1 \ldots e_N \tag{82}$$

so that

$$f(I) = \sum_{s_1,\ldots,s_N} \varepsilon_{1\ldots N}^{s_1\ldots s_N} a_{1s_1} \ldots a_{Ns_N} I \tag{83}$$

or

$$\det(f) = \sum_{s_1,\ldots,s_N} \varepsilon_{1\ldots N}^{s_1\ldots s_N} a_{1s_1} \ldots a_{Ns_N} \tag{84}$$

which is the standard definition. Now compute the determinant of the product of the linear transformations f and g

$$\begin{aligned} \det(fg)\, I &= fg(I) \\ &= f(g(I)) \\ &= f(\det(g)\, I) \\ &= \det(g)\, f(I) \\ &= \det(g)\det(f)\, I \end{aligned} \tag{85}$$

or

$$\det(fg) = \det(f)\det(g) \tag{86}$$

Do you have any idea of how miserable that is to prove from the standard definition of determinant?

Adjoint

If F is linear transformation and a and b are two arbitrary vectors the adjoint function, $\overline{F}$, is defined by

$$a \cdot \overline{F}(b) = b \cdot F(a) \tag{87}$$

From the definition the adjoint is also a linear transformation. For an arbitrary frame $\mathbf{e}_1, \ldots, \mathbf{e}_n$ we have

$$\mathbf{e}_i \cdot \overline{F}(a) = a \cdot F(\mathbf{e}_i) \tag{88}$$

So that we can explicitly construct the adjoint as

$$\begin{aligned} \overline{F}(a) &= \mathbf{e}^i a \cdot F(\mathbf{e}_i) && (89) \\ &= \mathbf{e}^i \left(F(\mathbf{e}_i) \cdot \mathbf{e}^j\right) a_j && (90) \end{aligned}$$

so that $\overline{F}_{ij} = F(\mathbf{e}_i)\cdot\mathbf{e}^j$ is the matrix representation of $\overline{F}$ for the $\mathbf{e}_1,\ldots,\mathbf{e}_n$ frame. However

$$F(a) = \mathbf{e}^i\left(F\left(\mathbf{e}^j\right)\cdot\mathbf{e}_i\right)a_j \tag{91}$$

so that the matrix representation of F is $F_{ij} = F\left(\mathbf{e}^j\right)\cdot\mathbf{e}_i$. If the $\mathbf{e}_1,\ldots,\mathbf{e}_n$ are orthonormal then $\mathbf{e}_j = \mathbf{e}^j$ for all j and $\overline{F}_{ij} = F_{ji}$ exactly the same as the adjoint in matrices.

Other basic properties of the adjoint are:

$$\overline{\overline{F}}(a) = \mathbf{e}^i a\cdot\overline{F}(\mathbf{e}_i) = \mathbf{e}^i\mathbf{e}_i\cdot F(a) = F(a) \tag{92}$$

and

$$\begin{aligned}\overline{FG}(a) &= \mathbf{e}^i a\cdot F(G(\mathbf{e}_i)) = \overline{F}(a)\cdot G(\mathbf{e}_i)\,\mathbf{e}^i \\ &= \overline{G}\left(\overline{F}(a)\right)\cdot\mathbf{e}_i\mathbf{e}^i = \overline{G}\,\overline{F}(a)\end{aligned} \tag{93}$$

so that $\overline{\overline{F}} = F$ and $\overline{FG} = \overline{G}\,\overline{F}$.

A symmetric function is one where $F = \overline{F}$. As an example consider $F\overline{F}$

$$\overline{F\overline{F}} = \overline{\overline{F}}\,\overline{F} = F\overline{F} \tag{94}$$

Inverse

Another linear algebraic relation in geometric algebra is

$$f^{-1}(A) = \frac{I\overline{f}\left(I^{-1}A\right)}{\det(f)} \quad \forall A \in \mathcal{G}(\mathcal{V}) \tag{95}$$

where $\overline{f}$ is the adjoint transformation defined by $a \cdot \overline{f}(b) = b \cdot f(a)$ $\forall a, b \in \mathcal{V}$ and you have an explicit formula for the inverse of a linear transformation!

Quaternions

Any multivector $A \in \mathcal{G}(3,0)$ may be written as

$$A = \alpha + a + B + \beta I \tag{96}$$

where $\alpha, \beta \in \Re$, $a \in \mathcal{V}(3,0)$, B is a bivector, and I is the unit pseudoscalar. The quaternions are the multivectors of even grades

$$A = \alpha + B \tag{97}$$

B can be represented as

$$B = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k} \tag{98}$$

where $\mathbf{i} = e_2e_3$, $\mathbf{j} = e_1e_3$, and $\mathbf{k} = e_1e_2$, and

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \tag{99}$$

The quaternions form a subalgebra of $\mathcal{G}(3,0)$ since the geometric product of any two quaternions is also a quaternion since the geometric product of two even grade multivector components is a even grade multivector. For example the product of two grade 2 multivectors can only consist of grades 0, 2, and 4, but in $\mathcal{G}(3,0)$ we can only have grades 0 and 2 since the highest possible grade is 3.

Spinors

The general definition of a spinor is a multivector, $\psi \in \mathcal{G}(p,q)$, such that $\psi v \psi^\dagger \in \mathcal{V}(p,q) \;\; \forall v \in \mathcal{V}(p,q)$. Practically speaking a spinor is the composition of a rotation and a dilation (stretching or shrinking) of a vector. Thus we can write

$$\psi v \psi^\dagger = \rho R v R^\dagger \tag{100}$$

where R is a rotor $\left(RR^\dagger = 1\right)$. Letting $U = R^\dagger \psi$ we must solve

$$U v U^\dagger = \rho v \tag{101}$$

U must generate a pure dilation. The most general form for U based on the fact that the l.h.s of equation 101 must be a vector is

$$U = \alpha + \beta I \tag{102}$$

so that

$$UvU^{\dagger} = \alpha^2 v + \alpha\beta \left(Iv + vI^{\dagger}\right) + \beta^2 IvI^{\dagger} = \rho v \tag{103}$$

Using $vI^{\dagger} = (-1)^{\frac{(n-1)(n-2)}{2}} Iv$, $vI^{\dagger} = (-1)^{n-1} I^{\dagger}v$, and $II^{\dagger} = (-1)^{q}$ we get

$$\alpha^2 v + \alpha\beta \left(1 + (-1)^{\frac{(n-1)(n-2)}{2}}\right) Iv + (-1)^{n+q-1} \beta^2 v = \rho v \tag{104}$$

If $\dfrac{(n-1)(n-2)}{2}$ is even $\beta = 0$ and $\alpha \neq 0$, otherwise $\alpha, \beta \neq 0$. For the odd case

$$\psi = R(\alpha + \beta I) \tag{105}$$

where $\rho = \alpha^2 + (-1)^{n+q-1} \beta^2$. In the case of $\mathcal{G}(1,3)$ (relativistic space time) we have $\rho = \alpha^2 + \beta^2$, $\rho > 0$.

