An Introduction to Geometric Algebra and Calculus

Alan Bromborsky
Army Research Lab (Retired)
abrombo@verizon.net

May 25, 2010

History

Geometric algebra is the Clifford algebra of a finite dimensional vector space over real scalars cast in a form most appropriate for physics and engineering. This was done by David Hestenes (Arizona State University) in the 1960's. From this start he developed the geometric calculus whose fundamental theorem includes the generalized Stokes theorem, the residue theorem, and new integral theorems not realized before. Hestenes likes to say he was motivated by the fact that physicists and engineers did not know how to multiply vectors.

Researchers at Arizona State and Cambridge have applied these developments to classical mechanics,quantum mechanics, general relativity (gauge theory of gravity), projective geometry, conformal geometry, etc.

Axioms of Geometric Algebra

Let $\mathcal{V}\left(p,q\right)$ be a finite dimensional vector space of signature (p,q)[1] over $\Re$. Then $\forall a,b,c \in \mathcal{V}$ there exists a geometric product with the properties -

$$\begin{gathered}(ab)c = a(bc)\\ a(b+c) = ab+ac\\ (a+b)c = ac+bc\\ aa \in \Re\end{gathered}$$

If $a^2 \neq 0$ then $a^{-1} = \dfrac{1}{a^2}a$.

---

[1] To be completely general we would have to consider $\mathcal{V}\left(p,q,r\right)$ where the dimension of the vector space is $n = p+q+r$ and p, q, and r are the number of basis vectors respectively with positive, negative and zero squares.

Why Learn This Stuff?

The geometric product of two (or more) vectors produces something "new" like the $\sqrt{-1}$ with respect to real numbers or vectors with respect to scalars. It must be studied in terms of its effect on vectors and in terms of its symmetries. It is worth the effort. Anything that makes understanding rotations in a N dimensional space simple is worth the effort! Also, if one proceeds on to geometric calculus many diverse areas in mathematics are unified and many areas of physics and engineering are greatly simplified.

Inner, $\cdot$, and outer, $\wedge$, product of two vectors and their basic properties

$$a \cdot b \equiv \frac{1}{2}\left(ab + ba\right) \tag{1}$$

$$a \wedge b \equiv \frac{1}{2}\left(ab - ba\right) \tag{2}$$

$$ab = a \cdot b + a \wedge b \tag{3}$$

$$a \wedge b = -b \wedge a \tag{4}$$

$$\begin{gathered}
c = a + b \\
c^2 = \left(a + b\right)^2 \\
c^2 = a^2 + ab + ba + b^2 \\
2a \cdot b = c^2 - a^2 - b^2 \\
a \cdot b \in \Re
\end{gathered} \tag{5}$$

$$a \cdot b = \left|a\right| \left|b\right| \cos\left(\theta\right) \text{ if } a^2, b^2 > 0 \tag{6}$$

Orthogonal vectors are defined by $a \cdot b = 0$.
For orthogonal vectors $a \wedge b = ab$.

Now compute $(a \wedge b)^2$

$$\begin{aligned}
(a \wedge b)^2 &= -(a \wedge b)(b \wedge a) && (7)\\
&= -(ab - a \cdot b)(ba - a \cdot b) && (8)\\
&= -\Big(abba - (a \cdot b)(ab + ba) + (a \cdot b)^2\Big) && (9)\\
&= -\Big(a^2b^2 - (a \cdot b)^2\Big) && (10)\\
&= -a^2b^2\left(1 - \cos^2(\theta)\right) && (11)\\
&= -a^2b^2\sin^2(\theta) && (12)
\end{aligned}$$

Thus in a Euclidean space, $a^2, b^2 > 0$, $(a \wedge b)^2 \leq 0$ and $a \wedge b$ is proportional to $\sin(\theta)$. If $e_{\|}$ and $e_{\perp}$ are any two orthonormal unit vectors in a Euclidean space then $\left(e_{\|}e_{\perp}\right)^2 = -1$. Who needs the $\sqrt{-1}$?

Outer, $\wedge$, product for r Vectors in terms of the geometric product

We define the outer product of r vectors to be

$$a_1 \wedge \ldots \wedge a_r \equiv \frac{1}{r!} \sum_{i_1,\ldots,i_r} \varepsilon^{i_1 \ldots i_r}_{1 \ldots r} a_{i_1} \ldots a_{i_r} \tag{13}$$

Thus

$$\begin{aligned} a_1 \wedge \ldots \wedge (a_j + b_j) \wedge \ldots \wedge a_r &= \\ a_1 \wedge \ldots \wedge a_j \wedge \ldots \wedge a_r &+ a_1 \wedge \ldots \wedge b_j \wedge \ldots \wedge a_r \end{aligned} \tag{14}$$

and

$$\begin{aligned} a_1 \wedge \ldots \wedge a_j \wedge a_{j+1} \wedge \ldots \wedge a_r &= \\ -a_1 \wedge \ldots \wedge a_{j+1} \wedge a_j \wedge \ldots \wedge a_r & \end{aligned} \tag{15}$$

The outer product of r vectors is called a blade of grade r.

Alternate Definition of Outer, $\wedge$, product for r Vectors

Let $e_1, e_2, \ldots, e_r$ be an orthogonal basis for the set of linearly independent vectors $a_1, a_2, \ldots, a_r$ so that we can write

$$a_i = \sum_j \alpha_{ij} e_j \tag{16}$$

Then

$$\begin{aligned} a_1 a_2 \ldots a_r &= \left(\sum_{j_1} \alpha_{1j_1} e_{j_1}\right)\left(\sum_{j_2} \alpha_{2j_2} e_{j_2}\right) \ldots \left(\sum_{j_r} \alpha_{rj_r} e_{j_r}\right) \\ &= \sum_{j_1,\ldots,j_r} \alpha_{1j_1}\alpha_{2j_2}\ldots\alpha_{rj_r} e_{j_1} e_{j_2} \ldots e_{j_r} \end{aligned} \tag{17}$$

Now define a blade of grade n as the geometric product of n orthogonal vectors. Thus the product $e_{j_1}e_{j_2}\ldots e_{j_r}$ in equation 17 could be a blade of grade r, $r-2$, $r-4$, etc. depending upon the number of repeated factors.

If there are no repeated factors in the product we have that

$$e_{j_1}\ldots e_{j_r} = \varepsilon^{j_1\ldots j_r}_{1\ldots r} e_1\ldots e_r \tag{18}$$

Due to the fact that interchanging two adjacent orthogonal vectors in the geometric product will reverse the sign of the product and we can define the outer product of r vectors as

$$\begin{aligned} a_1\wedge\ldots\wedge a_r &= \sum_{j_1,\ldots,j_r} \varepsilon^{j_1\ldots j_r}_{1\ldots r}\alpha_{1j_1}\ldots\alpha_{rj_r}e_1\ldots e_r && (19)\\ &= \det(\alpha)\, e_1\ldots e_r && (20)\end{aligned}$$

Thus the outer product of r independent vectors is the part of the geometric product of the r vectors that is of grade r. Equation 19 is equivalent to equation 13.

This can be proved by substituting equation 17 into equation 13 to get

$$
\begin{aligned}
a_1 \wedge \ldots \wedge a_r &= \frac{1}{r!} \sum_{i_1,\ldots,i_r} \sum_{j_1,\ldots,j_r} \varepsilon^{i_1\ldots i_r}_{1\ldots r} \alpha_{i_1 j_1} \ldots \alpha_{i_r j_r} e_{j_1} \ldots e_{j_r} && (21) \\
&= \frac{1}{r!} \sum_{i_1,\ldots,i_r} \sum_{j_1,\ldots,j_r} \varepsilon^{i_1\ldots i_r}_{1\ldots r} \varepsilon^{j_1\ldots j_r}_{1\ldots r} \alpha_{i_1 j_1} \ldots \alpha_{i_r j_r} e_1 \ldots e_r && (22) \\
&= \frac{1}{r!} \sum_{j_1,\ldots,j_r} \varepsilon^{j_1\ldots j_r}_{1\ldots r} \varepsilon^{j_1\ldots j_r}_{1\ldots r} \det(\alpha)\, e_1 \ldots e_r && (23) \\
&= \det(\alpha)\, e_1 \ldots e_r && (24)
\end{aligned}
$$

We go from equation 22 to equation 23 by noting that $\sum\limits_{i_1,\ldots,i_r} \varepsilon^{i_1\ldots i_r}_{1\ldots r} \alpha_{i_1 j_1} \ldots \alpha_{i_r j_r}$ is just $\det(\alpha)$ with the columns permuted.

Multiplying $\det(\alpha)$ by $\varepsilon^{j_1 \ldots j_r}_{1 \ldots r}$ gives the correct sign for the determinant with the columns permuted.

If $e_1, \ldots, e_n$ is an orthonormal basis for vector space the unit psuedoscalar is defined as

$$I = e_1 \ldots e_n \tag{25}$$

In equation 24 let $r = n$ and the $a_1, \ldots, a_n$ be another orthonormal basis for the vector space. Then we may write

$$a_1 \ldots a_n = \det(\alpha)\, e_1 \ldots e_n \tag{26}$$

Since both the a's and the e's form orthonormal bases the matrix α is orthogonal and $\det(\alpha) = \pm 1$. All psuedoscalars for the vector space are identical to within a scale factor of ± 1.[2]

Likewise $a_1 \wedge \ldots \wedge a_n$ is equal to I times a scale factor.

---

[2] It depends only upon the ordering of the basis vectors.

Useful Relation's

1. For a set of r orthogonal vectors, $e_1, \ldots, e_r$

$$e_1 \wedge \ldots \wedge e_r = e_1 \ldots e_r \tag{27}$$

2. For a set of r linearly independent vectors, $a_1, \ldots, a_r$, there exists a set of r orthogonal vectors, $e_1, \ldots, e_r$, such that

$$a_1 \wedge \ldots \wedge a_r = e_1 \ldots e_r \tag{28}$$

   If the vectors, $a_1, \ldots, a_r$, are not linearly independent then

$$a_1 \wedge \ldots \wedge a_r = 0 \tag{29}$$

The product $a_1 \wedge \ldots \wedge a_r$ is call a "blade" of grade r. The dimension of the vector space is the highest grade any blade can have.

Projection Operator

A multivector, the basic element of the geometric algebra, is made of of a sum of scalars, vectors, blades. A multivector is homogeneous (pure) if all the blades in it are of the same grade. The grade of a scalar is 0 and the grade of a vector is 1. The general multivector A is decomposed with the grade projection operator $\langle A \rangle_r$ as (N is dimension of the vector space):

$$A = \sum_{r=0}^{N} \langle A \rangle_r \tag{30}$$

As an example consider ab, the product of two vectors. Then

$$ab = \langle ab \rangle_0 + \langle ab \rangle_2 \tag{31}$$

We define $\langle A \rangle \equiv \langle A \rangle_0$ for any multivector A

Basis Blades

The geometric algebra of a vector space, $\mathcal{V}(p,q)$, is denoted $\mathcal{G}(p,q)$ or $\mathcal{G}(\mathcal{V})$ where (p,q) is the signature of the vector space (first p unit vectors square to $+1$ and next q unit vectors square to -1, dimension of the space is $p+q$). Examples are:

p	q	Type of Space
3	0	3D Euclidean
1	3	Relativistic Space Time
4	1	3D Conformal Geometry

If the orthonormal basis set of the vector space is $e_1, \ldots, e_N$, the basis of the geometric algebra (multivector space) is formed from the geometric products (since we have chosen an orthonormal basis) of the basis vectors. For grade r multivectors the basis blades are all the combinations of basis vectors products taken r at a time from the set of N vectors. Thus the number basis blades of r rank are $\binom{N}{r}$, the binomial expansion coefficient and the total dimension of the multivector space is the sum of $\binom{N}{r}$ over r which is 2^N. Thus the basis blades for $\mathcal{G}(3,0)$ are:

Grade			
0	1	2	3
1	e_1	e_1e_2	$e_1e_2e_3$
	e_2	e_1e_3	
	e_3	e_2e_3	

The multiplication table for the $\mathcal{G}(3,0)$ basis blades is

	1	e_1	e_2	e_3	e_1e_2	e_1e_3	e_2e_3	$e_1e_2e_3$
1	1	e_1	e_2	e_3	e_1e_2	e_1e_3	e_2e_3	$e_1e_2e_3$
e_1	e_1	1	e_1e_2	e_1e_3	e_2	e_3	$e_1e_2e_3$	e_2e_3
e_2	e_2	$-e_1e_2$	1	e_2e_3	$-e_1$	$-e_1e_2e_3$	e_3	$-e_1e_3$
e_3	e_3	$-e_1e_3$	$-e_2e_3$	1	$e_1e_2e_3$	$-e_1$	$-e_2$	e_1e_2
e_1e_2	e_1e_2	$-e_2$	e_1	$e_1e_2e_3$	-1	$-e_2e_3$	e_1e_3	$-e_3$
e_1e_3	e_1e_3	$-e_3$	$-e_1e_2e_3$	e_1	e_2e_3	-1	$-e_1e_2$	e_2
e_2e_3	e_2e_3	$e_1e_2e_3$	$-e_3$	e_2	$-e_1e_3$	e_1e_2	-1	$-e_1$
$e_1e_2e_3$	$e_1e_2e_3$	e_2e_3	$-e_1e_3$	e_1e_2	$-e_3$	e_2	$-e_1$	-1

Note that the squares of all the grade 2 and 3 basis blades are -1. The highest rank basis blade (in this case $e_1e_2e_3$) is ususally denoted by I and is called the pseudoscalar.