Geometric Algebra of the Minkowski Plane

Because of Relativity and QM the Geometric Algebra of the Minkowski Plane is very important for physical applications of Geometric Algebra so we will treat it in detail.

Let the orthonormal basis vectors for the plane be γ_0 and γ_1 where $\gamma_0^2 = -\gamma_1^2 = 1$.[5] Then the geometric product of two vectors $a = a_0\gamma_0 + a_1\gamma_1$ and $b = b_0\gamma_0 + b_1\gamma_1$ is

$$\begin{aligned} ab &= (a_0\gamma_0 + a_1\gamma_1)(b_0\gamma_0 + b_1\gamma_1) && (106) \\ &= a_0b_0\gamma_0^2 + a_1b_1\gamma_1^2 + (a_0b_1 - a_1b_0)\gamma_0\gamma_1 && (107) \\ &= a_0b_0 - a_1b_1 + (a_0b_1 - a_1b_0) I && (108) \end{aligned}$$

[5] $I = \gamma_0\gamma_1$

so that

$$a \cdot b = a_0 b_0 - a_1 b_1 \tag{109}$$

and

$$a \wedge b = (a_0 b_1 - a_1 b_0)\, I \tag{110}$$

and

$$I^2 = \gamma_0 \gamma_1 \gamma_0 \gamma_1 = -\gamma_0^2 \gamma_1^2 = 1 \tag{111}$$

Thus

$$\begin{aligned} e^{\alpha I} &= \sum_{i=0}^{\infty} \frac{\alpha^i I^i}{i!} && (112) \\ &= \sum_{i=0}^{\infty} \frac{\alpha^{2i}}{(2i)!} + \sum_{i=0}^{\infty} \frac{\alpha^{2i+1} I}{(2i+1)!} && (113) \\ &= \cosh(\alpha) + \sinh(\alpha)\, I && (114) \end{aligned}$$

since $I^{2i} = 1$.

In the Minkowski plane all vectors of the form $a_\pm = \alpha\left(\gamma_0 \pm \gamma_1\right)$ are null $\left(a_\pm^2 = 0\right)$. One question to answer are there any vectors $b_\pm$ such that $a_\pm \cdot b_\pm = 0$ that are not parallel to $a_\pm$.

$$\begin{gathered} a_\pm \cdot b_\pm = \alpha\left(b_0^\pm \mp b_1^\pm\right) = 0 \\ b_0^\pm \mp b_1^\pm = 0 \\ b_0^\pm = \pm b_1^\pm \end{gathered}$$

Thus $b_\pm$ must be proportional to $a_\pm$ and the are no vectors in the space that can be constructed that are normal to $a_\pm$. Thus there are no non-zero bivectors, $a \wedge b$, such that $(a \wedge b)^2 = 0$. Conversely, if $a \wedge b \neq 0$ then $(a \wedge b)^2 > 0$.

Finally for the condition that there always exist two orthogonal vectors e_1 and e_2 such that

$$a \wedge b = e_1 e_2 \tag{115}$$

we can state that neither e_1 nor e_2 can be null.

Lorentz Transformation

We now have all the tools needed to derive the Lorentz transformation with Geometric Algebra. Consider a two dimensional time-like plane with with coordinates t[6] and x_1 and basis vectors γ_0 and γ_1. Then a general space-time vector in the plane is given by

$$x = t\gamma_0 + x_1\gamma_1 = t'\gamma_0' + x_1'\gamma_1' \tag{116}$$

where the basis vectors of the two coordinate systems are related by

$$\gamma_\mu' = R\gamma_\mu R^\dagger \;\; \mu = 0, 1 \tag{117}$$

[6]We let the speed of light $c = 1$.

and R is a Minkowski plane rotor

$$R = \sinh\left(\frac{\alpha}{2}\right) + \cosh\left(\frac{\alpha}{2}\right)\gamma_1\gamma_0 \tag{118}$$

so that

$$R\gamma_0 R^\dagger = \cosh\left(\alpha\right)\gamma_0 + \sinh\left(\alpha\right)\gamma_1 \tag{119}$$

and

$$R\gamma_1 R^\dagger = \cosh\left(\alpha\right)\gamma_1 + \sinh\left(\alpha\right)\gamma_0 \tag{120}$$

Now consider the special case that the primed coordinate system is moving with velocity β in the direction of γ_1 and the two coordinate systems were coincident at time $t = 0$. Then $x_1 = \beta t$ and $x_1' = 0$ so we may write

$$t\gamma_0 + \beta t\gamma_1 = t' R\gamma_0 R^\dagger \tag{121}$$

$$\frac{t}{t'}\left(\gamma_0 + \beta\gamma_1\right) = \cosh\left(\alpha\right)\gamma_0 + \sinh\left(\alpha\right)\gamma_1 \tag{122}$$

Equating components gives

$$\cosh(\alpha) = \frac{t}{t'} \tag{123}$$

$$\sinh(\alpha) = \frac{t}{t'}\beta \tag{124}$$

Solving for α and $\dfrac{t}{t'}$ in equations 123 and 124 gives

$$\tanh(\alpha) = \beta \tag{125}$$

$$\frac{t}{t'} = \gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{126}$$

Now consider the general case of x, t and x', t' giving

$$\begin{aligned} t\gamma_0 + x\gamma_1 &= t'R\gamma_0R^\dagger + x'R\gamma_1R^\dagger && (127)\\ &= t'\gamma(\gamma_0 + \beta\gamma_1) + x'\gamma(\gamma_1 + \beta\gamma_0) && (128)\end{aligned}$$

Equating basis vector coefficients recovers the Lorentz transformation

$$\begin{aligned} t &= \gamma\left(t' + \beta x'\right) \\ x &= \gamma\left(x' + \beta t'\right) \end{aligned} \tag{129}$$

Commutator Product

The commutator product of two multivectors A and B is defined as

$$A \times B \equiv \frac{1}{2}(AB - BA) \tag{130}$$

An important theorem for the commutator product is that for a grade 2 multivector, $A_2 = \langle A \rangle_2$, and a grade r multivector $B_r = \langle B \rangle_r$ we have

$$A_2 B_r = A_2 \wedge B_r + A_2 \times B_r + A_2 \cdot B_r \tag{131}$$

From the geometric product grade expansion for multivectors we have

$$A_2 B_r = \langle A_2 B_r \rangle_{r+2} + \langle A_2 B_r \rangle_r + \langle A_2 B_r \rangle_{|r-2|} \tag{132}$$

Thus we must show that

$$\langle A_2 B_r \rangle_r = A_2 \times B_r \tag{133}$$

Let $e_1, \ldots, e_n$ be an orthogonal set for the vector space where $B_r = e_1 \ldots e_r$ and $A_2 = \sum_{l<m=2}^{n} \alpha_{lm} e_l e_m$ so we can write

$$A_2 \times B_r = \left(\sum_{l<m=2}^{n} \alpha_{lm} e_l e_m \right) \times (e_1 \ldots e_r) \tag{134}$$

Now consider the following three cases

1. l and $m > r$ where $e_l e_m e_1 \ldots e_r = e_1 \ldots e_r e_l e_m$

2. $l \leq r$ and $m > r$ where $e_l e_m e_1 \ldots e_r = -e_1 \ldots e_r e_l e_m$

3. l and $m \leq r$ where $e_l e_m e_1 \ldots e_r = e_1 \ldots e_r e_l e_m$

For case 1 and 3 $e_l e_m$ commute with B_r and the contribution to the commutator product is zero. In case 3 $e_l e_m$ anticommutes with B_r and thus are the only terms that contribute to the commutator. All these terms are of grade r and the theorem is proved.

Additionally, the commutator product obeys the Jacobi identity

$$A\times (B\times C) = (A\times B) \times C + B\times (A\times C) \tag{135}$$

This is important for the geometric algebra treatment of Lie groups and algebras.

Differentiation

If $F\left(a\right)$ if a multivector valued function of the vector a, and a and b are any vectors in the space then the derivative of F is defined by

$$b\cdot\nabla F \equiv \lim_{\epsilon\rightarrow 0}\frac{F\left(a+\epsilon b\right)-F\left(a\right)}{\epsilon} \tag{136}$$

then letting $a=\mathbf{e}_k$ be the components of a coordinate frame with $x=x^k\mathbf{e}_k$ we have

$$\mathbf{e}_k\cdot\nabla F = \lim_{\epsilon\rightarrow 0}\frac{F\left(x^j\mathbf{e}_j+\epsilon\mathbf{e}_k\right)-F\left(x^j\mathbf{e}_j\right)}{\epsilon} \tag{137}$$

Using what we know about coordinates gives

$$\nabla F = \mathbf{e}^j\frac{\partial F}{\partial x^j} = \mathbf{e}^j\partial_j F \tag{138}$$

or looking at ∇ as a symbolic operator we may write

$$\nabla = \mathbf{e}^j \partial_j \tag{139}$$

Due to the properties of coordinate frame expansions ∇F is independent of the choice of the $\mathbf{e}_k$ frame. If we consider x to be a position vector then $F(x)$ is in general a multivector field.

Dervatives of Scalar Functions

If $f(x)$ is scalar valued function of the vector x then the derivative is

$$\nabla f = \mathbf{e}^k \partial_k f \tag{140}$$

which is the standard definition of the gradient of a scalar function (remember that in an orthonormal coordinate system $\mathbf{e}_k = \mathbf{e}^k$).

Using equation 140 we can show the following results for the gradient of some specific scalar functions

$$\begin{array}{ccccc} f & = & x \cdot a & x^k & xx \\ \nabla f & = & a & \mathbf{e}^k & 2x \end{array} \tag{141}$$

Product Rule

Let $\circ$ represent a bilinear product operator such as the geometric, inner, or outer product and note that for the multivector fields F and G we have

$$\partial_k (F \circ G) = (\partial_k F) \circ G + F \circ (\partial_k G) \tag{142}$$

so that

$$\begin{aligned} \nabla (F \circ G) &= \mathbf{e}^k ((\partial_k F) \circ G + F \circ (\partial_k G)) \\ &= \mathbf{e}^k (\partial_k F) \circ G + \mathbf{e}^k F \circ (\partial_k G) \end{aligned} \tag{143}$$

However since the geometric product is not communicative, in general

$$\nabla (F \circ G) \neq (\nabla F) \circ G + F \circ (\nabla G) \tag{144}$$

The notation adopted by Hestenes is

$$\nabla (F \circ G) = \nabla F \circ G + \dot{\nabla} F \circ \dot{G} \tag{145}$$

The convention of the overdot notation is

i. In the absence of brackets, ∇ acts on the object to its immediate right

ii. When the ∇ is followed by brackets, the derivative acts on all the the terms in the brackets.

iii. When the ∇ acts on a multivector to which it is not adjacent, we use overdots to describe the scope.

Note that with the overdot notation the expression $\dot{A}\dot{\nabla}$ makes sense!