The multiplication table for the $\mathcal{G}\left(1,3\right)$ basis blades is (Part I)

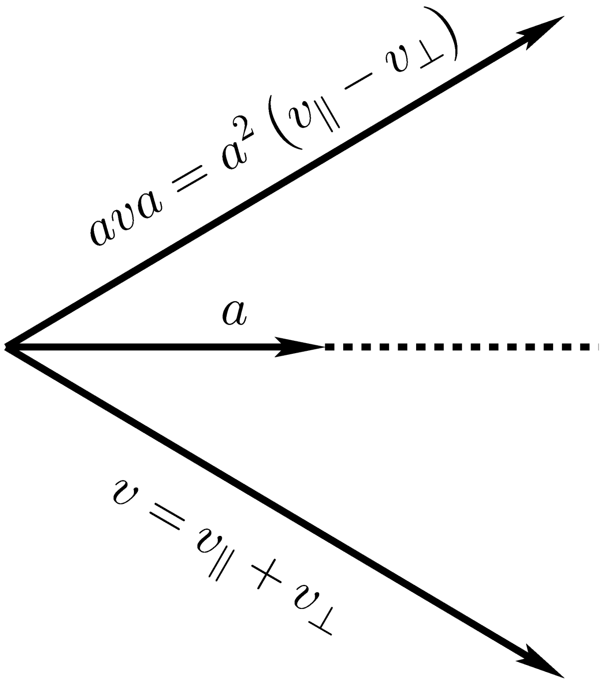
	1	γ_0	γ_1	γ_2	γ_3	$\gamma_0\gamma_1$	$\gamma_0\gamma_2$	$\gamma_1\gamma_2$
1	1	γ_0	γ_1	γ_2	γ_3	$\gamma_0\gamma_1$	$\gamma_0\gamma_2$	$\gamma_1\gamma_2$
γ_0	γ_0	1	$\gamma_0\gamma_1$	$\gamma_0\gamma_2$	$\gamma_0\gamma_3$	γ_1	γ_2	$\gamma_0\gamma_1\gamma_2$
γ_1	γ_1	$-\gamma_0\gamma_1$	-1	$\gamma_1\gamma_2$	$\gamma_1\gamma_3$	γ_0	$-\gamma_0\gamma_1\gamma_2$	$-\gamma_2$
γ_2	γ_2	$-\gamma_0\gamma_2$	$-\gamma_1\gamma_2$	-1	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2$	γ_0	γ_1
γ_3	γ_3	$-\gamma_0\gamma_3$	$-\gamma_1\gamma_3$	$-\gamma_2\gamma_3$	-1	$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$
$\gamma_0\gamma_1$	$\gamma_0\gamma_1$	$-\gamma_1$	$-\gamma_0$	$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_3$	1	$-\gamma_1\gamma_2$	$-\gamma_0\gamma_2$
$\gamma_0\gamma_2$	$\gamma_0\gamma_2$	$-\gamma_2$	$-\gamma_0\gamma_1\gamma_2$	$-\gamma_0$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2$	1	$\gamma_0\gamma_1$
$\gamma_1\gamma_2$	$\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2$	γ_2	$-\gamma_1$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_2$	$-\gamma_0\gamma_1$	-1
$\gamma_0\gamma_3$	$\gamma_0\gamma_3$	$-\gamma_3$	$-\gamma_0\gamma_1\gamma_3$	$-\gamma_0\gamma_2\gamma_3$	$-\gamma_0$	$\gamma_1\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$
$\gamma_1\gamma_3$	$\gamma_1\gamma_3$	$\gamma_0\gamma_1\gamma_3$	γ_3	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_1$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_2\gamma_3$
$\gamma_2\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$	γ_3	$-\gamma_2$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_3$	$\gamma_1\gamma_3$
$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2$	$\gamma_1\gamma_2$	$\gamma_0\gamma_2$	$-\gamma_0\gamma_1$	$\gamma_0\gamma_1\gamma_2\gamma_3$	γ_2	$-\gamma_1$	$-\gamma_0$
$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_1\gamma_3$	$\gamma_1\gamma_3$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_1$	γ_3	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_2\gamma_3$
$\gamma_0\gamma_2\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_2$	$\gamma_1\gamma_2\gamma_3$	γ_3	$\gamma_0\gamma_1\gamma_3$
$\gamma_1\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_2\gamma_3$	$\gamma_1\gamma_3$	$-\gamma_1\gamma_2$	$\gamma_0\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_3$	$-\gamma_3$
$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_3$	$-\gamma_0\gamma_1\gamma_2$	$\gamma_2\gamma_3$	$-\gamma_1\gamma_3$	$-\gamma_0\gamma_3$

The multiplication table for the $\mathcal{G}\left(1,3\right)$ basis blades is (Part II)

	$\gamma_0\gamma_3$	$\gamma_1\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$
1	$\gamma_0\gamma_3$	$\gamma_1\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$
γ_0	γ_3	$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2$	$\gamma_1\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$
γ_1	$-\gamma_0\gamma_1\gamma_3$	$-\gamma_3$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_2$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_2\gamma_3$	$\gamma_0\gamma_2\gamma_3$
γ_2	$-\gamma_0\gamma_2\gamma_3$	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_3$	$-\gamma_0\gamma_1$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_3$	$\gamma_1\gamma_3$	$-\gamma_0\gamma_1\gamma_3$
γ_3	γ_0	γ_1	γ_2	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_1$	$-\gamma_0\gamma_2$	$-\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2$
$\gamma_0\gamma_1$	$-\gamma_1\gamma_3$	$-\gamma_0\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$	γ_2	γ_3	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_2\gamma_3$	$\gamma_2\gamma_3$
$\gamma_0\gamma_2$	$-\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_3$	$-\gamma_1$	$\gamma_1\gamma_2\gamma_3$	γ_3	$\gamma_0\gamma_1\gamma_3$	$-\gamma_1\gamma_3$
$\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_2\gamma_3$	$-\gamma_1\gamma_3$	$-\gamma_0$	$\gamma_0\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_3$	$-\gamma_3$	$-\gamma_0\gamma_3$
$\gamma_0\gamma_3$	1	$\gamma_0\gamma_1$	$\gamma_0\gamma_2$	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_1$	$-\gamma_2$	$-\gamma_0\gamma_1\gamma_2$	$\gamma_1\gamma_2$
$\gamma_1\gamma_3$	$-\gamma_0\gamma_1$	-1	$\gamma_1\gamma_2$	$-\gamma_0\gamma_2\gamma_3$	$-\gamma_0$	$\gamma_0\gamma_1\gamma_2$	γ_2	$\gamma_0\gamma_2$
$\gamma_2\gamma_3$	$-\gamma_0\gamma_2$	$-\gamma_1\gamma_2$	-1	$\gamma_0\gamma_1\gamma_3$	$-\gamma_0\gamma_1\gamma_2$	$-\gamma_0$	$-\gamma_1$	$-\gamma_0\gamma_1$
$\gamma_0\gamma_1\gamma_2$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_3$	-1	$\gamma_2\gamma_3$	$-\gamma_1\gamma_3$	$-\gamma_0\gamma_3$	$-\gamma_3$
$\gamma_0\gamma_1\gamma_3$	$-\gamma_1$	$-\gamma_0$	$\gamma_0\gamma_1\gamma_2$	$-\gamma_2\gamma_3$	-1	$\gamma_1\gamma_2$	$\gamma_0\gamma_2$	γ_2
$\gamma_0\gamma_2\gamma_3$	$-\gamma_2$	$-\gamma_0\gamma_1\gamma_2$	$-\gamma_0$	$\gamma_1\gamma_3$	$-\gamma_1\gamma_2$	-1	$-\gamma_0\gamma_1$	$-\gamma_1$
$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2$	γ_2	$-\gamma_1$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_2$	$\gamma_0\gamma_1$	1	$-\gamma_0$
$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_1\gamma_2$	$\gamma_0\gamma_2$	$-\gamma_0\gamma_1$	γ_3	$-\gamma_2$	γ_1	γ_0	-1

Reflections

We wish to show that $a, v \in \mathcal{V} \to ava \in \mathcal{V}$ and v is reflected about a if $a^2 = 1$.



1. Decompose $v = v_\parallel + v_\perp$ where $v_\parallel$ is the part of v parallel to a and $v_\perp$ is the part perpendicular to a.

2. $av = av_{\|} + av_{\perp} = v_{\|}a - v_{\perp}a$ since a and $v_{\perp}$ are orthogonal.

3. $ava = a^2 \left(v_{\|} - v_{\perp}\right)$ is a vector since a^2 is a scalar.

4. ava is the reflection of v about the direction of a if $a^2 = 1$.

5. Thus $a_1 \dots a_r v a_r \dots a_1 \in \mathcal{V}$ and produces a composition of reflections of v if $a_1^2 = \dots = a_r^2 = 1$.

Rotations, Part 1

First define the reverse of a product of vectors. If $R = a_1 \ldots a_s$ then the reverse is $R^\dagger = (a_1 \ldots a_s)^\dagger = a_r \ldots a_1$, the order of multiplication is reversed. Then let $R = ab$ so that

$$RR^\dagger = (ab)(ba) = ab^2a = a^2b^2 = R^\dagger R \tag{32}$$

Let $RR^\dagger = 1$ and calculate $\left(RvR^\dagger\right)^2$, where v is an arbitrary vector.

$$\left(RvR^\dagger\right)^2 = RvR^\dagger RvR^\dagger = Rv^2R^\dagger = v^2RR^\dagger = v^2 \tag{33}$$

Thus $RvR^\dagger$ leaves the length of v unchanged.

Now we must also prove $Rv_1R^\dagger \cdot Rv_2R^\dagger = v_1 \cdot v_2$. Since $Rv_1R^\dagger$ and $Rv_2R^\dagger$ are both vectors we can use the definition of the dot product for two vectors

$$\begin{aligned}
Rv_1R^\dagger \cdot Rv_2R^\dagger &= \frac{1}{2}\left(Rv_1R^\dagger Rv_2R^\dagger + Rv_2R^\dagger Rv_1R^\dagger\right) \\
&= \frac{1}{2}\left(Rv_1v_2R^\dagger + Rv_2v_1R^\dagger\right) \\
&= \frac{1}{2}R\left(v_1v_2 + v_2v_1\right)R^\dagger \\
&= R\left(v_1 \cdot v_2\right)R^\dagger \\
&= v_1 \cdot v_2RR^\dagger \\
&= v_1 \cdot v_2
\end{aligned}$$

Thus the transformation $RvR^\dagger$ preserves both length and angle and must be a rotation. The normal designation for R is a rotor.

If we have a series of successive rotations $R_1, R_2, \ldots, R_k$ to be applied to a vector v then the result of the k rotations will be

$$R_k R_{k-1} \dots R_1 v R_1^\dagger R_2^\dagger \dots R_k^\dagger$$

Since each individual rotation can be written as the geometric product of two vectors, the composition of k rotations can be written as the geometric product of $2k$ vectors. The multivector that results from the geometric product of r vectors is called a **versor** of order r. A composition of rotations is always a versor of even order.

Rotations, Part 2

The general rotation can be represented by $R = e^{\frac{\theta}{2}u}$ where u is a unit bivector in the plane of the rotation and θ is the rotation angle in the plane.[3] The two possible non-degenerate cases are $u^2 = \pm 1$

$$e^{\frac{\theta}{2}u} = \left\{ \begin{array}{lrl} \text{(Euclidean plane)} & u^2 = -1: & \cos\left(\frac{\theta}{2}\right) + u\sin\left(\frac{\theta}{2}\right) \\ \text{(Minkowski plane)} & u^2 = 1: & \cosh\left(\frac{\theta}{2}\right) + u\sinh\left(\frac{\theta}{2}\right) \end{array} \right\} \tag{34}$$

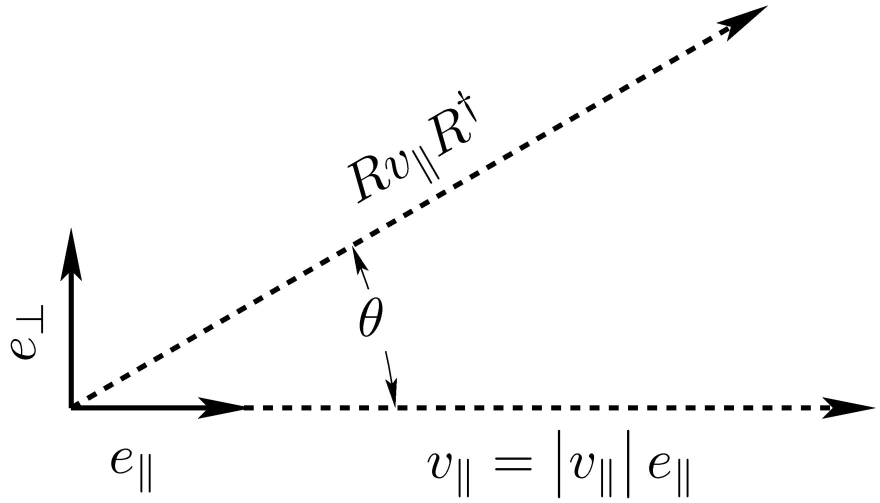
Decompose $v = v_{\|} + \left(v - v_{\|}\right)$ where $v_{\|}$ is the projection of v into the plane defined by u. Note the $v - v_{\|}$ is orthogonal to all vectors in the u plane. Now let $u = e_{\perp}e_{\|}$ where $e_{\|}$ is parallel to $v_{\|}$ and of course $e_{\perp}$ is in the plane u and orthogonal to $e_{\|}$. $v - v_{\|}$ anticommutes with $e_{\|}$ and $e_{\perp}$ and $v_{\|}$ anticommutes with $e_{\perp}$ (it is left to the viewer to show $RR^{\dagger} = 1$).

---

[3] e^A is defined as the Taylor series expansion $e^A = \sum_{j=0}^{\infty} \frac{A^j}{j!}$ where A is any multivector.

Euclidean Case

For the case of $u^2 = -1$



$$RvR^{\dagger} = \left(\cos\left(\frac{\theta}{2}\right) + e_{\perp}e_{\parallel}\sin\left(\frac{\theta}{2}\right)\right)\left(v_{\parallel} + (v - v_{\parallel})\right)\left(\cos\left(\frac{\theta}{2}\right) + e_{\parallel}e_{\perp}\sin\left(\frac{\theta}{2}\right)\right)$$

Since $v - v_{\|}$ anticommutes with $e_{\|}$ and $e_{\perp}$ it commutes with R and

$$RvR^{\dagger} = Rv_{\|}R^{\dagger} + \left(v - v_{\|}\right) \tag{35}$$

So that we only have to evaluate

$$Rv_{\|}R^{\dagger} = \left(\cos\left(\frac{\theta}{2}\right) + e_{\perp}e_{\|}\sin\left(\frac{\theta}{2}\right)\right) v_{\|} \left(\cos\left(\frac{\theta}{2}\right) + e_{\|}e_{\perp}\sin\left(\frac{\theta}{2}\right)\right) \tag{36}$$

Since $v_{\|} = \left|v_{\|}\right| e_{\|}$

$$Rv_{\|}R^{\dagger} = \left|v_{\|}\right| \left(\cos\left(\theta\right) e_{\|} + \sin\left(\theta\right) e_{\perp}\right) \tag{37}$$

and the component of v in the u plane is rotated correctly.

Minkowski Case

For the case of $u^2 = 1$ there are two possibilities, $v_{\|}^2 > 0$ or $v_{\|}^2 < 0$. In the first case $e_{\|}^2 = 1$ and $e_{\perp}^2 = -1$. In the second case $e_{\|}^2 = -1$ and $e_{\perp}^2 = 1$. Again $v - v_{\|}$ is not affected by the rotation so that we need only evaluate

$$Rv_{\|}R^{\dagger} = \left(\cosh\left(\frac{\theta}{2}\right) + e_{\perp}e_{\|}\sinh\left(\frac{\theta}{2}\right)\right) v_{\|} \left(\cosh\left(\frac{\theta}{2}\right) + e_{\|}e_{\perp}\sinh\left(\frac{\theta}{2}\right)\right)$$

Note that in this case $\left|v_{\|}\right| = \sqrt{\left|v_{\|}^2\right|}$ and

$$Rv_{\|}R^{\dagger} = \left\{ \begin{array}{c} v_{\|}^2 > 0 : \left|v_{\|}\right| \left(\cosh\left(\theta\right) e_{\|} + \sinh\left(\theta\right) e_{\perp}\right) \\ v_{\|}^2 < 0 : \left|v_{\|}\right| \left(\cosh\left(\theta\right) e_{\|} - \sinh\left(\theta\right) e_{\perp}\right) \end{array} \right\} \tag{38}$$

Expansion of geometric product and generalization of $\cdot$ and $\wedge$

If A_r and B_s are respectively grade r and s pure grade multivectors then

$$A_rB_s = \langle A_rB_s\rangle_{|r-s|} + \langle A_rB_s\rangle_{|r-s|+2} + \cdots + \langle A_rB_s\rangle_{\min(r+s,2N-(r+s))} \tag{39}$$

$$A_r \cdot B_s \equiv \langle A_rB_s\rangle_{|r-s|} \tag{40}$$

$$A_r \wedge B_s \equiv \langle A_rB_s\rangle_{r+s} \tag{41}$$

Thus if $r + s > N$ then $A_r \wedge B_s = 0$, also note that these formulas are the most efficient way of calculating $A_r \cdot B_s$ and $A_r \wedge B_s$.