Interior and Exterior Derivative

The interior and exterior derivatives of an r-grade multivector field are simply defined as

$$\nabla \cdot A_r \equiv \langle \nabla A_r \rangle_{r-1} = \mathbf{e}^k \cdot \partial_k A_r \tag{146}$$

and

$$\nabla \wedge A_r \equiv \langle \nabla A_r \rangle_{r+1} = \mathbf{e}^k \wedge \partial_k A_r \tag{147}$$

Note that

$$\begin{aligned} \nabla \wedge (\nabla \wedge A_r) &= \mathbf{e}^i \partial_i \left(\mathbf{e}^j \wedge \partial_j A_r \right) \\ &= \mathbf{e}^i \wedge \mathbf{e}^j \wedge (\partial_i \partial_j A_r) \\ &= 0 \end{aligned} \tag{148}$$

since $\mathbf{e}^i \wedge \mathbf{e}^j = -\mathbf{e}^j \wedge \mathbf{e}^i$, but $\partial_i \partial_j A_r = \partial_j \partial_i A_r$.

$$\begin{aligned}
\nabla \cdot (\nabla \cdot A_r) &= \mathbf{e}^i \cdot \partial_i \left(\mathbf{e}^j \cdot \partial_j A_r \right) \\
&= \mathbf{e}^i \cdot \left(\mathbf{e}^j \cdot (\partial_i \partial_j A_r) \right) \\
&= \pm \mathbf{e}^i \cdot \left(\mathbf{e}^j \cdot (\partial_i \partial_j A_r^* I) \right) \\
&= \pm \mathbf{e}^i \cdot \left(\left(\mathbf{e}^j \wedge (\partial_i \partial_j A_r^*) \right) I \right) \\
&= \pm \left(\mathbf{e}^i \wedge \left(\mathbf{e}^j \wedge (\partial_i \partial_j A_r^*) \right) \right) I \\
&= 0 \qquad (149)
\end{aligned}$$

Where * indicates the dual of a multivector, $A^* = AI$ (I is the pseudoscalar and $A = \pm A^* I$ since $I^2 = \pm 1$), and we use equation 53 to exchange the inner and outer products.

Thus for the general multivector field A (built from sums of A_r's) we have $\nabla \wedge (\nabla \wedge A) = 0$ and $\nabla \cdot (\nabla \cdot A) = 0$. If ϕ is a scalar function we also have

$$\begin{aligned}\nabla \wedge (\nabla \phi) &= \mathbf{e}^i \wedge \partial_i \left(\mathbf{e}^j \partial_j \phi\right) \\ &= \mathbf{e}^i \wedge \mathbf{e}^j \partial_i \partial_j \phi \\ &= 0 \end{aligned} \tag{150}$$

Another use for the overdot notation would in the case where $f(a)$ is a linear function of a and a is a general function of position. Then the meaning of the overdot notation would be

$$\dot{\nabla} \dot{f}(a) = \nabla f(a) - \mathbf{e}^k f(\partial_k a) \tag{151}$$

Other basic results (examples) are

$$\nabla x \cdot A_r = r A_r \tag{152}$$

$$\nabla x \wedge A_r = (n-r)\, A_r \tag{153}$$

$$\dot{\nabla} A_r \dot{x} = (-1)^r\,(n-2r)\, A_r \tag{154}$$

The basic identities for the case of a scalar field α and multivector field F are

$$\nabla (\alpha F) = (\nabla \alpha)\, F + \alpha \nabla F \tag{155}$$

$$\nabla \cdot (\alpha F) = (\nabla \alpha) \cdot F + \alpha \nabla \cdot F \tag{156}$$

$$\nabla \wedge (\alpha F) = (\nabla \alpha) \wedge F + \alpha \nabla \wedge F \tag{157}$$

if f_1 and f_2 are vector fields

$$\nabla \wedge (f_1 \wedge f_2) = (\nabla \wedge f_1) \wedge f_2 - (\nabla \wedge f_2) \wedge f_1 \tag{158}$$

and finally if F_r is a grade r multivector field

$$\nabla \cdot (F_r I) = (\nabla \wedge F_r)\, I \tag{159}$$

where I is the psuedoscalar for the geometric algebra.

Curvilinear Coordinates

A general set of coordinates consists of a set of scalar functions $\left\{x^i\left(x\right), i = 1, \ldots, n\right\}$ defined over some region. In this region the functions are invertible so that we may write $x\left(x^1, \ldots, x^n\right)$ (x is defined parametrically in terms of the x^i's). The frame vectors at each point x are defined by

$$\mathbf{e}_i\left(x\right) \equiv \frac{\partial x}{\partial x^i} \tag{160}$$

and

$$\mathbf{e}_i \cdot \nabla = \mathbf{e}_i \cdot \mathbf{e}^k \partial_k = \delta_i^k \partial_k = \partial_i \tag{161}$$

In order that the coordinate system be valid over the region we also require that

$$\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n \neq 0 \tag{162}$$

or that the orientation of the coordinate system does not change of the region of interest. The reciprocal frame to $\mathbf{e}_i$ can be constructed from

$$\mathbf{e}^i = \nabla x^i \tag{163}$$

since

$$\mathbf{e}_i \cdot \mathbf{e}^j = \mathbf{e}_i \cdot \nabla x^i = \frac{\partial x^j}{\partial x^i} = \delta_i^j \tag{164}$$

what we are saying is that equation 163 gives exactly the same set of vectors as does the geometric construction of the reciprocal frame from the $\mathbf{e}_i$. Also note that

$$\nabla \wedge \left(\mathbf{e}^i\right) = \nabla \wedge \left(\nabla x^i\right) = 0 \tag{165}$$

Now assume we have a function $F(x) = F\left(x^1, \ldots, x^n\right)$ of x expressed in terms of the coordinates $x^1, \ldots, x^n$ and we use the chain rule to calculate ∇F as

$$\nabla F = \nabla x^i \partial_i F = e^i \partial_i F \tag{166}$$

Tensor Analysis

A consequence of curvilinear frame vectors is that one has to be careful when working entirely in terms of coordinates, as in the case of tensor analysis, since the way the tensor components change also depend on the way the frame vectors change as a function of position. When calculating the derivatives of vector and tensor fields in curvilinear coordinate systems one must in general use connection coefficients (christoffel symbols). Two cases where connection coefficients are not needed are the exterior and interior derivatives of a vector field $J = J^i \mathbf{e}_i = J_i \mathbf{e}^i$.

$$\begin{aligned}
\nabla \wedge J &= \nabla \wedge \left(J_i \mathbf{e}^i \right) \\
&= \mathbf{e}^j \wedge \partial_j \left(J_i \mathbf{e}^i \right) \\
&= \mathbf{e}^j \wedge \left(\partial_j J_i \mathbf{e}^i + J_i \partial_j \mathbf{e}^i \right) \\
&= \left(\mathbf{e}^j \partial_j J_i \right) \wedge \mathbf{e}^i + J_i \left(\mathbf{e}_j \wedge \partial_j \mathbf{e}^i \right)
\end{aligned}$$

$$
\begin{aligned}
&= (\nabla J_i) \wedge \mathbf{e}^i + J_i \nabla \wedge \mathbf{e}^i \\
&= (\nabla J_i) \wedge \mathbf{e}^i
\end{aligned} \tag{167}
$$

$$
\begin{aligned}
\nabla \cdot J &= \nabla \cdot \left(J^i \mathbf{e}_i\right) \\
&= \left(\nabla J^i\right) \cdot \mathbf{e}_i + J^i \nabla \cdot \mathbf{e}_i
\end{aligned} \tag{168}
$$

To reduce equation 168 further we must use the following relationships between the frame vectors and their reciprocal vectors.

$$
\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_n = IV \tag{169}
$$

where V is a volume factor function of position that insures that I is constant.

$$
\mathbf{e}_i = (-1)^{i-1} \mathbf{e}^n \wedge \mathbf{e}^{n-1} \wedge \cdots \wedge \breve{\mathbf{e}}^i \wedge \cdots \wedge \mathbf{e}^1 IV = \mathcal{E}^i IV \tag{170}
$$

With these expressions we can write

$$\begin{aligned}\nabla \cdot \mathbf{e}_i &= \nabla \cdot \left(\mathcal{E}^i I V\right) \\ &= \nabla \cdot \left(\mathcal{E}^i I\right) V + \left(\mathcal{E}^i I\right) \cdot \nabla V \\ &= \left(\nabla \wedge \mathcal{E}^i\right) I V + \left(\mathcal{E}^i I\right) \cdot \nabla V \end{aligned} \tag{171}$$

from the properties of $\mathcal{E}^i$ and $\nabla \wedge \mathbf{e}^i = 0$ we have $\nabla \wedge \mathcal{E}^i = 0$. Also $\mathcal{E}^i I = \dfrac{\mathbf{e}_i}{V}$ so that

$$\begin{aligned}\nabla \cdot \mathbf{e}_i &= \frac{\mathbf{e}_i}{V} \cdot \nabla V \\ &= \frac{\mathbf{e}_i \cdot \mathbf{e}^j}{V} \partial_j V \\ &= \frac{\partial_i V}{V} \end{aligned} \tag{172}$$

substituting back into equation 168 gives

$$\begin{aligned}\nabla \cdot J &= \left(\nabla J^i\right) \cdot \mathbf{e}_i + \frac{J^i}{V}\partial_i V \\ &= \left(\mathbf{e}^j \partial_j J^i\right) \cdot \mathbf{e}_i + \frac{J^i}{V}\partial_i V \\ &= \frac{1}{V}\frac{\partial}{\partial x^i}\left(V J^i\right) \qquad (173)\end{aligned}$$

if $J = \nabla\phi$ we have

$$\nabla^2\phi = \frac{1}{V}\frac{\partial}{\partial x^i}\left(V g^{ij}\frac{\partial\phi}{\partial x^j}\right) \qquad (174)$$

where $g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j$ and $V = \sqrt{g} = \sqrt{\det\left(g_{ij}\right)}$ where $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$.

Analytic Functions

Starting with $\mathcal{G}(2,0)$ and orthonormal basis vectors $\mathbf{e}_x$ and $\mathbf{e}_y$ so that $I = \mathbf{e}_x\mathbf{e}_y$ and $I^2 = -1$. Then we have

$$\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y \tag{175}$$

$$\nabla = \mathbf{e}_x\frac{\partial}{\partial x} + \mathbf{e}_y\frac{\partial}{\partial y} \tag{176}$$

Map $\mathbf{r}$ onto the complex number z via

$$z = x + Iy = \mathbf{e}_x\mathbf{r} \tag{177}$$

Define the multivector field $\psi = u + Iv$ where u and v are scalar fields. Then

$$\nabla\psi = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\mathbf{e}_x + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\mathbf{e}_y \tag{178}$$

Thus the statement that ψ is an analytic function is equivalent to

$$\nabla\psi = 0 \tag{179}$$

This is the fundamental equation that can be generalized to higher dimensions remembering that in general that ψ is a multivector rather than a scalar function!

To complete the connection with complex analysis we define $(z^\dagger = x - Iy)$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - I\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial z^\dagger} = \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) \tag{180}$$

so that

$$\begin{aligned} \frac{\partial z}{\partial z} = 1, \quad \frac{\partial z^\dagger}{\partial z} = 0 \\ \frac{\partial z}{\partial z^\dagger} = 0, \quad \frac{\partial z^\dagger}{\partial z^\dagger} = 1 \end{aligned} \tag{181}$$

An analytic function is one that depends on z alone so that we can write $\psi(x + Iy) = \psi(z)$ and

$$\frac{\partial \psi(z)}{\partial z^{\dagger}} = 0 \tag{182}$$

equivalently

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right)\psi = \frac{1}{2}\mathbf{e}_x \nabla \psi = 0 \tag{183}$$

Now it is simple to show why solutions to $\nabla \psi = 0$ can be written as a power series in z. First

$$\begin{aligned}
\nabla z &= \nabla(\mathbf{e}_x \mathbf{r}) \\
&= \mathbf{e}_x \mathbf{e}_x \frac{\partial \mathbf{r}}{\partial x} + \mathbf{e}_y \mathbf{e}_x \frac{\partial \mathbf{r}}{\partial y} \\
&= \mathbf{e}_x \mathbf{e}_x \mathbf{e}_x + \mathbf{e}_y \mathbf{e}_x \mathbf{e}_y \\
&= \mathbf{e}_x - \mathbf{e}_x \\
&= 0
\end{aligned} \tag{184}$$

so that

$$\nabla \left(z - z_0\right)^k = k\nabla \left(\mathbf{e}_x\mathbf{r} - z_0\right)\left(z - z_0\right)^{k-1} = 0 \tag{185}$$