Using equations 28 and 39 we can prove that for a vector a and a grade r multivector B_r

$$a \cdot B_r = \frac{1}{2}\left(aB_r - (-1)^r B_ra\right) \tag{42}$$

$$a \wedge B_r = \frac{1}{2}\left(aB_r + (-1)^r B_r a\right) \tag{43}$$

If equations 42 and 43 are true for a grade r blade they are also true for a grade r multivector (superposition of grade r blades). By equation 28 let $B_r = e_1 \ldots e_r$ where the $e's$ are orthogonal and expand a

$$a = a_\perp + \sum_{j=1}^{r} \alpha_j e_j \tag{44}$$

where $a_\perp$ is orthogonal to all the $e's$. Then[4]

$$\begin{aligned} aB_r &= \sum_{j=1}^{r} (-1)^{j-1} \alpha_j e_j^2 e_1 \cdots \breve{e}_j \cdots e_r + a_\perp e_1 \ldots e_r \\ &= a \cdot B_r + a \wedge B_r \end{aligned} \tag{45}$$

---

[4] $e_1 \ldots e_{j-1} \breve{e}_j e_{j+1} \ldots e_r = e_1 \ldots e_{j-1} e_{j+1} \ldots e_r$

Now calculate

$$\begin{aligned} B_r a &= \sum_{j=1}^{r} (-1)^{r-j} \alpha_j e_j^2 e_1 \cdots \breve{e}_j \cdots e_r - (-1)^{r-1} a_\perp e_1 \ldots e_r \\ &= (-1)^{r-1} \left(\sum_{j=1}^{r} (-1)^{j-1} \alpha_j e_j^2 e_1 \cdots \breve{e}_j \cdots e_r - a_\perp e_1 \ldots e_r \right) \\ &= (-1)^{r-1} \left(a \cdot B_r - a \wedge B_r \right) \qquad (46) \end{aligned}$$

Adding and subtracting equations 45 and 46 gives equations 42 and 43.

Duality and the Pseudoscalar

If $e_1, \ldots, e_n$ is an orthonormal basis for the vector space the the pseudoscalar I is defined by

$$I = e_1 \ldots e_n \tag{47}$$

Since one can transform one orthonormal basis to another by an orthogonal transformation the I's for all orthonormal bases are equal to within a ± 1 scale factor with depends on the ordering of the basis vectors.

If A_r is a pure r grade multivector $(A_r = \langle A_r \rangle_r)$ then

$$A_r I = \langle A_r I \rangle_{n-r} \tag{48}$$

or $A_r I$ is a pure $n - r$ grade multivector. Further by the symmetry properties of I we have

$$IA_r = (-1)^{(n-1)r} A_r I \tag{49}$$

I can also be used to exchange the $\cdot$ and $\wedge$ products as follows using equations 42 and 43

$$\begin{aligned}
a \cdot (A_r I) &= \frac{1}{2}\Big(aA_r I - (-1)^{n-r} A_r I a\Big) && (50)\\
&= \frac{1}{2}\Big(aA_r I - (-1)^{n-r} (-1)^{n-1} A_r a I\Big) && (51)\\
&= \frac{1}{2}\left(aA_r + (-1)^{r} A_r a\right) I && (52)\\
&= \left(a \wedge A_r\right) I && (53)
\end{aligned}$$

More generally if A_r and B_s are pure grade multivectors with $r + s \leq n$ we have using equation 40 and 48

$$\begin{aligned} A_r \cdot (B_s I) &= \left\langle A_r B_s I \right\rangle_{|r-(n-s)|} && (54) \\ &= \left\langle A_r B_s I \right\rangle_{n-(r+s)} && (55) \\ &= \left\langle A_r B_s \right\rangle_{r+s} I && (56) \\ &= (A_r \wedge B_s)\, I && (57) \end{aligned}$$

Finally we can relate I to $I^\dagger$ by

$$I^\dagger = (-1)^{\frac{n(n-1)}{2}} I \qquad (58)$$

Reciprocal Frames

Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be a set of linearly independent vectors that span the vector space that are not necessarily orthogonal. These vectors define the frame (frame vectors are shown in bold face since they are almost always associated with a particular coordinate system) with volume element

$$E_n \equiv \mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_n \tag{59}$$

So that $E_n \propto I$. The reciprocal frame is the set of vectors $\mathbf{e}^1, \ldots, \mathbf{e}^n$ that satisfy the relation

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j, \quad \forall i, j = 1, \ldots, n \tag{60}$$

The $\mathbf{e}^i$ are constructed as follows

$$\mathbf{e}^j = (-1)^{j-1} \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \ldots \wedge \breve{\mathbf{e}}_j \wedge \ldots \wedge \mathbf{e}_n E_n^{-1} \tag{61}$$

So that the dot product is (using equation 53 since $E_n^{-1} \propto I$)

$$\begin{aligned}
\mathbf{e}_i \cdot \mathbf{e}^j &= (-1)^{j-1}\,\mathbf{e}_i \cdot \left(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \ldots \wedge \breve{\mathbf{e}}_j \wedge \ldots \wedge \mathbf{e}_n E_n^{-1}\right) && (62)\\
&= (-1)^{j-1}\left(\mathbf{e}_i \wedge \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \ldots \wedge \breve{\mathbf{e}}_j \wedge \ldots \wedge \mathbf{e}_n\right) E_n^{-1} && (63)\\
&= 0, \quad \forall i \neq j && (64)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{e}_1 \cdot \mathbf{e}^1 &= \mathbf{e}_1 \cdot \left(\mathbf{e}_2 \wedge \ldots \wedge \mathbf{e}_n E_n^{-1}\right) && (65)\\
&= \left(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \ldots \wedge \mathbf{e}_n\right) E_n^{-1} && (66)\\
&= 1 && (67)
\end{aligned}$$

Coordinates

The reciprocal frame can be used to develop a coordinate representation for multivectors in an arbitrary frame $\mathbf{e}_1, \ldots, \mathbf{e}_n$ with reciprocal frame $\mathbf{e}^1, \ldots, \mathbf{e}^n$.

Since both the frame and it's reciprocal span the base vector space we can write any vector a in the vector space as

$$a = a^i \mathbf{e}_i = a_i \mathbf{e}^i \tag{68}$$

where if an index such as i is repeated it is assumes that the terms with the repeated index will be summed from 1 to n. Using that $\mathbf{e}_i \cdot \mathbf{e}^j = \delta_i^j$ we have

$$\begin{aligned} a_i &= a \cdot \mathbf{e}_i & (69) \\ a^i &= a \cdot \mathbf{e}^i & (70) \end{aligned}$$

In tensor notation a_i would be the covarient representation and a^i the contravarient representation of the vector a.

Now consider the case of grade 2 and grade 3 blades:

$$
\begin{aligned}
\mathbf{e}^i \cdot (a \wedge b) &= a \cdot \mathbf{e}^i b - b \cdot \mathbf{e}^i a \\
\mathbf{e}_i \left(a \cdot \mathbf{e}^i b - b \cdot \mathbf{e}^i a \right) &= ab - ba = 2a \wedge b \\
\mathbf{e}^i \cdot (a \wedge b \wedge c) &= \\
& a \cdot \mathbf{e}^i b \wedge c - b \cdot \mathbf{e}^i a \wedge c + c \cdot \mathbf{e}^i a \wedge b \\
\mathbf{e}_i \left(a \cdot \mathbf{e}^i b \wedge c - b \cdot \mathbf{e}^i a \wedge c + c \cdot \mathbf{e}^i a \wedge b \right) &= \\
& ab \wedge c - ba \wedge c + ca \wedge b = 3a \wedge b \wedge c
\end{aligned}
$$

for an r-blade A_r we have (the proof is left to the student)

$$\mathbf{e}_i \mathbf{e}^i \cdot A_r = r A_r \tag{71}$$

Since $\mathbf{e}_i\mathbf{e}^i = n$ we have

$$\mathbf{e}_i\mathbf{e}^i \wedge A_r = \mathbf{e}_i\left(\mathbf{e}^i A_r - \mathbf{e}^i \cdot A_r\right) = (n - r)\, A_r \tag{72}$$

Flipping $\mathbf{e}^i$ and A_r in equations 71 and 72 and subtracting equation 71 from 72 gives

$$\mathbf{e}_i A_r \mathbf{e}^i = (-1)^r\,(n - 2r)\, A_r \tag{73}$$

In Hestenes and Sobczyk (3.14) it is proved that

$$\left(\mathbf{e}^{k_r} \wedge \ldots \wedge \mathbf{e}^{k_1}\right) \cdot \left(\mathbf{e}_{j_1} \wedge \ldots \wedge \mathbf{e}_{j_r}\right) = \delta^{j_1}_{k_1}\delta^{j_2}_{k_2}\ldots\delta^{j_r}_{k_r} \tag{74}$$

so that the general multivector A can be expanded in terms of the blades of the frame and reciprocal frame as

$$A = \sum_{i<j<\cdots<k} A_{ij\cdots k}\mathbf{e}^i \wedge \mathbf{e}^j \wedge \cdots \wedge \mathbf{e}^k \tag{75}$$

where

$$A_{ij\cdots k} = (\mathbf{e}_k \wedge \cdots \wedge \mathbf{e}_j \wedge \mathbf{e}_i) \cdot A \tag{76}$$

The components $A_{ij\cdots k}$ are totally antisymmetric on all indices and are usually referred to as the components of an *antisymmetric tensor*.

Linear Transformations

Let f be a linear transformation $f : \mathcal{V} \rightarrow \mathcal{V}$ with $f\left(\alpha a + \beta b\right) = \alpha f\left(a\right) + \beta f\left(b\right)\ \forall a, b \in \mathcal{V}$ and $\alpha, \beta \in \Re$. Then define the action of f on a blade of the geometric algebra by

$$f\left(a_1 \wedge \ldots \wedge a_r\right) = f\left(a_1\right) \wedge \ldots \wedge f\left(a_1\right) \tag{77}$$

and the action of f on any two $A, B \in \mathcal{G}\left(\mathcal{V}\right)$ by

$$f\left(\alpha A + \beta B\right) = \alpha f\left(A\right) + \beta f\left(B\right) \tag{78}$$

Since any multivector A can be expanded as a sum of blades $f\left(A\right)$ is defined. This has many consequences. Consider the following definition for the determinant of f, $\det\left(f\right)$.

$$f\left(I\right) = \det\left(f\right) I \tag{79}$$

First show that this definition is equivalent to the standard definition of the determinant (again $e_1, \ldots, e_N$ is an orthonormal basis for $\mathcal{V}$).

$$f\left(e_r\right) = \sum_{s=1}^{N} a_{rs} e_s \tag{80}$$

Then

$$\begin{aligned} f\left(I\right) &= \left(\sum_{s_1=1}^{N} a_{1s_1} e_s\right) \wedge \ldots \wedge \left(\sum_{s_N=1}^{N} a_{Ns_N} e_s\right) \\ &= \sum_{s_1,\ldots,s_N} a_{1s_1} \ldots a_{Ns_N} e_{s_1} \ldots e_{s_N} \end{aligned} \tag{81}$$

But

$$e_{s_1} \ldots e_{s_N} = \varepsilon_{1\ldots N}^{s_1 \ldots s_N} e_1 \ldots e_N \tag{82}$$

so that

$$f\left(I\right)=\sum_{s_1,\ldots,s_N}\varepsilon_{1\ldots N}^{s_1\ldots s_N}a_{1s_1}\ldots a_{Ns_N}I \tag{83}$$

or

$$\det\left(f\right)=\sum_{s_1,\ldots,s_N}\varepsilon_{1\ldots N}^{s_1\ldots s_N}a_{1s_1}\ldots a_{Ns_N} \tag{84}$$

which is the standard definition. Now compute the determinant of the product of the linear transformations f and g

$$\begin{aligned}\det\left(fg\right)I &= fg\left(I\right)\\ &= f\left(g\left(I\right)\right)\\ &= f\left(\det\left(g\right)I\right)\\ &= \det\left(g\right)f\left(I\right)\\ &= \det\left(g\right)\det\left(f\right)I \qquad (85)\end{aligned}$$

or

$$\det(fg) = \det(f)\det(g) \tag{86}$$

Do you have any idea of how miserable that is to prove from the standard definition of determinant?

Adjoint

If F is linear transformation and a and b are two arbitrary vectors the adjoint function, $\overline{F}$, is defined by

$$a \cdot \overline{F}(b) = b \cdot F(a) \tag{87}$$

From the definition the adjoint is also a linear transformation. For an arbitrary frame $\mathbf{e}_1, \ldots, \mathbf{e}_n$ we have

$$\mathbf{e}_i \cdot \overline{F}(a) = a \cdot F(\mathbf{e}_i) \tag{88}$$

So that we can explicitly construct the adjoint as

$$\begin{aligned} \overline{F}(a) &= \mathbf{e}^i a \cdot F(\mathbf{e}_i) && (89) \\ &= \mathbf{e}^i \left(F(\mathbf{e}_i) \cdot \mathbf{e}^j\right) a_j && (90) \end{aligned}$$

so that $\overline{F}_{ij} = F\left(\mathbf{e}_i\right)\cdot\mathbf{e}^j$ is the matrix representation of $\overline{F}$ for the $\mathbf{e}_1,\ldots,\mathbf{e}_n$ frame. However

$$F\left(a\right) = \mathbf{e}^i\left(F\left(\mathbf{e}^j\right)\cdot\mathbf{e}_i\right)a_j \tag{91}$$

so that the matrix representation of F is $F_{ij} = F\left(\mathbf{e}^j\right)\cdot\mathbf{e}_i$. If the $\mathbf{e}_1,\ldots,\mathbf{e}_n$ are orthonormal then $\mathbf{e}_j = \mathbf{e}^j$ for all j and $\overline{F}_{ij} = F_{ji}$ exactly the same as the adjoint in matrices.

Other basic properties of the adjoint are:

$$\overline{\overline{F}}\left(a\right) = \mathbf{e}^i a\cdot\overline{F}\left(\mathbf{e}_i\right) = \mathbf{e}^i\mathbf{e}_i\cdot F\left(a\right) = F\left(a\right) \tag{92}$$

and

$$\begin{aligned}\overline{FG}\left(a\right) &= \mathbf{e}^i a\cdot F\left(G\left(\mathbf{e}_i\right)\right) = \overline{F}\left(a\right)\cdot G\left(\mathbf{e}_i\right)\mathbf{e}^i \\ &= \overline{G}\left(\overline{F}\left(a\right)\right)\cdot\mathbf{e}_i\mathbf{e}^i = \overline{G}\,\overline{F}\left(a\right)\end{aligned} \tag{93}$$

so that $\overline{\overline{F}} = F$ and $\overline{FG} = \overline{G}\,\overline{F}$.

A symmetric function is one where $F = \overline{F}$. As an example consider $F\overline{F}$

$$\overline{F\overline{F}} = \overline{\overline{F}}\,\overline{F} = F\overline{F} \tag{94}$$

Inverse

Another linear algebraic relation in geometric algebra is

$$f^{-1}(A) = \frac{I\overline{f}\left(I^{-1}A\right)}{\det(f)} \quad \forall A \in \mathcal{G}(\mathcal{V}) \tag{95}$$

where $\overline{f}$ is the adjoint transformation defined by $a \cdot \overline{f}(b) = b \cdot f(a)$ $\forall a, b \in \mathcal{V}$ and you have an explicit formula for the inverse of a linear transformation!

Quaternions

Any multivector $A \in \mathcal{G}(3,0)$ may be written as

$$A = \alpha + a + B + \beta I \tag{96}$$

where $\alpha, \beta \in \Re$, $a \in \mathcal{V}(3,0)$, B is a bivector, and I is the unit pseudoscalar. The quaternions are the multivectors of even grades

$$A = \alpha + B \tag{97}$$

B can be represented as

$$B = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k} \tag{98}$$

where $\mathbf{i} = e_2e_3$, $\mathbf{j} = e_1e_3$, and $\mathbf{k} = e_1e_2$, and

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \tag{99}$$

The quaternions form a subalgebra of $\mathcal{G}\left(3,0\right)$ since the geometric product of any two quaternions is also a quaternion since the geometric product of two even grade multivector components is a even grade multivector. For example the product of two grade 2 multivectors can only consist of grades 0, 2, and 4, but in $\mathcal{G}\left(3,0\right)$ we can only have grades 0 and 2 since the highest possible grade is 3.

Spinors

The general definition of a spinor is a multivector, $\psi \in \mathcal{G}(p,q)$, such that $\psi v \psi^{\dagger} \in \mathcal{V}(p,q) \;\; \forall v \in \mathcal{V}(p,q)$. Practically speaking a spinor is the composition of a rotation and a dilation (stretching or shrinking) of a vector. Thus we can write

$$\psi v \psi^{\dagger} = \rho R v R^{\dagger} \tag{100}$$

where R is a rotor $\left(RR^{\dagger} = 1\right)$. Letting $U = R^{\dagger}\psi$ we must solve

$$U v U^{\dagger} = \rho v \tag{101}$$

U must generate a pure dilation. The most general form for U based on the fact that the l.h.s of equation 101 must be a vector is

$$U = \alpha + \beta I \tag{102}$$

so that

$$UvU^{\dagger} = \alpha^2 v + \alpha\beta\left(Iv + vI^{\dagger}\right) + \beta^2 IvI^{\dagger} = \rho v \tag{103}$$

Using $vI^{\dagger} = (-1)^{\frac{(n-1)(n-2)}{2}} Iv$, $vI^{\dagger} = (-1)^{n-1} I^{\dagger}v$, and $II^{\dagger} = (-1)^{q}$ we get

$$\alpha^2 v + \alpha\beta\left(1 + (-1)^{\frac{(n-1)(n-2)}{2}}\right) Iv + (-1)^{n+q-1}\beta^2 v = \rho v \tag{104}$$

If $\dfrac{(n-1)(n-2)}{2}$ is even $\beta = 0$ and $\alpha \neq 0$, otherwise $\alpha, \beta \neq 0$. For the odd case

$$\psi = R(\alpha + \beta I) \tag{105}$$

where $\rho = \alpha^2 + (-1)^{n+q-1}\beta^2$. In the case of $\mathcal{G}(1,3)$ (relativistic space time) we have $\rho = \alpha^2 + \beta^2$, $\rho > 0$.

Geometric Algebra of the Minkowski Plane

Because of Relativity and QM the Geometric Algebra of the Minkowski Plane is very important for physical applications of Geometric Algebra so we will treat it in detail.

Let the orthonormal basis vectors for the plane be γ_0 and γ_1 where $\gamma_0^2 = -\gamma_1^2 = 1$.[5] Then the geometric product of two vectors $a = a_0\gamma_0 + a_1\gamma_1$ and $b = b_0\gamma_0 + b_1\gamma_1$ is

$$
\begin{aligned}
ab &= (a_0\gamma_0 + a_1\gamma_1)(b_0\gamma_0 + b_1\gamma_1) && (106)\\
&= a_0b_0\gamma_0^2 + a_1b_1\gamma_1^2 + (a_0b_1 - a_1b_0)\gamma_0\gamma_1 && (107)\\
&= a_0b_0 - a_1b_1 + (a_0b_1 - a_1b_0) I && (108)
\end{aligned}
$$

---

[5] $I = \gamma_0\gamma_1$

so that

$$a \cdot b = a_0 b_0 - a_1 b_1 \tag{109}$$

and

$$a \wedge b = (a_0 b_1 - a_1 b_0)\, I \tag{110}$$

and

$$I^2 = \gamma_0\gamma_1\gamma_0\gamma_1 = -\gamma_0^2\gamma_1^2 = 1 \tag{111}$$

Thus

$$\begin{aligned} e^{\alpha I} &= \sum_{i=0}^{\infty} \frac{\alpha^i I^i}{i!} && (112) \\ &= \sum_{i=0}^{\infty} \frac{\alpha^{2i}}{(2i)!} + \sum_{i=0}^{\infty} \frac{\alpha^{2i+1} I}{(2i+1)!} && (113) \\ &= \cosh(\alpha) + \sinh(\alpha)\, I && (114) \end{aligned}$$

since $I^{2i} = 1$.

In the Minkowski plane all vectors of the form $a_\pm = \alpha\left(\gamma_0 \pm \gamma_1\right)$ are null $\left(a_\pm^2 = 0\right)$. One question to answer are there any vectors $b_\pm$ such that $a_\pm \cdot b_\pm = 0$ that are not parallel to $a_\pm$.

$$\begin{gathered} a_\pm \cdot b_\pm = \alpha\left(b_0^\pm \mp b_1^\pm\right) = 0 \\ b_0^\pm \mp b_1^\pm = 0 \\ b_0^\pm = \pm b_1^\pm \end{gathered}$$

Thus $b_\pm$ must be proportional to $a_\pm$ and the are no vectors in the space that can be constructed that are normal to $a_\pm$. Thus there are no non-zero bivectors, $a \wedge b$, such that $\left(a \wedge b\right)^2 = 0$. Conversely, if $a \wedge b \neq 0$ then $\left(a \wedge b\right)^2 > 0$.

Finally for the condition that there always exist two orthogonal vectors e_1 and e_2 such that

$$a \wedge b = e_1 e_2 \tag{115}$$

we can state that neither e_1 nor e_2 can be null.

Lorentz Transformation

We now have all the tools needed to derive the Lorentz transformation with Geometric Algebra. Consider a two dimensional time-like plane with with coordinates t[6] and x_1 and basis vectors γ_0 and γ_1. Then a general space-time vector in the plane is given by

$$x = t\gamma_0 + x_1\gamma_1 = t'\gamma_0' + x_1'\gamma_1' \tag{116}$$

where the basis vectors of the two coordinate systems are related by

$$\gamma_\mu' = R\gamma_\mu R^\dagger \;\; \mu = 0, 1 \tag{117}$$

---

[6] We let the speed of light $c = 1$.

and R is a Minkowski plane rotor

$$R = \sinh\left(\frac{\alpha}{2}\right) + \cosh\left(\frac{\alpha}{2}\right)\gamma_1\gamma_0 \tag{118}$$

so that

$$R\gamma_0 R^\dagger = \cosh\left(\alpha\right)\gamma_0 + \sinh\left(\alpha\right)\gamma_1 \tag{119}$$

and

$$R\gamma_1 R^\dagger = \cosh\left(\alpha\right)\gamma_1 + \sinh\left(\alpha\right)\gamma_0 \tag{120}$$

Now consider the special case that the primed coordinate system is moving with velocity β in the direction of γ_1 and the two coordinate systems were coincident at time $t = 0$. Then $x_1 = \beta t$ and $x_1' = 0$ so we may write

$$t\gamma_0 + \beta t\gamma_1 = t' R\gamma_0 R^\dagger \tag{121}$$

$$\frac{t}{t'}\left(\gamma_0 + \beta\gamma_1\right) = \cosh\left(\alpha\right)\gamma_0 + \sinh\left(\alpha\right)\gamma_1 \tag{122}$$

Equating components gives

$$\cosh(\alpha) = \frac{t}{t'} \tag{123}$$

$$\sinh(\alpha) = \frac{t}{t'}\beta \tag{124}$$

Solving for α and $\dfrac{t}{t'}$ in equations 123 and 124 gives

$$\tanh(\alpha) = \beta \tag{125}$$

$$\frac{t}{t'} = \gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{126}$$

Now consider the general case of x, t and x', t' giving

$$\begin{aligned} t\gamma_0 + x\gamma_1 &= t'R\gamma_0R^\dagger + x'R\gamma_1R^\dagger && (127)\\ &= t'\gamma(\gamma_0+\beta\gamma_1) + x'\gamma(\gamma_1+\beta\gamma_0) && (128) \end{aligned}$$

Equating basis vector coefficients recovers the Lorentz transformation

$$\begin{aligned} t &= \gamma\,(t' + \beta x') \\ x &= \gamma\,(x' + \beta t') \end{aligned} \tag{129}$$

Commutator Product

The commutator product of two multivectors A and B is defined as

$$A{\times}B \equiv \frac{1}{2}(AB - BA) \tag{130}$$

An important theorem for the commutator product is that for a grade 2 multivector, $A_2 = \left\langle A \right\rangle_2$, and a grade r multivector $B_r = \left\langle B \right\rangle_r$ we have

$$A_2B_r = A_2 \wedge B_r + A_2{\times}B_r + A_2 \cdot B_r \tag{131}$$

From the geometric product grade expansion for multivectors we have

$$A_2B_r = \left\langle A_2B_r \right\rangle_{r+2} + \left\langle A_2B_r \right\rangle_r + \left\langle A_2B_r \right\rangle_{|r-2|} \tag{132}$$

Thus we must show that

$$\left\langle A_2 B_r \right\rangle_r = A_2 \times B_r \tag{133}$$

Let $e_1, \ldots, e_n$ be an orthogonal set for the vector space where $B_r = e_1 \ldots e_r$ and $A_2 = \sum\limits_{l<m=2}^{n} \alpha_{lm} e_l e_m$ so we can write

$$A_2 \times B_r = \left(\sum_{l<m=2}^{n} \alpha_{lm} e_l e_m \right) \times (e_1 \ldots e_r) \tag{134}$$

Now consider the following three cases

1. l and $m > r$ where $e_l e_m e_1 \ldots e_r = e_1 \ldots e_r e_l e_m$
2. $l \leq r$ and $m > r$ where $e_l e_m e_1 \ldots e_r = -e_1 \ldots e_r e_l e_m$

3. l and $m \leq r$ where $e_l e_m e_1 \ldots e_r = e_1 \ldots e_r e_l e_m$

For case 1 and 3 $e_l e_m$ commute with B_r and the contribution to the commutator product is zero. In case 3 $e_l e_m$ anticommutes with B_r and thus are the only terms that contribute to the commutator. All these terms are of grade r and the theorem is proved.

Additionally, the commutator product obeys the Jacobi identity

$$A\times (B\times C) = (A\times B)\times C + B\times (A\times C) \tag{135}$$

This is important for the geometric algebra treatment of Lie groups and algebras.

Differentiation

If $F(a)$ if a multivector valued function of the vector a, and a and b are any vectors in the space then the derivative of F is defined by

$$b \cdot \nabla F \equiv \lim_{\epsilon \rightarrow 0} \frac{F(a+\epsilon b) - F(a)}{\epsilon} \tag{136}$$

then letting $b = \mathbf{e}_k$ be the components of a coordinate frame with $x = x^k \mathbf{e}_k$ (we are using the summation convention that the same upper and lower indicies are summed over 1 to N) we have

$$\mathbf{e}_k \cdot \nabla F = \lim_{\epsilon \rightarrow 0} \frac{F\left(x^j \mathbf{e}_j + \epsilon \mathbf{e}_k\right) - F\left(x^j \mathbf{e}_j\right)}{\epsilon} \tag{137}$$

Using what we know about coordinates gives

$$\nabla F = \mathbf{e}^j \frac{\partial F}{\partial x^j} = \mathbf{e}^j \partial_j F \tag{138}$$

or looking at ∇ as a symbolic operator we may write

$$\nabla = \mathbf{e}^j \partial_j \tag{139}$$

Due to the properties of coordinate frame expansions ∇F is independent of the choice of the $\mathbf{e}_k$ frame. If we consider x to be a position vector then $F(x)$ is in general a multivector field.

Dervatives of Scalar Functions

If $f\left(x\right)$ is scalar valued function of the vector x then the derivative is

$$\nabla f = \mathbf{e}^k \partial_k f \tag{140}$$

which is the standard definition of the gradient of a scalar function (remember that in an orthonormal coordinate system $\mathbf{e}_k = \mathbf{e}^k$).

Using equation 140 we can show the following results for the gradient of some specific scalar functions

$$\begin{array}{ccccc} f & = & x \cdot a, & x^k, & xx \\ \nabla f & = & a, & \mathbf{e}^k, & 2x \end{array} \tag{141}$$

Product Rule

Let $\circ$ represent a bilinear product operator such as the geometric, inner, or outer product and note that for the multivector fields F and G we have

$$\partial_k \left(F \circ G \right) = \left(\partial_k F \right) \circ G + F \circ \left(\partial_k G \right) \tag{142}$$

so that

$$\begin{aligned} \nabla \left(F \circ G \right) &= \mathbf{e}^k \left(\left(\partial_k F \right) \circ G + F \circ \left(\partial_k G \right) \right) \\ &= \mathbf{e}^k \left(\partial_k F \right) \circ G + \mathbf{e}^k F \circ \left(\partial_k G \right) \end{aligned} \tag{143}$$

However since the geometric product is not communicative, in general

$$\nabla \left(F \circ G \right) \neq \left(\nabla F \right) \circ G + F \circ \left(\nabla G \right) \tag{144}$$

The notation adopted by Hestenes is

$$\nabla (F \circ G) = \nabla F \circ G + \dot{\nabla} F \circ \dot{G} \tag{145}$$

The convention of the overdot notation is

$i.$ In the absence of brackets, ∇ acts on the object to its immediate right

$ii.$ When the ∇ is followed by brackets, the derivative acts on all the the terms in the brackets.

$iii.$ When the ∇ acts on a multivector to which it is not adjacent, we use overdots to describe the scope.

Note that with the overdot notation the expression $\dot{A}\dot{\nabla}$ makes sense!

Interior and Exterior Derivative

The interior and exterior derivatives of an r-grade multivector field are simply defined as (don't forget the summation convention)

$$\nabla \cdot A_r \equiv \langle \nabla A_r \rangle_{r-1} = \mathbf{e}^k \cdot \partial_k A_r \tag{146}$$

and

$$\nabla \wedge A_r \equiv \langle \nabla A_r \rangle_{r+1} = \mathbf{e}^k \wedge \partial_k A_r \tag{147}$$

Note that

$$\begin{aligned} \nabla \wedge (\nabla \wedge A_r) &= \mathbf{e}^i \partial_i \left(\mathbf{e}^j \wedge \partial_j A_r \right) \\ &= \mathbf{e}^i \wedge \mathbf{e}^j \wedge (\partial_i \partial_j A_r) \\ &= 0 \end{aligned} \tag{148}$$

since $\mathbf{e}^i \wedge \mathbf{e}^j = -\mathbf{e}^j \wedge \mathbf{e}^i$, but $\partial_i\partial_j A_r = \partial_j\partial_i A_r$.

$$\begin{aligned}
\nabla \cdot (\nabla \cdot A_r) &= \mathbf{e}^i \cdot \partial_i \left(\mathbf{e}^j \cdot \partial_j A_r\right) \\
&= \mathbf{e}^i \cdot \left(\mathbf{e}^j \cdot (\partial_i\partial_j A_r)\right) \\
&= \pm\mathbf{e}^i \cdot \left(\mathbf{e}^j \cdot (\partial_i\partial_j A_r^* I)\right) \\
&= \pm\mathbf{e}^i \cdot \left(\left(\mathbf{e}^j \wedge (\partial_i\partial_j A_r^*)\right) I\right) \\
&= \pm \left(\mathbf{e}^i \wedge \left(\mathbf{e}^j \wedge (\partial_i\partial_j A_r^*)\right)\right) I \\
&= 0 \qquad (149)
\end{aligned}$$

Where * indicates the dual of a multivector, $A^* = AI$ (I is the pseudoscalar and $A = \pm A^* I$ since $I^2 = \pm 1$), and we use equation 53 to exchange the inner and outer products.