Directed Intergration - Line Integrals

If $F(x)$ is a multivector field and $x(\lambda)$ is a parametric representation of a vector path (curve) then the line intergral of F along the path x is defined to be

$$\int F(x)\frac{dx}{d\lambda}d\lambda = \int F\,dx = \lim_{n\mapsto\infty}\sum_{i=1}^{n}\bar{F}^i\Delta x^i \tag{186}$$

where

$$\Delta x^i = x_i - x_{i-1}, \quad \bar{F}^i = \frac{1}{2}\left(F(x_{i-1}) + F(x_i)\right) \tag{187}$$

if $x_n = x_1$ the path is closed. Since dx is a vector, that is

$F(x)\dfrac{dx}{d\lambda} \neq \dfrac{dx}{d\lambda}F(x)$, a more general line integral would be

$$\int F(x)\frac{dx}{d\lambda}G(x) = \int F(x)\,dx\,G(x) \tag{188}$$

The most general form of line integral would be

$$\int \mathsf{L}(\partial_\lambda x; x)\ d\lambda = \int \mathsf{L}(dx) \tag{189}$$

where $\mathsf{L}(a) = \mathsf{L}(a;x) =$ is a multivector-valued linear function of a. The position dependence in L can often be suppressed to streamline the notation.

Directed Intergration - Surface Integrals

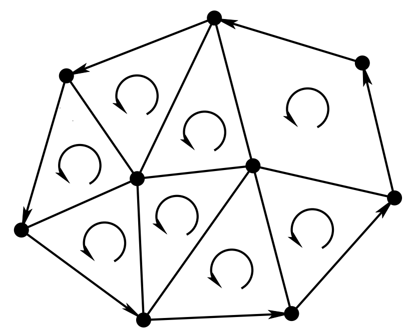
The next step is a directed surface integral. Let $F(x)$ be a multivector field and let a surface be parametrized by two coordinates $x\left(x^1, x^2\right)$. Then we can define a directed surface measure by

$$dX = \frac{\partial x}{\partial x^1} \wedge \frac{\partial x}{\partial x^2}\, dx^1 dx^2 = \mathbf{e}_1 \wedge \mathbf{e}_2\, dx^1 dx^2 \tag{190}$$

A directed surface integral takes the form

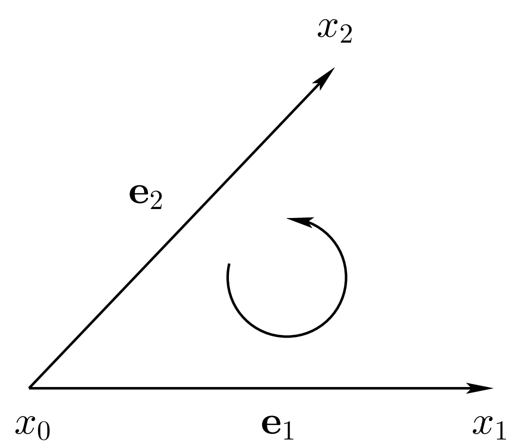
$$\int F\, dX = \int F \mathbf{e}_1 \wedge \mathbf{e}_2\, dx^1 dx^2 \tag{191}$$

In order to construct some of the more important proof it is necessary to express the surface integral as the limit of a sum. This requires the concept of a triangulated surface as shown



Triangulated Surface

Each triangle in the surface is described by a planar simplex as shown



Planar Simplex

The three vertices of the planar simplex are x_0, x_1, and x_2 with the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ defined by

$$\mathbf{e}_1 = x_1 - x_0, \quad \mathbf{e}_2 = x_2 - x_0 \tag{192}$$

so that the surface measure of the simplex is

$$\Delta X = \frac{1}{2}\mathbf{e}_1 \wedge \mathbf{e}_2 = \frac{1}{2}\left(x_1 \wedge x_2 + x_2 \wedge x_0 + x_0 \wedge x_1\right) \tag{193}$$

with this definition of ΔX we have

$$\int F\, dX = \lim_{n \mapsto \infty} \sum_{k=1}^{n} \bar{F}^k \Delta X^k \tag{194}$$

where $\bar{F}^k$ is the average of F over the k^{th} simplex.

Directed Intergration - n-dimensional Surfaces

The definition of the simplex can be extended to an n-dimensional surface by defining vertices $x_0, x_1, \ldots, x_n$, with the order of the vertices defining the orientation of the simplex. Then define the vectors

$$e_i = x_i - x_0, \quad i = 1, \ldots, n, \tag{195}$$

and the directed volume element

$$\Delta X = \frac{1}{n!} e_1 \wedge \cdots \wedge e_n \tag{196}$$

Any point in the simplex can be written in terms of the coordinates λ^i as

$$x = x_0 + \sum_{i=1}^{n} \lambda^i e_i \tag{197}$$

with restrictions

$$0 \leq \lambda^i \leq 1 \quad \text{and} \quad \sum_{i=1}^{n} \lambda^i \leq 1 \tag{198}$$

First we show that

$$\int_{\text{simplex}} dX = \int_{\text{simplex}} e_1 \wedge \cdots \wedge e_n \, d\lambda^1 \cdots d\lambda^n = \Delta X \tag{199}$$

or

$$\int_{\text{simplex}} d\lambda^1 \cdots d\lambda^n = \frac{1}{n!} \tag{200}$$

define $\Lambda_j = 1 - \sum_{i=1}^{j} \lambda^i$. From the restrictions on the λ^i's we have

$$\int_{\text{simplex}} d\lambda^1 \cdots d\lambda^n = \int_0^{\Lambda_0} d\lambda^1 \int_0^{\Lambda_1} d\lambda^2 \cdots \int_0^{\Lambda_{n-1}} d\lambda^n \tag{201}$$