Thus for the general multivector field A (built from sums of A_r's) we have $\nabla \wedge (\nabla \wedge A) = 0$ and $\nabla \cdot (\nabla \cdot A) = 0$. If ϕ is a scalar function we also have

$$\begin{aligned} \nabla \wedge (\nabla \phi) &= \mathbf{e}^i \wedge \partial_i \left(\mathbf{e}^j \partial_j \phi \right) \\ &= \mathbf{e}^i \wedge \mathbf{e}^j \partial_i \partial_j \phi \\ &= 0 \end{aligned} \tag{150}$$

Another use for the overdot notation would in the case where $f\,(x,a)$ is a linear function of its second argrument $(f(x,\alpha a+\beta b)=\alpha f(x,a)+\beta f(x,b))$ and a is a general function of position $\left(a(x)=a^i(x)\,\mathbf{e}_i\right)$. Now calculate

$$\nabla f(x,a)=\mathbf{e}^k\frac{\partial}{\partial x^k}f(x,a)=\mathbf{e}^k\frac{\partial}{\partial x^k}f\left(x,a^i(x)\,\mathbf{e}_i\right) \tag{151}$$

$$=\mathbf{e}^k\frac{\partial}{\partial x^k}\left(a^i(x)\,f(x,\mathbf{e}_i)\right) \tag{152}$$

$$=\mathbf{e}^k\frac{\partial a^i}{\partial x^k}f(x,\mathbf{e}_i)+a^i\mathbf{e}^k\frac{\partial}{\partial x^k}f(x,\mathbf{e}_i) \tag{153}$$

$$=\mathbf{e}^k f\left(x,\frac{\partial a}{\partial x^k}\right)+a^i\mathbf{e}^k\frac{\partial}{\partial x^k}f(x,\mathbf{e}_i) \tag{154}$$

Defining

$$\dot{\nabla}\dot{f}\,(a) \equiv a^i \mathbf{e}^k \frac{\partial}{\partial x^k} f(x,\mathbf{e}_i) = \mathbf{e}^k \,\frac{\partial}{\partial x^k} f(x,a)\bigg|_{a=\text{constant}} \tag{155}$$

Then supressing the explicit x dependence of f we get

$$\dot{\nabla}\dot{f}\,(a) = \nabla f\,(a) - \mathbf{e}^k f\left(\frac{\partial a}{\partial x^k}\right) \tag{156}$$

Other basic results (examples) are

$$\nabla x \cdot A_r = r A_r \tag{157}$$

$$\nabla x \wedge A_r = (n-r)\,A_r \tag{158}$$

$$\dot{\nabla} A_r \dot{x} = (-1)^r\,(n-2r)\,A_r \tag{159}$$

The basic identities for the case of a scalar field α and multivector field F are

$$\nabla(\alpha F) = (\nabla\alpha)F + \alpha\nabla F \tag{160}$$

$$\nabla\cdot(\alpha F) = (\nabla\alpha)\cdot F + \alpha\nabla\cdot F \tag{161}$$

$$\nabla\wedge(\alpha F) = (\nabla\alpha)\wedge F + \alpha\nabla\wedge F \tag{162}$$

if f_1 and f_2 are vector fields

$$\nabla\wedge(f_1\wedge f_2) = (\nabla\wedge f_1)\wedge f_2 - (\nabla\wedge f_2)\wedge f_1 \tag{163}$$

and finally if F_r is a grade r multivector field

$$\nabla\cdot(F_r I) = (\nabla\wedge F_r)I \tag{164}$$

where I is the psuedoscalar for the geometric algebra.

Geometric Derivative of a Multivector Function

For a vector space of dimension N spanned by the vectors $\boldsymbol{u}_i$ the coordinates of a vector x are the $x^i = x \cdot \boldsymbol{u}^i$ so that $x = x^i \boldsymbol{u}_i$ (summation convention is from 1 to N).

Curvilinear coordinates for that space are generated by a one to one invertible differentiable mapping from $\left(x^1, \ldots, x^N\right) \leftrightarrow \left(\theta^1, \ldots, \theta^N\right)$ where the θ^i are called the curvilinear coordinates.

If the mapping is given by $x\left(\theta^1, \ldots, \theta^N\right) = x^i\left(\theta^1, \ldots, \theta^N\right) \boldsymbol{u}_i$ then the basis vectors associated with the transformation are given by

$$\mathbf{e}_k = \frac{\partial x}{\partial \theta^k} = \frac{\partial x^i}{\partial \theta^k} \boldsymbol{u}_i \tag{165}$$

The one critical relationship that is required to express the geometric derivative in curvilinear coordinated is

$$\mathbf{e}^k = \frac{\partial \theta^k}{\partial x^i} \boldsymbol{u}^i \tag{166}$$

The proof is

$$\begin{aligned}
\mathbf{e}_j \cdot \mathbf{e}^k &= \frac{\partial x^m}{\partial \theta^j} \frac{\partial \theta^k}{\partial x^n} \boldsymbol{u}_m \cdot \boldsymbol{u}^n && (167) \\
&= \frac{\partial x^m}{\partial \theta^j} \frac{\partial \theta^k}{\partial x^n} \delta_m^n && (168) \\
&= \frac{\partial x^m}{\partial \theta^j} \frac{\partial \theta^k}{\partial x^m} && (169) \\
&= \frac{\partial \theta^k}{\partial \theta^j} = \delta_j^k && (170)
\end{aligned}$$

We wish to express the geometric derivative of an R-grade multivector field F_R in terms of the curvilinear coordinates. Thus

$$\nabla F_R = \boldsymbol{u}^i \frac{\partial F_R}{\partial x^i} = \left(\boldsymbol{u}^i \frac{\partial \theta^k}{\partial x^i} \right) \frac{\partial F_R}{\partial \theta^k} = \boldsymbol{e}^k \frac{\partial F_R}{\partial \theta^k} \tag{171}$$

Note that if we start by defining the $\boldsymbol{e}_k$'s the reciprocal frame vectors $\boldsymbol{e}^k$ can be calculated geometrically (we do not need the inverse partial derivatives).

Now define a new blade symbol by

$$\boldsymbol{e}_{[i_1,\ldots,i_R]} = \mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_R} \tag{172}$$

and represent an R-grade multivector function F by

$$F = F^{i_1 \ldots i_R} \boldsymbol{e}_{[i_1,\ldots,i_R]} \tag{173}$$

Then

$$\nabla F = \frac{\partial F^{i_1 \dots i_R}}{\partial \theta^k} \boldsymbol{e}^k \boldsymbol{e}_{[i_1,\dots,i_R]} + F^{i_1 \dots i_R} \boldsymbol{e}^k \frac{\partial}{\partial \theta^k} \boldsymbol{e}_{[i_1,\dots,i_R]} \tag{174}$$

Define

$$C\left\{\boldsymbol{e}_{[i_1,\dots,i_R]}\right\} \equiv \boldsymbol{e}^k \frac{\partial}{\partial \theta^k} \boldsymbol{e}_{[i_1,\dots,i_R]} \tag{175}$$

Where $C\left\{\boldsymbol{e}_{[i_1,\dots,i_R]}\right\}$ are the connection multivectors for each base of the geoemetric algebra and we can write

$$\nabla F = \frac{\partial F^{i_1 \dots i_R}}{\partial \theta^k} \boldsymbol{e}^k \boldsymbol{e}_{[i_1,\dots,i_R]} + F^{i_1 \dots i_R} C\left\{\boldsymbol{e}_{[i_1,\dots,i_R]}\right\} \tag{176}$$

Note that all the quantities in the equation not dependent upon the $F^{i_1 \dots i_R}$ can be directily calculated if the $\boldsymbol{e}_k\left(\theta^1, \dots, \theta^N\right)$ is known so further simplification is not needed.

In general the $\mathbf{e}_k$'s we have defined are not normalized so define

$$|\mathbf{e}_k| = \sqrt{|\mathbf{e}_k^2|} \tag{177}$$

$$\hat{\boldsymbol{e}}_k = \frac{\mathbf{e}_k}{|\mathbf{e}_k|} \tag{178}$$

and note that $\hat{\boldsymbol{e}}_k^2 = \pm 1$ depending upon the metric. Note also that

$$\hat{\boldsymbol{e}}^k = |\mathbf{e}_k|\,\mathbf{e}^k \tag{179}$$

since

$$\hat{\boldsymbol{e}}^j \cdot \hat{\boldsymbol{e}}_k = \big(|\mathbf{e}_j|\,\mathbf{e}^j\big) \cdot \left(\frac{\mathbf{e}_k}{|\mathbf{e}_k|}\right) = \delta^j_k \frac{|\mathbf{e}_j|}{|\mathbf{e}_k|} = \delta^j_k \tag{180}$$

so that if F_R is represented in terms of the normalized basis vectors we have

$$F_R = F_R^{i_1 \ldots i_R} \hat{\boldsymbol{e}}_{[i_1,\ldots,i_R]} \tag{181}$$

and the geometric derivative is now

$$\nabla F = \frac{\partial F^{i_1 \ldots i_R}}{\partial \theta^k} \frac{\hat{\boldsymbol{e}}^k}{|\mathbf{e}_k|} \hat{\boldsymbol{e}}_{[i_1,\ldots,i_R]} + F^{i_1 \ldots i_R} \hat{C} \left\{ \hat{\boldsymbol{e}}_{[i_1,\ldots,i_R]} \right\} \tag{182}$$

and

$$\hat{C} \left\{ \hat{\boldsymbol{e}}_{[i_1,\ldots,i_R]} \right\} = \frac{\hat{\boldsymbol{e}}^k}{|\mathbf{e}_k|} \frac{\partial}{\partial \theta^k} \hat{\boldsymbol{e}}_{[i_1,\ldots,i_R]} \tag{183}$$

Example: Spherical Coordinates

For spherical coordinates the coordinate generating function is:

$$x = r \left(\sin(\theta) \, \boldsymbol{u}_z + \cos(\theta) \left(\cos(\phi) \, \boldsymbol{u}_x + \sin(\phi) \, \boldsymbol{u}_y \right) \right) \tag{184}$$

so that

$$\mathbf{e}_r = \cos(\theta)\left(\cos(\phi)\,\boldsymbol{u}_x + \sin(\phi)\,\boldsymbol{u}_y\right) + \sin(\theta)\,\boldsymbol{u}_z \tag{185}$$

$$\mathbf{e}_\theta = r\left(-\sin(\theta)\left(\cos(\phi)\,\boldsymbol{u}_x + \sin(\phi)\,\boldsymbol{u}_y\right) + \cos(\theta)\,\boldsymbol{u}_z\right) \tag{186}$$

$$\mathbf{e}_\phi = r\cos(\theta)\left(-\sin(\phi)\,\boldsymbol{u}_x + \cos(\phi)\,\boldsymbol{u}_y\right) \tag{187}$$

where

$$|\mathbf{e}_r| = 1 \quad |\mathbf{e}_\theta| = r \quad |\mathbf{e}_\phi| = r\cos(\theta) \tag{188}$$

and

$$\hat{\boldsymbol{e}}_r = \cos(\theta)\left(\cos(\phi)\,\boldsymbol{u}_x + \sin(\phi)\,\boldsymbol{u}_y\right) + \sin(\theta)\,\boldsymbol{u}_z \tag{189}$$

$$\hat{\boldsymbol{e}}_\theta = -\sin(\theta)\left(\cos(\phi)\,\boldsymbol{u}_x + \sin(\phi)\,\boldsymbol{u}_y\right) + \cos(\theta)\,\boldsymbol{u}_z \tag{190}$$

$$\hat{\boldsymbol{e}}_\phi = -\sin(\phi)\,\boldsymbol{u}_x + \cos(\phi)\,\boldsymbol{u}_y \tag{191}$$

the connection mulitvectors for the normalize basis vectors are

$$\hat{C}\left\{\hat{\boldsymbol{e}}_{r}\right\} = \frac{2}{r} \tag{192}$$

$$\hat{C}\left\{\hat{\boldsymbol{e}}_{\theta}\right\} = \frac{\cos\left(\theta\right)}{r\sin\left(\theta\right)} + \frac{1}{r}\hat{\boldsymbol{e}}_{r}\wedge\hat{\boldsymbol{e}}_{\theta} \tag{193}$$

$$\hat{C}\left\{\hat{\boldsymbol{e}}_{\phi}\right\} = \frac{1}{r}\hat{\boldsymbol{e}}_{r}\wedge\hat{\boldsymbol{e}}_{\phi} + \frac{\cos\left(\theta\right)}{r\sin\left(\theta\right)}\hat{\boldsymbol{e}}_{\theta}\wedge\hat{\boldsymbol{e}}_{\phi} \tag{194}$$

$$\hat{C}\left\{\hat{\boldsymbol{e}}_{r}\wedge\hat{\boldsymbol{e}}_{\theta}\right\} = -\frac{\cos\left(\theta\right)}{r\sin\left(\theta\right)}\hat{\boldsymbol{e}}_{r} + \frac{1}{r}\hat{\boldsymbol{e}}_{\theta} \tag{195}$$

$$\hat{C}\left\{\hat{\boldsymbol{e}}_{r}\wedge\hat{\boldsymbol{e}}_{\phi}\right\} = \frac{1}{r}\hat{\boldsymbol{e}}_{\phi} - \frac{\cos\left(\theta\right)}{r\sin\left(\theta\right)}\hat{\boldsymbol{e}}_{r}\wedge\hat{\boldsymbol{e}}_{\theta}\wedge\hat{\boldsymbol{e}}_{\phi} \tag{196}$$

$$\hat{C}\left\{\hat{\boldsymbol{e}}_{\theta}\wedge\hat{\boldsymbol{e}}_{\phi}\right\} = \frac{2}{r}\hat{\boldsymbol{e}}_{r}\wedge\hat{\boldsymbol{e}}_{\theta}\wedge\hat{\boldsymbol{e}}_{\phi} \tag{197}$$

$$\hat{C}\left\{\hat{\boldsymbol{e}}_{r}\wedge\hat{\boldsymbol{e}}_{\theta}\wedge\hat{\boldsymbol{e}}_{\phi}\right\} = 0 \tag{198}$$

For a vector function A using equation 176 and that $\nabla A = \nabla \cdot A + \nabla \wedge A$

$$\begin{aligned}\nabla \cdot A &= \frac{1}{r\sin\left(\theta\right)}\left(A^{\theta}\cos\left(\theta\right) + \partial_{\phi}A^{\phi}\right) + \frac{1}{r}\left(2A^{r} + \partial_{\theta}A^{\theta}\right) + \partial_{r}A^{r} \quad (199)\\ &= \frac{1}{r^{2}}\partial_{r}\left(r^{2}A^{r}\right) + \frac{1}{r\sin\left(\theta\right)}\left(\partial_{\theta}\left(\sin\left(\theta\right)A^{\theta}\right) + \partial_{\phi}A^{\phi}\right) \quad (200)\end{aligned}$$

$$\begin{aligned}\nabla \times A &= -I\left(\nabla \wedge A\right) \quad (201)\\ &= \left(\frac{\partial_{\theta}A^{\phi}}{r} + \frac{1}{r\sin\left(\theta\right)}\left(A^{\phi}\cos\left(\theta\right) - \partial_{\phi}A^{\theta}\right)\right)\boldsymbol{\hat{e}}_{r} \quad (202)\\ &\quad + \left(\frac{\partial_{\phi}A^{r}}{r\sin\left(\theta\right)} - \frac{A^{\phi}}{r} - \partial_{r}A^{\phi}\right)\boldsymbol{\hat{e}}_{\theta} \quad (203)\\ &\quad + \left(\frac{A^{\theta}}{r} + \partial_{r}A^{\theta} - \frac{\partial_{\theta}A^{r}}{r}\right)\boldsymbol{\hat{e}}_{\phi} \quad (204)\end{aligned}$$

$$\nabla \times A = \frac{1}{r\sin(\theta)}\left(\partial_\theta\left(\sin(\theta)\,A^\phi\right) - \partial_\phi A^\theta\right)\hat{\boldsymbol{e}}_r \tag{205}$$

$$+\frac{1}{r}\left(\frac{1}{\sin(\theta)}\partial_\phi A^r - \partial_r\left(rA^\phi\right)\right)\hat{\boldsymbol{e}}_\theta \tag{206}$$

$$+\frac{1}{r}\left(\partial_r\left(rA^\theta\right) - \partial_\theta A^r\right)\hat{\boldsymbol{e}}_\phi \tag{207}$$

These are the standard formulas for div and curl in spherical coordinates.