To prove that the r.h.s of equation 201 is $1/n!$ we form the following sequence and use induction to prove that V_j is the result of the first j partial intergrations of equation 201

$$V_j = \frac{1}{j!}\left(\Lambda_{n-j}\right)^j \tag{202}$$

Then

$$\begin{aligned} V_{j+1} &= \int_0^{\Lambda_{n-j-1}} d\lambda^{n-j} V_j \\ &= \int_0^{\Lambda_{n-j-1}} d\lambda^{n-j} \frac{1}{j!}\left(\Lambda_{n-j-1} - \lambda^{n-j}\right)^j \\ &= \frac{-1}{(j+1)\,j!}\left[\left(\Lambda_{n-j-1} - \lambda^{n-j}\right)^{j+1}\right]_0^{\Lambda_{n-j-1}} \\ &= \frac{1}{(j+1)!}\left(\Lambda_{n-j-1}\right)^{j+1} \end{aligned} \tag{203}$$

so that $V_n = 1/n!$ and the assertion is proved. Now let there be a multivector field $F(x)$ that assumes the values $F_i = F(x_i)$ at the vectices of the simplex and define the interpolating function

$$f(x) = F_0 + \sum_{i=1}^{n} \lambda^i (F_i - F_0) \tag{204}$$

We now wish to show that

$$\int_{\text{simplex}} f \, dX = \frac{1}{n+1} \left(\sum_{i=0}^{n} F_i \right) \Delta X = \bar{F} \, dX \tag{205}$$

To prove this we must show that

$$\int_{\text{simplex}} \lambda^i \, dX = \frac{1}{n+1} \Delta X, \quad \forall \lambda^i \tag{206}$$

To do this consider the integral

$$\int_0^{\Lambda_{n-j-1}} d\lambda^{n-j}\, \lambda^{n-j} V_j = \int_0^{\Lambda_{n-j-1}} d\lambda^{n-j} \frac{1}{j!} \lambda^{n-j} \left(\Lambda_{n-j-1} - \lambda^{n-j}\right)^j$$

$$= \frac{1}{(j+2)!} \left(\Lambda_{n-j-1}\right)^{j+2} \qquad (207)$$

Note that since the extra λ^i factor occurs in exactly one of the subintegrals for each different λ^i the final result of the total integral is mulipliedby a factor of $\frac{1}{(n+1)}$ since the weight of the total integral is now $\frac{1}{(n+1)!}$ and the assertion (equation 206 and hence equation 205) is proved.

Now summing over all the simplices making up the surface gives

$$\int_{\text{surface}} F\, dX = \lim_{n \mapsto \infty} \sum_{i=1}^{n} \bar{F}^i dX^k \qquad (208)$$

The most general statement of equation 208 is

$$\int_{\mathsf{surface}} \mathsf{L}\left(dX\right) = \lim_{n \mapsto \infty} \sum_{i=1}^{n} \bar{\mathsf{L}}^{i}\left(dX^{i}\right) \tag{209}$$

where $\mathsf{L}\left(F_n; x\right)$ is a position dependent linear function of a grade-n multivector F_n and $\bar{\mathsf{L}}^{i}$ is the average value of $\mathsf{L}\left(dX\right)$ over the vertices of each simplex.

Directed Intergration - The Fundamental Theorem of Geometric Calculus

We let $(x_0, x_1, \ldots, x_k)$ denote the k-simplex defined by the $k+1$ points $x_0, x_1, \ldots, x_k$. This is abbreviated by

$$(x)_{(k)} = (x_0, x_1, \ldots, x_k) \tag{210}$$

The order of the points is important for a simplex, since it specified the orientation of the simplex. If any two adjacent points are swaped the simplex changes sign. The boundary operator for the simplex is denoted by ∂ and defined by

$$\partial\,(x)_{(k)} \equiv \sum_{i=0}^{k} (-1)^i\,(x_0, \ldots, \breve{x}_i, \ldots, x_k)_{(k-1)} \tag{211}$$

To see that this make sense cosider a triangle $(x)_{(3)} = (x_0, x_1, x_2)$. Then

$$\begin{aligned} \partial\,(x)_{(3)} &= (x_1, x_2)_{(2)} - (x_0, x_2)_{(2)} + (x_0, x_1)_{(2)} \\ &= (x_1, x_2)_{(2)} + (x_2, x_0)_{(2)} + (x_0, x_1)_{(2)} \end{aligned} \tag{212}$$

each simplex in the boudary connects head to tail with the same sign. Now consider the boundary of the boundary

$$\begin{aligned} \partial^2\,(x)_{(3)} &= \partial\,(x_1, x_2)_{(2)} + \partial\,(x_2, x_0)_{(2)} + \partial\,(x_0, x_1)_{(2)} \\ &= (x_1)_{(1)} - (x_2)_{(1)} + (x_2)_{(1)} - (x_0)_{(1)} + (x_0)_{(1)} - (x_1)_{(1)} \\ &= 0 \end{aligned} \tag{213}$$

What we need to prove is that in general $\partial^2\,(x)_{(k)} = 0$. To do this consider the boundary of the i^{th} term on thr r.h.s. of equation 211 letting

$A_{ij}^{(k-2)} = (x_0, \ldots, \breve{x}_i, \ldots, \breve{x}_i, \ldots, x_k)_{(k-1)}$, noting $A_{ij}^{(k-2)} = A_{ji}^{(k-2)}$

$$\partial\left(x_0, \ldots, \breve{x}_i, \ldots, x_k\right)_{(k-1)} = \sum_{j=1}^{i-1} (-1)^j A_{ij}^{(k-2)} + \sum_{j=i+1}^{k} (-1)^{j-1} A_{ij}^{(k-2)} \tag{214}$$

The critical point in equation 214 is that the exponent of -1 in the second term on the r.h.s. is not j, but $j-1$. The reason for this is that when x_i was removed from the simplex the vertices were **not** renumbered. We can now express the boundary of the boundary as ($B_{ij}^{(k-2)} = (-1)^{i+j} A_{ij}^{(k-2)}$)

$$\begin{aligned} \partial^2 (x)_{(k)} &= \sum_{i=1}^{k} (-1)^i \left(\sum_{j=1}^{i-1} (-1)^j A_{ij}^{(k-2)} + \sum_{j=i+1}^{k} (-1)^{j-1} A_{ij}^{(k-2)} \right) \\ &= \sum_{i=1}^{k} \sum_{j=1}^{i-1} B_{ij}^{(k-2)} - \sum_{i=1}^{k} \sum_{j=i+1}^{k} B_{ij}^{(k-2)} = 0 \end{aligned} \tag{215}$$

Now add some geometry to the simplex by defining the Δ operator which returns the directed content of the simplex

$$\Delta\left(x\right)_{(k)} \equiv \frac{1}{k!}\left(x_1 - x_0\right) \wedge \left(x_2 - x_0\right) \wedge \cdots \wedge \left(x_k - x_0\right) \tag{216}$$

This is the result of integrating the directed measure over a simplex

$$\int_{(x)_{(k)}} dX = \Delta\left(x\right)_{(k)} = \Delta X \tag{217}$$

We now must prove that

$$\Delta\left(\partial\left(x\right)_{(k)}\right) = 0 \tag{218}$$

Consider a planar simplex of three points

$$\partial\left(x_0, x_1, x_2\right)_{(3)} = \left(x_1, x_2\right)_{(2)} - \left(x_0, x_2\right)_{(2)} + \left(x_0, x_1\right)_{(2)} \tag{219}$$

so that

$$\Delta\left(\partial\left(x_0, x_1, x_2\right)_{(3)}\right) = \left(x_2 - x_1\right) - \left(x_2 - x_0\right) + \left(x_1 - x_0\right) = 0 \qquad (220)$$

We shall now prove equation 218 via induction. First note that

$$\Delta\left(\breve{x}_i\right)_{(k-1)} = \left\{\begin{array}{cc} i = 0: & \frac{1}{k-1}\Delta\left(\breve{x}_0\right)_{(k-2)} \wedge \left(x_k - x_1\right) \\ 0 < i \leq k-1: & \frac{1}{k-1}\Delta\left(\breve{x}_i\right)_{(k-2)} \wedge \left(x_k - x_0\right) \end{array}\right\} \qquad (221)$$

so that

$$\Delta\left(\partial\left(x\right)_{(k)}\right) = \frac{1}{k-1}\sum_{i=1}^{k-1}\left(-1\right)^i \Delta\left(\breve{x}_i\right)_{(k-2)} \wedge \left(x_k - x_0\right) + \mathcal{C} \qquad (222)$$

where

$$\mathcal{C} = \frac{1}{k-1}\Delta\left(\breve{x}_0\right)_{(k-2)} \wedge \left(x_k - x_1\right) + (-1)^k\,\Delta\left(\breve{x}_k\right)_{(k-1)} \tag{223}$$

if we let $\delta = x_0 - x_1$ we can write

$$\mathcal{C} = \frac{1}{k-1}\Delta\left(\breve{x}_0\right)_{(k-2)} \wedge \left(x_k - x_0\right) + \frac{1}{k-1}\Delta\left(\breve{x}_0\right)_{(k-2)} \wedge \delta + (-1)^k\,\Delta\left(\breve{x}_k\right)_{(k-1)} \tag{224}$$

However

$$(-1)^k\,\Delta\left(\breve{x}_k\right)_{(k-1)} = \frac{-1}{k-1}\Delta\left(\breve{x}_0\right)_{(k-2)} \wedge \delta \tag{225}$$

so that

$$\mathcal{C} = \frac{1}{k-1}\Delta\left(\breve{x}_0\right)_{(k-2)} \wedge \left(x_k - x_1\right) \tag{226}$$

and

$$
\begin{aligned}
\Delta\left(\partial\left(x\right)_{(k)}\right) &= \frac{1}{k-1}\left(\sum_{i=0}^{k-1}\left(-1\right)^{i}\Delta\left(\breve{x}_{i}\right)_{(k-2)}\right)\wedge\left(x_{k}-x_{0}\right) \\
&= \frac{1}{k-1}\left(\Delta\left(\partial\left(x\right)_{(k-1)}\right)\right)\wedge\left(x_{k}-x_{0}\right) \\
&= 0 \qquad (227)
\end{aligned}
$$

We have proved equatin 218. Note that to reduce equation 226 we had to use that for any set of vectors $\delta, y_1, \ldots, y_k$ we have

$$\delta\wedge\left(y_{1}+\delta\right)\wedge\cdots\wedge\left(y_{k}+\delta\right)=\delta\wedge y_{1}\wedge\cdots\wedge y_{k} \qquad (228)$$

Think about equation 228. It's easy to prove!

Equation 218 is sufficient to prove that the directed integral over the surface of simplex is zero

$$\oint_{\partial(x)_{(k)}} dS = \sum_{i=0}^{k} (-1)^i \int_{(\breve{x}_i)_{(k-1)}} dX = \Delta\left(\partial\left(x\right)_{(k)}\right) = 0 \qquad (229)$$

The characteristics of a geneneral volume are:

1. A general volume is built up from a chain of simplicies.
2. Simplices in the chain are defined so that at any common boundary the directed areas of the bounding faces are equal and opposite.
3. Surface integrals over two simplices cancel over their common face.
4. The surface integral over the boundary of the volume can be replaced by the sum of the surface integrals over each simplex in the chain.

If the boundary of the volume is closed we have

$$\oint dS = \lim_{n \mapsto \infty} \sum_{a=1}^{n} \oint dS^a = 0 \tag{230}$$

Where $\oint dS^a$ is the surface intergral over the a^{th} simplex. Implicit in equation 230 is that the surface is orientated, simply connected, and closed.

References

1. Cambridge site: www.mrao.cam.ac.uk/∼clifford

2. Arizona State site: modelingnts.la.asu.edu

3. Chris Doran and Anthony Lasenby, "Geometric Algebra for Physicists," Cambridge University Press, 2003.

4. David Hestenes and Garret Sobczyk, "Clifford Algebra to Geometric Calculus," Kluwer Academic Publishers, 1984.

5. David Hestenes, "New Foundations for Classical Mechanics (2^{nd} edition)," Kluwer Academic Publishers, 1999.