Analytic Functions

Starting with $\mathcal{G}(2,0)$ and orthonormal basis vectors $\mathbf{e}_x$ and $\mathbf{e}_y$ so that $I = \mathbf{e}_x\mathbf{e}_y$ and $I^2 = -1$. Then we have

$$\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y \tag{208}$$

$$\nabla = \mathbf{e}_x\frac{\partial}{\partial x} + \mathbf{e}_y\frac{\partial}{\partial y} \tag{209}$$

Map $\mathbf{r}$ onto the complex number z via

$$z = x + Iy = \mathbf{e}_x\mathbf{r} \tag{210}$$

Define the multivector field $\psi = u + Iv$ where u and v are scalar fields. Then

$$\nabla\psi = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\mathbf{e}_x + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\mathbf{e}_y \tag{211}$$

Thus the statement that ψ is an analytic function is equivalent to

$$\nabla\psi = 0 \tag{212}$$

This is the fundamental equation that can be generalized to higher dimensions remembering that in general that ψ is a multivector rather than a scalar function!

To complete the connection with complex analysis we define ($z^\dagger = x - Iy$)

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - I\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial z^\dagger} = \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) \tag{213}$$

so that

$$\begin{array}{ll} \dfrac{\partial z}{\partial z} = 1, & \dfrac{\partial z^\dagger}{\partial z} = 0 \\ \dfrac{\partial z}{\partial z^\dagger} = 0, & \dfrac{\partial z^\dagger}{\partial z^\dagger} = 1 \end{array} \tag{214}$$

An analytic function is one that depends on z alone so that we can write $\psi(x+Iy)=\psi(z)$ and

$$\frac{\partial\psi(z)}{\partial z^{\dagger}}=0 \tag{215}$$

equivalently

$$\frac{1}{2}\left(\frac{\partial}{\partial x}+I\frac{\partial}{\partial y}\right)\psi=\frac{1}{2}\mathbf{e}_x\nabla\psi=0 \tag{216}$$

Now it is simple to show why solutions to $\nabla\psi=0$ can be written as a power series in z. First

$$\begin{aligned}
\nabla z &= \nabla(\mathbf{e}_x\mathbf{r}) \\
&= \mathbf{e}_x\mathbf{e}_x\frac{\partial\mathbf{r}}{\partial x}+\mathbf{e}_y\mathbf{e}_x\frac{\partial\mathbf{r}}{\partial y} \\
&= \mathbf{e}_x\mathbf{e}_x\mathbf{e}_x+\mathbf{e}_y\mathbf{e}_x\mathbf{e}_y \\
&= \mathbf{e}_x-\mathbf{e}_x \\
&= 0
\end{aligned} \tag{217}$$

so that

$$\nabla \left(z - z_0\right)^k = k \nabla \left(\mathbf{e}_x \mathbf{r} - z_0\right) \left(z - z_0\right)^{k-1} = 0 \tag{218}$$

Directed Integration - Line Integrals

If $F(x)$ is a multivector field and $x(\lambda)$ is a parametric representation of a vector path (curve) then the line Integral of F along the path x is defined to be

$$\int F(x)\frac{dx}{d\lambda}d\lambda = \int F\,dx = \lim_{n\mapsto\infty}\sum_{i=1}^{n}\bar{F}^{i}\Delta x^{i} \tag{219}$$

where

$$\Delta x^{i} = x_i - x_{i-1}, \quad \bar{F}^{i} = \frac{1}{2}\left(F(x_{i-1}) + F(x_i)\right) \tag{220}$$

if $x_n = x_1$ the path is closed. Since dx is a vector, that is $F(x)\dfrac{dx}{d\lambda} \neq \dfrac{dx}{d\lambda}F(x)$, a more general line integral would be

$$\int F(x)\frac{dx}{d\lambda}G(x)\ d\lambda = \int F(x)\,dx\,G(x) \tag{221}$$

The most general form of line integral would be

$$\int \mathsf{L}\left(\partial_\lambda x; x\right)\, d\lambda = \int \mathsf{L}\left(dx\right) \tag{222}$$

where $\mathsf{L}\left(a\right) = \mathsf{L}\left(a; x\right) =$ is a multivector-valued linear function of a. The position dependence in L can often be suppressed to streamline the notation.

Directed Integration - Surface Integrals

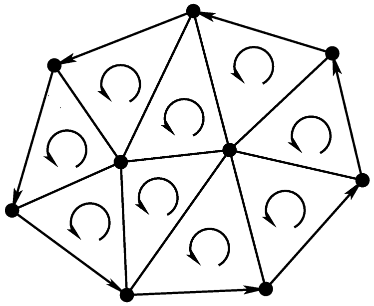
The next step is a directed surface integral. Let $F\left(x\right)$ be a multivector field and let a surface be parametrized by two coordinates $x\left(x^1, x^2\right)$. Then we can define a directed surface measure by

$$dX = \frac{\partial x}{\partial x^1} \wedge \frac{\partial x}{\partial x^2}\, dx^1 dx^2 = \mathbf{e}_1 \wedge \mathbf{e}_2\, dx^1 dx^2 \tag{223}$$

A directed surface integral takes the form

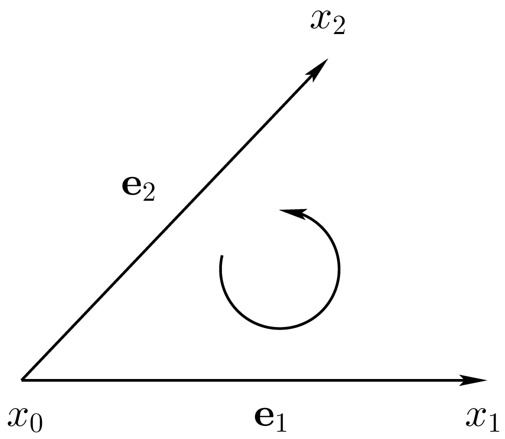
$$\int F\, dX = \int F \mathbf{e}_1 \wedge \mathbf{e}_2\, dx^1 dx^2 \tag{224}$$

In order to construct some of the more important proof it is necessary to express the surface integral as the limit of a sum. This requires the concept of a triangulated surface as shown



Triangulated Surface

Each triangle in the surface is described by a planar simplex as shown



Planar Simplex

The three vertices of the planar simplex are x_0, x_1, and x_2 with the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ defined by

$$\mathbf{e}_1 = x_1 - x_0, \quad \mathbf{e}_2 = x_2 - x_0 \tag{225}$$

so that the surface measure of the simplex is

$$\Delta X \equiv \frac{1}{2}\mathbf{e}_1 \wedge \mathbf{e}_2 = \frac{1}{2}\left(x_1 \wedge x_2 + x_2 \wedge x_0 + x_0 \wedge x_1\right) \tag{226}$$

with this definition of ΔX we have

$$\int F\, dX = \lim_{n \mapsto \infty} \sum_{k=1}^{n} \bar{F}^k \Delta X^k \tag{227}$$

where $\bar{F}^k$ is the average of F over the k^{th} simplex.

Directed Integration - n-dimensional Surfaces

k-Simplex Definition

In geometry, a simplex or k-simplex is an k-dimensional analogue of a triangle. Specifically, a simplex is the convex hull of a set of $(k+1)$ affinely independent points in some Euclidean space of dimension k or higher.

For example, a 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, and a 4-simplex is a pentachoron (in each case with interior).

A regular simplex is a simplex that is also a regular polytope. A regular k-simplex may be constructed from a regular $(k-1)$-simplex by connecting a new vertex to all original vertices by the common edge length.

k-Chain Definition (Algebraic Topology)

A finite set of k-simplexes embedded in an open subset of $\Re^n$ is called an affine k-chain. The simplexes in a chain need not be unique; they may occur with multiplicity. Rather than using standard set notation to denote an affine chain, it is instead the standard practice to use plus signs to separate each member in the set. If some of the simplexes have the opposite orientation, these are prefixed by a minus sign. If some of the simplexes occur in the set more than once, these are prefixed with an integer count. Thus, an affine chain takes the symbolic form of a sum with integer coefficients.

Simplex Notation

If $(x_0, x_1, \ldots, x_k)$ is the k-simplex defined by the $k+1$ points $x_0, x_1, \ldots, x_k$. This is abbreviated by

$$(x)_{(k)} = (x_0, x_1, \ldots, x_k) \tag{228}$$

The order of the points is important for a simplex, since it specifies the orientation of the simplex. If any two adjacent points are swaped the simplex orientation changes sign. The boundary operator for the simplex is denoted by ∂ and defined by

$$\partial\left(x\right)_{(k)} \equiv \sum_{i=0}^{k} \left(-1\right)^i \left(x_0, \ldots, \breve{x}_i, \ldots, x_k\right)_{(k-1)} \tag{229}$$

To see that this make sense consider a triangle $\left(x\right)_{(3)} = \left(x_0, x_1, x_2\right)$. Then

$$\begin{aligned} \partial\left(x\right)_{(3)} &= \left(x_1, x_2\right)_{(2)} - \left(x_0, x_2\right)_{(2)} + \left(x_0, x_1\right)_{(2)} \\ &= \left(x_1, x_2\right)_{(2)} + \left(x_2, x_0\right)_{(2)} + \left(x_0, x_1\right)_{(2)} \end{aligned} \tag{230}$$

each 2-simplex in the boundary 2-chain connects head to tail with the same sign.

Now consider the boundary of the boundary

$$\begin{aligned}\partial^2 (x)_{(3)} &= \partial (x_1, x_2)_{(2)} + \partial (x_2, x_0)_{(2)} + \partial (x_0, x_1)_{(2)} \\ &= (x_1)_{(1)} - (x_2)_{(1)} + (x_2)_{(1)} - (x_0)_{(1)} + (x_0)_{(1)} - (x_1)_{(1)} \\ &= 0 \qquad (231)\end{aligned}$$

What we need to prove is that in general $\partial^2 (x)_{(k)} = 0$. To do this consider the boundary of the i^{th} term on thr r.h.s. of equation 229 letting $A_{ij}^{(k-2)} = (x_0, \ldots, \breve{x}_i, \ldots, \breve{x}_j, \ldots, x_k)_{(k-1)}$, noting

Then

$$\partial\,(x_0,\ldots,\breve{x}_i,\ldots,x_k)_{(k-1)} =$$

$$\left\{\begin{array}{cc} i=0: & \displaystyle\sum_{j=1}^{k}(-1)^{j-1}\,A_{ij}^{(k-2)} \\ 0<i<k: & \displaystyle\sum_{j=0}^{i-1}(-1)^{j}\,A_{ij}^{(k-2)}+\sum_{j=i+1}^{k}(-1)^{j-1}\,A_{ij}^{(k-2)} \\ i=k: & \displaystyle\sum_{j=0}^{k-1}(-1)^{j}\,A_{ij}^{(k-2)} \end{array}\right\} \quad (232)$$

The critical point in equation 232 is that the exponent of -1 in the second term on the r.h.s. is not j, but $j-1$. The reason for this is that when x_i was removed from the simplex the vertices were **not** renumbered. We can now express the boundary of the boundary in terms of the following matrix elements ($B_{ij}^{(k-2)} = (-1)^{i+j}\, A_{ij}^{(k-2)}$) as

$$\begin{aligned}
\partial^2 (x)_{(k)} =& \sum_{j=1}^{k} (-1)^{j-1} A_{0j}^{(k-2)} + (-1)^k \sum_{j=0}^{k-1} (-1)^j A_{kj}^{(k-2)} \\
& + \sum_{i=1}^{k-1} (-1)^i \left(\sum_{j=0}^{i-1} (-1)^j A_{ij}^{(k-2)} + \sum_{j=i+1}^{k} (-1)^{j-1} A_{ij}^{(k-2)} \right) \\
=& - \sum_{j=1}^{k} B_{0j}^{(k-2)} + \sum_{j=0}^{k-1} B_{kj}^{(k-2)} \\
& + \sum_{i=1}^{k-1} \sum_{j=0}^{i-1} B_{ij}^{(k-2)} - \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} B_{ij}^{(k-2)} = 0 \qquad (233)
\end{aligned}$$

The consider $B_{ij}^{(k-2)}$ as a matrix (i-row index, j-column index). The matrix is symmetrical and in equation 233 you are subtracting all the elements above the main diagonal from the elements below the main diagonal so that $\partial^2 (x)_{(k)} = 0$ and the boundary of a boundary of a simplex is zero.

Now add geometry to the simplex by defining the vectors

$$e_i = x_i - x_0, \quad i = 1, \ldots, k, \tag{234}$$

and the directed volume element

$$\Delta X = \frac{1}{k!} e_1 \wedge \cdots \wedge e_k \tag{235}$$

We now wish to prove that

$$\int_{(x)_{(k)}} dX = \Delta X \tag{236}$$

Any point in the simplex can be written in terms of the coordinates λ^i as

$$x = x_0 + \sum_{i=1}^{k} \lambda^i e_i \tag{237}$$

with restrictions

$$0 \leq \lambda^i \leq 1 \quad \text{and} \quad \sum_{i=1}^{k} \lambda^i \leq 1 \tag{238}$$

First we show that

$$\int_{(x)_{(k)}} dX = \int_{(x)_{(k)}} e_1 \wedge \cdots \wedge e_k \, d\lambda^1 \cdots d\lambda^k = \Delta X \tag{239}$$

or

$$\int_{(x)_{(k)}} d\lambda^1 \cdots d\lambda^k = \frac{1}{k!} \tag{240}$$

define $\Lambda_j = 1 - \sum_{i=1}^{j} \lambda^i$ (Note that $\Lambda_0 = 1$). From the restrictions on the λ^i's we have

$$\int_{(x)_{(k)}} d\lambda^1 \cdots d\lambda^k = \int_0^{\Lambda_0} d\lambda^1 \int_0^{\Lambda_1} d\lambda^2 \cdots \int_0^{\Lambda_{k-1}} d\lambda^k \tag{241}$$

To prove that the r.h.s of equation 241 is $1/k!$ we form the following sequence and use induction to prove that V_j is the result of the first j partial Integrations of equation 241

$$V_j = \frac{1}{j!} \left(\Lambda_{k-j}\right)^j \tag{242}$$

Then

$$\begin{aligned} V_{j+1} &= \int_0^{\Lambda_{k-j-1}} d\lambda^{k-j} V_j \\ &= \int_0^{\Lambda_{k-j-1}} d\lambda^{k-j} \frac{1}{j!} \left(\Lambda_{k-j-1} - \lambda^{k-j}\right)^j \\ &= \frac{-1}{(j+1)\, j!} \left[\left(\Lambda_{k-j-1} - \lambda^{k-j}\right)^{j+1}\right]_0^{\Lambda_{k-j-1}} \\ &= \frac{1}{(j+1)!} \left(\Lambda_{k-j-1}\right)^{j+1} \qquad (243) \end{aligned}$$

so that $V_k = 1/k!$ and the assertion is proved. Now let there be a multivector field $F(x)$ that assumes the values $F_i = F(x_i)$ at the vectices of the simplex and define the interpolating function

$$f\left(x\right)=F_0+\sum_{i=1}^{k}\lambda^i\left(F_i-F_0\right) \tag{244}$$

We now wish to show that

$$\int_{(x)_{(k)}} f\,dX=\frac{1}{k+1}\left(\sum_{i=0}^{k}F_i\right)\Delta X=\bar{F}\,\Delta X \tag{245}$$

To prove this we must show that

$$\int_{(x)_{(k)}} \lambda^i\,dX=\frac{1}{k+1}\Delta X,\quad \forall\,\lambda^i \tag{246}$$

To do this consider the integral (equation 243 with V_j replaced by $\lambda^{k-j}V_j$)

$$\begin{aligned}
\int_0^{\Lambda_{k-j-1}} d\lambda^{k-j}\,\lambda^{k-j}V_j &= \int_0^{\Lambda_{k-j-1}} d\lambda^{k-j}\frac{1}{j!}\lambda^{k-j}\left(\Lambda_{k-j-1}-\lambda^{k-j}\right)^j \\
&= \frac{1}{(j+2)!}\left(\Lambda_{k-j-1}\right)^{j+2} \qquad (247)
\end{aligned}$$

Note that since the extra λ^i factor occurs in exactly one of the subintegrals for each different λ^i the final result of the total integral is muliplied by a factor of $\frac{1}{(k+1)}$ since the weight of the total integral is now $\frac{1}{(k+1)!}$ and the assertion (equation 246 and hence equation 245) is proved.

Now summing over all the simplices making up the directed volume gives

$$\int_{\text{volume}} F\,dX = \lim_{n\mapsto\infty}\sum_{i=1}^{n}\bar{F}^i\Delta X^i \qquad (248)$$

The most general statement of equation 248 is

$$\int_{\mathsf{volume}} \mathsf{L}\left(dX\right) = \lim_{n\mapsto\infty}\sum_{i=1}^{n}\bar{\mathsf{L}}^{i}\left(\Delta X^{i}\right) \tag{249}$$

where $\mathsf{L}\left(F_n;x\right)$ is a position dependent linear function of a grade-n multivector F_n and $\bar{\mathsf{L}}^{i}$ is the average value of $\mathsf{L}\left(dX\right)$ over the vertices of each simplex.

An example of this would be

$$\mathsf{L}\left(F_n;x\right) = G\left(x\right)F_nH\left(x\right) \tag{250}$$

where $G\left(x\right)$ and $H\left(x\right)$ are multivector functions of x.

Directed Integration - The Fundamental Theorem of Geometric Calculus

We now must prove that the directed measure of a simplex boundary is zero

$$\Delta\left(\partial\left(x\right)_{(k)}\right)=\Delta\left(\partial\left(x_0,\ldots,x_k\right)_{(k)}\right)=0 \tag{251}$$

Start with a planar simplex of three points

$$\partial\left(x_0,x_1,x_2\right)_{(2)}=\left(x_1,x_2\right)_{(1)}-\left(x_0,x_2\right)_{(1)}+\left(x_0,x_1\right)_{(1)} \tag{252}$$

so that

$$\Delta\left(\partial\left(x_0,x_1,x_2\right)_{(2)}\right)=\left(x_2-x_1\right)-\left(x_2-x_0\right)+\left(x_1-x_0\right)=0 \tag{253}$$

We shall now prove equation 251 via induction. First note that

$$\Delta\left(\breve{x}_i\right)_{(k-1)} = \left\{ \begin{array}{cc} i = 0 : & \dfrac{1}{k-1}\Delta\left(\breve{x}_0\right)_{(k-2)} \wedge \left(x_k - x_1\right) \\ 0 < i \leq k-1 : & \dfrac{1}{k-1}\Delta\left(\breve{x}_i\right)_{(k-2)} \wedge \left(x_k - x_0\right) \end{array} \right\} \quad (254)$$

and

$$\Delta\left(\breve{x}_k\right)_{(k-1)} = \frac{1}{(k-1)!}\left(x_1 - x_0\right) \wedge \cdots \wedge \left(x_{k-1} - x_0\right) \quad (255)$$

so that

$$\Delta\left(\partial\left(x\right)_{(k)}\right) = \frac{1}{k-1}\sum_{i=1}^{k-1}\left(-1\right)^i \Delta\left(\breve{x}_i\right)_{(k-2)} \wedge \left(x_k - x_0\right) + \mathcal{C}$$

where

$$\mathcal{C} = \frac{1}{k-1}\Delta\left(\breve{x}_0\right)_{(k-2)} \wedge \left(x_k - x_1\right) + \left(-1\right)^k \Delta\left(\breve{x}_k\right)_{(k-1)} \tag{256}$$

if we let $\delta = x_0 - x_1$ we can write

$$\mathcal{C} = \frac{1}{k-1}\Delta\left(\breve{x}_0\right)_{(k-2)} \wedge \left(x_k - x_0\right) + \frac{1}{k-1}\Delta\left(\breve{x}_0\right)_{(k-2)} \wedge \delta + \left(-1\right)^k \Delta\left(\breve{x}_k\right)_{(k-1)} \tag{257}$$

Then

$$\Delta\left(\breve{x}_0\right)_{(k-2)} \wedge \delta = \frac{1}{(k-2)!}\left(x_2 - x_1\right) \wedge \cdots \wedge \left(x_{k-1} - x_1\right) \wedge \delta \tag{258}$$

$$= \frac{1}{(k-2)!}\left(x_2 - x_0 + \delta\right) \wedge \cdots \wedge \left(x_{k-1} - x_0 + \delta\right) \wedge \delta \tag{259}$$

$$= \frac{1}{(k-2)!}\left(x_2 - x_0\right) \wedge \cdots \wedge \left(x_{k-1} - x_0\right) \wedge \delta \tag{260}$$

$$= \frac{(-1)^{k-2}}{(k-2)!}\delta \wedge (x_2 - x_0) \wedge \cdots \wedge (x_{k-1} - x_0) \tag{261}$$

$$= \frac{(-1)^{k-1}}{(k-2)!}(x_1 - x_0) \wedge (x_2 - x_0) \wedge \cdots \wedge (x_{k-1} - x_0) \tag{262}$$

Thus

$$\frac{-1}{k-1}\Delta\left(\breve{x}_0\right)_{(k-2)} \wedge \delta = \tag{263}$$

$$= \frac{(-1)^{k}}{(k-1)!}(x_1 - x_0) \wedge (x_2 - x_0) \wedge \cdots \wedge (x_{k-1} - x_0) \tag{264}$$

$$= (-1)^k \Delta\left(\breve{x}_k\right)_{(k-1)} \tag{265}$$

However

$$(-1)^k \Delta\left(\breve{x}_k\right)_{(k-1)} = \frac{-1}{k-1}\Delta\left(\breve{x}_0\right)_{(k-2)} \wedge \delta \tag{266}$$

so that

$$\mathcal{C} = \frac{1}{k-1}\Delta\left(\breve{x}_0\right)_{(k-2)} \wedge \left(x_k - x_0\right) \tag{267}$$

and

$$\begin{aligned}
\Delta\left(\partial\left(x\right)_{(k)}\right) &= \frac{1}{k-1}\left(\sum_{i=0}^{k-1}\left(-1\right)^i \Delta\left(\breve{x}_i\right)_{(k-2)}\right) \wedge \left(x_k - x_0\right) \\
&= \frac{1}{k-1}\left(\Delta\left(\partial\left(x\right)_{(k-1)}\right)\right) \wedge \left(x_k - x_0\right) \\
&= 0
\end{aligned} \tag{268}$$

We have proved equation 251. Note that to reduce equation 267 we had to use that for any set of vectors $\delta, y_1, \ldots, y_k$ we have

$$\delta \wedge \left(y_1 + \delta\right) \wedge \cdots \wedge \left(y_k + \delta\right) = \delta \wedge y_1 \wedge \cdots \wedge y_k \tag{269}$$

Think about equation 269. It's easy to prove ($\delta \wedge \delta = 0$)!

Equation 251 is sufficient to prove that the directed integral over the surface of simplex is zero

$$\oint_{\partial(x)_{(k)}} dS = \sum_{i=0}^{k} (-1)^{i} \int_{(\breve{x}_i)_{(k-1)}} dX = \Delta\left(\partial\left(x\right)_{(k)}\right) = 0 \tag{270}$$

The characteristics of a general volume are:

1. A general volume is built up from a chain of simplicies.
2. Simplices in the chain are defined so that at any common boundary the directed areas of the bounding faces are equal and opposite.
3. Surface integrals over two simplices cancel over their common face.
4. The surface integral over the boundary of the volume can be replaced by the sum of the surface integrals over each simplex in the chain.

If the boundary of the volume is closed we have

$$\oint dS = \lim_{n \mapsto \infty} \sum_{a=1}^{n} \oint dS^{a} = 0 \tag{271}$$

Where $\oint dS^a$ is the surface Integral over the a^{th} simplex. Implicit in equation 271 is that the surface is orientated, simply connected, and closed.

The next lemma to prove that if b is a constant vector on the simplex $(x)_{(k)}$ then

$$\oint_{\partial (x)_{(k)}} b \cdot x \, dS = b \cdot \Delta \left((x)_{(k)} \right) = b \cdot \Delta X \tag{272}$$

The starting point of the lemma is equation 246. First define

$$b = \sum_{i=1}^{k} b_i e^i, \tag{273}$$

where the e^i's are the reciprocal frame to $e_i = x_i - x_0$ so that

$$x - x_0 = \sum_{i=1}^{k} \lambda^i e_i, \tag{274}$$

$$b_i = b \cdot e_i, \tag{275}$$

and

$$\sum_{i=1}^{k} \lambda^i b_i = b \cdot (x - x_0) \,. \tag{276}$$

Substituting into equation 246 we get

$$\sum_{i=1}^{k} \int_{(x)_{(k)}} b_i \lambda^i \, dX = \int_{(x)_{(k)}} b \cdot (x - x_0) \, dX = \frac{1}{k+1} \sum_{i=1}^{k} b \cdot e_i \, \Delta X \tag{277}$$

Rearranging terms gives

$$\begin{aligned}\int_{(x)_{(k)}} b \cdot x \, dX &= \frac{1}{k+1} \left(\left(\sum_{i=1}^{k} b \cdot (x_i - x_0) \right) + (k+1)\, b \cdot x_0 \right) \Delta X \\ &= \frac{b}{k+1} \cdot \left(\sum_{i=0}^{k} x_i \right) \Delta X \\ &= b \cdot \bar{x} \Delta X \qquad (278)\end{aligned}$$

Now using the definition of a simplex boundary (equation 229) and equation 278 we get

$$\oint_{\partial (x)_{(k)}} b \cdot x \, dS = \frac{1}{k} \sum_{i=0}^{k} (-1)^i \, b \cdot (x_0 + \cdots + \breve{x}_i + \cdots + x_k) \, \Delta \left((\breve{x}_i)_{(k-1)} \right) \qquad (279)$$

The coefficient multiplying the r.h.s. of equation 279 is $\frac{1}{k}$ and not $\frac{1}{k+1}$ because both $(x_0+\cdots+\breve{x}_i+\cdots+x_k)$ and $(\breve{x}_i)_{(k-1)}$ refer to $k-1$ simplices (the boundary of $(x)_{(k)}$ is the sum of all the simplices $(\breve{x}_i)_{(k)}$ with proper sign assigned).

Now to prove equation 272 we need to prove one final purely algebraic lemma

$$\sum_{i=0}^{k}(-1)^i\, b\cdot(x_0+\cdots+\breve{x}_i+\cdots+x_k)\,\Delta\,(\breve{x}_i)_{(k-1)}=\frac{1}{(k-1)!}b\cdot(e_1\wedge\cdots\wedge e_k) \tag{280}$$

Begin with the definition of the l.h.s. of equation 280

$$\mathsf{C}=\sum_{i=0}^{k}(-1)^i\, b\cdot(x_0+\cdots+\breve{x}_i+\cdots+x_k)\,\Delta\,(\breve{x}_i)_{(k-1)}=\sum_{i=0}^{k}\mathsf{C}_i \tag{281}$$

where C_i is defined by

$$\mathsf{C}_i = \left\{ \begin{array}{cc} i = 0 : & b \cdot (x_1 + \cdots + x_k)\,\Delta\,(\breve{x}_0)_{(k-1)} \\ 0 < i \leq k : & (-1)^i\, b \cdot (x_0 + \cdots + \breve{x}_i + \cdots + x_k)\,\Delta\,(\breve{x}_i)_{(k-1)} \end{array} \right\} \tag{282}$$

now define $E_k = e_1 \wedge \cdots \wedge e_k$ so that (using equation 61 from the section on reciprocal frames)

$$(-1)^{i-1}\, e^i E_k = e_1 \wedge \cdots \wedge \breve{e}_i \wedge \cdots \wedge e_k, \quad \forall\, 0 < i \leq k \tag{283}$$

and

$$\mathsf{C}_{0<i\leq k} = \frac{-1}{(k-1)!} b \cdot (x_0 + \cdots + \breve{x}_i + \cdots + x_k)\, e^i E_k \tag{284}$$

The main problem is in evaluating C$_0$ since

$$\Delta\left(\breve{x}_0\right)_{(k-1)} = \frac{1}{(k-1)!}\left(x_2 - x_1\right)\wedge\cdots\wedge\left(x_k - x_1\right) \tag{285}$$

using $e_i = x_i - x_0$ reduces equation 285 to

$$\Delta\left(\breve{x}_0\right)_{(k-1)} = \frac{1}{(k-1)!}\left(e_2 - e_1\right)\wedge\cdots\wedge\left(e_k - e_1\right) \tag{286}$$

but equation 286 can be expanded into equation 287. The critical point in doing the expansion is that in generating the sum on the r.h.s. of the first line of equation 287 all products containing x_1 (of course all terms in the sum contain x_1 exactly once since we are using the outer product) are put in normal order by bringing the x_1 factor to the front of the product thus requiring the factor of $(-1)^i$ in each term in the sum.

$$\begin{aligned}(e_2 - e_1) \wedge \cdots \wedge (e_k - e_1) &= e_2 \wedge \cdots \wedge e_k - \sum_{i=2}^{k} (-1)^i e_1 \wedge e_2 \wedge \cdots \wedge \breve{e}_i \wedge \cdots \wedge e_k \\ &= \sum_{i=1}^{k} (-1)^{i-1} e_1 \wedge e_2 \wedge \cdots \wedge \breve{e}_i \wedge \cdots \wedge e_k \\ &= \sum_{i=1}^{k} e^i E_k \qquad (287)\end{aligned}$$

or

$$\Delta\left(\breve{x}_0\right)_{(k-1)} = \frac{1}{(k-1)!} \sum_{i=1}^{k} e^i E_k \qquad (288)$$

from equation 287 we have

$$\begin{aligned}
\mathsf{C} &= \frac{1}{(k-1)!}\sum_{i=1}^{k}\left(b\cdot\left(x_i - x_0\right)\right)e^iE_k \\
&= \frac{1}{(k-1)!}\sum_{i=1}^{k}\left(b\cdot e_i\right)e^iE_k \\
&= \frac{1}{(k-1)!}\sum_{i=1}^{k}b_ie^iE_k \\
&= \frac{1}{(k-1)!}bE_k \\
&= \frac{1}{(k-1)!}\left(b\cdot E_k + b\wedge E_k\right) \\
&= \frac{1}{(k-1)!}b\cdot E_k
\end{aligned} \tag{289}$$

and equation 280 is proved which means substituting equation 280 into equation 279 proves equation 272

The Fundamental Theorem At Last!

The simplicial coordinates, λ^i, can be expressed in terms of the position vector, x, and the frame vectors of the simplex, $e_i = x_i - x_0$. Let the vectors e^j be the reciprocal frame to e_i ($e_i \cdot e^j = \delta_i^j$). Then

$$\lambda^i = e^i \cdot (x - x_0) \tag{290}$$

and let $f(x)$ be an affine multivector function of x (equation 244) which interpolates, F, a differentiable multivector function of x on the simplex. Then

$$\oint_{\partial(x)_{(k)}} f(x)\; dS = \sum_{i=1}^{k} (F_i - F_0) \oint_{\partial(x)_{(k)}} e^i \cdot (x - x_0)$$

$$= \sum_{i=1}^{k} \left(F_i - F_0\right) e^i \cdot \left(\Delta X\right) \tag{291}$$

But

$$\frac{\partial f\left(x\right)}{\partial \lambda^i} = F_i - F_0 \tag{292}$$

so that the surface integral of equation 291 can be rewritten

$$\begin{aligned} \oint_{\partial(x)_{(k)}} f\left(x\right)\; dS &= \sum_{i=1}^{k} \left(F_i - F_0\right) e^i \cdot \left(\Delta X\right) \\ &= \sum_{i=1}^{k} \frac{\partial f}{\partial \lambda^i} e^i \cdot \left(\Delta X\right) = \dot{f}\dot{\nabla} \cdot \left(\Delta X\right) \end{aligned} \tag{293}$$

If we now sum equation 293 over a chain of simplices realizing that the interpolated function $f\left(x\right)$ takes on the same value over the common boundary of two adjacent simplices, since $f\left(x\right)$ is only defined by the values at the common vertices. In forming a sum over a chain, all of the internal faces cancel and only the surface integral over the boundary remains. Thus

$$\oint f\left(x\right)\ dS=\sum_{a}\dot{f}\dot{\nabla}\cdot\left(\Delta X^{a}\right) \tag{294}$$

with the sum running over all simplices in the chain. Taking the limit as more points are added and each simplex is shrunk in size we obtain the first realization of the fundmental theorem of geometric calculus

$$\oint_{\partial V} F\ dS=\int_{V}\dot{F}\dot{\nabla}\ dX \tag{295}$$

We can write ∇dX instead of $\nabla\cdot dX$ since the vector ∇ is totally within the vector space defined by dX so that $\nabla\wedge dX=0$. The method of proof used can also be applied to the form

$$\oint_{\partial V} dS\, G = \int_V \dot{\nabla}\, dX\, \dot{G} \tag{296}$$

A more general statement of the theorem is as follows:

Let $\mathsf{L}\left(A_{k-1}\right) = \mathsf{L}\left(A_{k-1}; x\right)$ be a linear functional of a multivector A_{k-1} of grade $k-1$ and a general fuction of position x which returns a general multivector. The linear interpolation (approximation) of L over a simplex is defined by:

$$L\left(A\right) \equiv \mathsf{L}\left(A; x_0\right) + \sum_{i=1}^{k} \lambda^i \left(\mathsf{L}\left(A; x_i\right) - \mathsf{L}\left(A; x_0\right)\right) \tag{297}$$

Then the integral over a simplex is (Note that since integration is a linear operation, a summation, the integral can be placed inside $L\left(A\right)$ since

$L(A)$ is linear in A)

$$\begin{aligned}\oint_{\partial(x)_{(k)}} L(dS) &= \mathsf{L}\left(\oint dS; x_0\right) + \sum_{i=1}^{k} \mathsf{L}\left(\oint \lambda^i dS; x_i\right) - \sum_{i=1}^{k} \mathsf{L}\left(\oint \lambda^i dS; x_0\right) \\ &= \sum_{i=1}^{k} \left(\mathsf{L}\left(e^i \Delta X; x_i\right) - \mathsf{L}\left(e^i \Delta X; x_0\right)\right) \\ &= \dot{L}\left(\dot{\nabla}\Delta X\right) \qquad (298)\end{aligned}$$

Taking the limit of a sum of simplicies gives

$$\oint_{\partial V} \mathsf{L}(dS) = \int_V \dot{\mathsf{L}}\left(\dot{\nabla} dX\right) \qquad (299)$$

Divergence and Green's Theorems

As a specific example consider $\mathsf{L}\left(A\right)=\left\langle JAI^{-1}\right\rangle$ where J is a vector, I is the unit pseudoscalar for a n-dimensional vector space, and A is a multivector of grade $n-1$. Then equation 299 gives

$$\int_V \left\langle \dot{J}\dot{\nabla} dXI^{-1}\right\rangle = \int_V \nabla\cdot J\left|dX\right| = \oint_{\partial V}\left\langle JdSI^{-1}\right\rangle \tag{300}$$

we have $dX=I\left|dX\right|$ where $\left|dX\right|$ is the scalar measure of the volume. The normal to the surface, n, is defined by

$$n\left|dS\right| = dSI^{-1} \tag{301}$$

where $\left|dS\right|$ is the scalar valued measure over the surface. With this definition we get

$$\int_V \nabla \cdot J \, |dX| = \oint_{\partial V} n \cdot J \, |dS| \tag{302}$$

Now using the form of the Fundamental Theorem of Geometric Calculus in equation 296 and let G be the vector $\boldsymbol{J}$ in two-dimensional Euclidian space and noting that since dA is a pseudoscalar (for 2-D ds is the boundary measure and dA is the volume measure) it anticommutes with vectors in two dimensions we get -

$$\oint_{\partial A} ds\boldsymbol{J} = \int_A \dot{\boldsymbol{\nabla}} dA \dot{\boldsymbol{J}} = -\int_A \boldsymbol{\nabla}\boldsymbol{J} dA \tag{303}$$

In 2-D Cartesian coordinates $dA = Idxdy$ and $ds = nI\,|ds|$ so that

$$\oint_{\partial A} ds\boldsymbol{J} = -\int_A \boldsymbol{\nabla}\boldsymbol{J} Idxdy. \tag{304}$$

or

$$\oint_{\partial A} nI\boldsymbol{J}\,|ds| = -\int_A \boldsymbol{\nabla}\boldsymbol{J}Idxdy$$

$$-\oint_{\partial A} n\boldsymbol{J}I\,|ds| = -\int_A \boldsymbol{\nabla}\boldsymbol{J}Idxdy \tag{305}$$

$$\oint_{\partial A} n\boldsymbol{J}\,|ds| = \int_A \boldsymbol{\nabla}\boldsymbol{J}dxdy \tag{306}$$

Letting $\boldsymbol{J} = P\boldsymbol{e}_x + Q\boldsymbol{e}_y$ and $n = n^x\boldsymbol{e}_x + n^y\boldsymbol{e}_y$ we get -

$$n\boldsymbol{J} = n \cdot \boldsymbol{J} + (n^x Q - n^y P)\,\boldsymbol{e}_x\boldsymbol{e}_y \tag{307}$$

$$\nabla\boldsymbol{J} = \nabla \cdot \boldsymbol{J} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\boldsymbol{e}_x\boldsymbol{e}_y \tag{308}$$

so that

$$\oint_{\partial A} n \cdot \boldsymbol{J} \, |ds| = \int_A \nabla \cdot \boldsymbol{J} dxdy \tag{309}$$

$$\oint_{\partial A} (n^x Q - n^y P) \, |ds| \, \boldsymbol{e}_x \boldsymbol{e}_y = \int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy \boldsymbol{e}_x \boldsymbol{e}_y \tag{310}$$

but $dy = n^x \, |ds|$ and $dx = -n^y \, |ds|$ so that

$$\oint_{\partial A} P dx + Q dy = \int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy \tag{311}$$

which is Green's theorem in the plane.

Cauchy's Integral Formula In Two Dimensions (Complex Plane)

Consider a two dimensional euclidian space with vectors $\boldsymbol{r} = x\boldsymbol{e}_x + y\boldsymbol{e}_y$. Then the complex number z corresponding to $\boldsymbol{r}$ is

$$z = \boldsymbol{e}_x\boldsymbol{r} = x + y\boldsymbol{e}_x\boldsymbol{e}_y = x + yI \tag{312}$$

$$z^\dagger = \boldsymbol{r}\boldsymbol{e}_x = x - y\boldsymbol{e}_x\boldsymbol{e}_y = x - yI \tag{313}$$

$$zz^\dagger = z^\dagger z = x^2 + y^2 = \boldsymbol{r}^2 \tag{314}$$

Thus even grade multivectors correspond to complex numbers since $I^2 = -1$ and the reverse of z, $z^\dagger$ correspondes to the conjugate of z. Even grade multivectors commute with dS, since dS is proportional to I.

Let $\psi\left(\boldsymbol{r}\right)$ be an even multivector function of $\boldsymbol{r}$, then

$$\int \boldsymbol{\nabla}\psi dS = \oint d\boldsymbol{s}\psi = \oint \frac{\partial \boldsymbol{r}}{\partial \lambda}\psi d\lambda \tag{315}$$

but the complex number z is given by $z = \boldsymbol{e}_x\boldsymbol{r}$ and

$$\boldsymbol{e}_x \oint d\boldsymbol{s}\psi = \oint \psi dz = \int \boldsymbol{e}_x\boldsymbol{\nabla}\psi dS. \tag{316}$$

Thus if a function ψ is analytic, $\boldsymbol{\nabla}\psi = 0$ and $\oint \psi dz = 0$. Now note that (this will be proved for the N-dimensional case in the next example)

$$\boldsymbol{\nabla}\frac{\boldsymbol{r}-\boldsymbol{a}}{\left(\boldsymbol{r}-\boldsymbol{a}\right)^2} = 2\pi\delta\left(\boldsymbol{r}-\boldsymbol{a}\right) \tag{317}$$

where $a = \boldsymbol{e}_x \boldsymbol{a}$. Now let

$$\psi = \frac{\boldsymbol{r} - \boldsymbol{a}}{(\boldsymbol{r} - \boldsymbol{a})^2} \boldsymbol{e}_x f(\boldsymbol{e}_x \boldsymbol{r}) \tag{318}$$

so that

$$\begin{aligned}
\boldsymbol{e}_x \oint d\boldsymbol{s}\psi &= \boldsymbol{e}_x \oint d\boldsymbol{s} \left(\frac{\boldsymbol{r} - \boldsymbol{a}}{(\boldsymbol{r} - \boldsymbol{a})^2} \boldsymbol{e}_x f(\boldsymbol{e}_x \boldsymbol{r}) \right) \\
&= \oint dz \left(\frac{\boldsymbol{r} - \boldsymbol{a}}{(\boldsymbol{r} - \boldsymbol{a})^2} \boldsymbol{e}_x f(\boldsymbol{e}_x \boldsymbol{r}) \right) \\
&= \oint dz \left(\frac{(z-a)^\dagger}{(z-a)(z-a)^\dagger} f(z) \right) \\
&= \oint \frac{f(z)}{z-a} dz \qquad (319)
\end{aligned}$$

and

$$\begin{aligned}
\oint \frac{f\left(z\right)}{z-a}dz &= \boldsymbol{e}_x \int \boldsymbol{\nabla}\left(\frac{\boldsymbol{r}-\boldsymbol{a}}{\left(\boldsymbol{r}-\boldsymbol{a}\right)^2}\boldsymbol{e}_x f\left(\boldsymbol{e}_x\boldsymbol{r}\right)\right) dS \\
&= \boldsymbol{e}_x \int \left(2\pi\delta\left(\boldsymbol{r}-\boldsymbol{a}\right)\boldsymbol{e}_x f\left(\boldsymbol{e}_x\boldsymbol{r}\right) + \boldsymbol{\nabla} f\left(\boldsymbol{e}_x\boldsymbol{r}\right)\frac{\boldsymbol{r}-\boldsymbol{a}}{\left(\boldsymbol{r}-\boldsymbol{a}\right)^2}\boldsymbol{e}_x\right) I\left|dS\right| \\
&= 2\pi I f(a) + \int \boldsymbol{e}_x\boldsymbol{\nabla} f\left(\boldsymbol{e}_x\boldsymbol{r}\right)\frac{z^\dagger - a^\dagger}{\left|z-a\right|^2} I\left|dS\right| \\
&= 2\pi I f(a) + \int \boldsymbol{e}_x\boldsymbol{\nabla} f\left(\boldsymbol{e}_x\boldsymbol{r}\right)\frac{1}{z-a} I\left|dS\right| \\
&= 2\pi I f(a) + \int \left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f\left(z\right)\frac{1}{z-a} I\left|dS\right| \qquad (320)
\end{aligned}$$

If $\boldsymbol{\nabla} f(z) = 0$ we have -

$$\oint \frac{f(z)}{z-a} dz = 2\pi I f(a) \tag{321}$$

which is the Cauchy integral formula. If $\boldsymbol{\nabla} f(z)$ is not zero we can write the more general relation -

$$2\pi I f(a) = \oint \frac{f}{z-a} dz - 2 \int \frac{\partial f}{\partial z^\dagger} \frac{1}{z-a} I \left| dS \right| \tag{322}$$

since

$$\frac{\partial}{\partial z^\dagger} = \frac{1}{2} \left(\frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) \tag{323}$$

and

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - I \frac{\partial}{\partial y} \right) \tag{324}$$

Green's Functions in N-dimensional Euclidian Spaces

Let ψ be an even multivector function or let N be even so that ψ commutes with I. The analog of an analytic function in N-dimensions is $\nabla\psi = 0$.

The Green's function of the ∇ operator is ($S_N = 2\pi^{N/2}/\Gamma(N/2)$ is the hyperarea of the unit radius sphere in N dimensions)

$$G\left(x;y\right) = \lim_{\epsilon\rightarrow 0} \frac{x-y}{S_N\left(\left|x-y\right|^N+\epsilon\right)} \tag{325}$$

So that

$$\lim_{\epsilon\rightarrow 0} \nabla_x G\left(x;y\right) = \delta\left(x-y\right). \tag{326}$$

To prove equation 326 we need to use

$$\nabla_x (x-y) = N \quad \text{and} \quad \nabla_x |x-y|^M = M(x-y)|x-y|^{M-2}$$

So that

$$\begin{aligned}
\nabla_x G &= \frac{1}{S_N}\left\{\nabla_x \left(|x-y|^N+\epsilon\right)^{-1}(x-y) + \left(|x-y|^N+\epsilon\right)^{-1}\nabla_x (x-y)\right\} \\
&= \frac{N}{S_N}\left\{\frac{-|x-y|^N}{\left(|x-y|^N+\epsilon\right)^2} + \frac{1}{\left(|x-y|^N+\epsilon\right)}\right\} \\
&= \frac{N}{S_N}\frac{\epsilon}{\left(|x-y|^N+\epsilon\right)^2} = \nabla_x \cdot G = \dot{G}\cdot\dot{\nabla}_x = \dot{G}\dot{\nabla}_x \qquad (327)
\end{aligned}$$

so what must be proved is that

$$\lim_{\epsilon \to 0} \frac{N}{S_N} \frac{\epsilon}{\left(|x-y|^N + \epsilon \right)^2} = \delta(x-y) \tag{328}$$

First define the volume B_τ $(\tau > 0)$ by

$$x \in B_\tau \iff |x| \le \tau \tag{329}$$

and let $y = 0$ and calculate (Note that we use $|dV|$ since in our notation $dV = I\,|dV|$ and the oriented dV is not needed in this proof. Also $r = |x|$.)

$$\int_{B_\infty} \frac{N}{S_N} \frac{\epsilon}{\left(|x|^N + \epsilon \right)^2} |dV| = \int_0^\infty \frac{N \epsilon r^{N-1}}{\left(r^N + \epsilon \right)^2} dr$$

$$
\begin{aligned}
&= \int_0^\infty \frac{\epsilon}{\left(r^N+\epsilon\right)^2} d\left(r^N\right) \\
&= -\left[\frac{\epsilon}{r^N+\epsilon}\right]_0^\infty \\
&= 1 \qquad (330)
\end{aligned}
$$

Thus the first requirement of a delta function is fulfilled. Now let $\phi(x)$ be a scalar test function on the N-dimensional space and S a point set in the space and define the functions

$$
\max\left(\phi, S\right) = \left\{\max\left(\phi(x)\right) \forall x \in S\right\} \quad \text{and} \quad \min\left(\phi, S\right) = \left\{\min\left(\phi(x)\right) \forall x \in S\right\}
$$

and calculate the integral

$$
\frac{N}{S_N} \int_{B_\tau} \frac{\epsilon}{\left(|x|^N+\epsilon\right)^2} |dV| = \int_0^\tau \frac{\epsilon}{\left(r^N+\epsilon\right)^2} d\left(r^N\right)
$$

$$= -\left[\frac{\epsilon}{r^N + \epsilon}\right]_0^\tau$$
$$= 1 - \frac{\epsilon}{\tau^N + \epsilon} \tag{331}$$

and note that

$$\frac{N}{S_N}\int_{B_\infty - B_\tau} \frac{\epsilon}{\left(|x|^N + \epsilon\right)^2}\,|dV| = \frac{\epsilon}{\tau^N + \epsilon} \tag{332}$$

Thus $\forall \tau > 0$ we have

$$\lim_{\epsilon \to 0} \frac{N}{S_N}\int_{B_\infty - B_\tau} \frac{\epsilon}{\left(|x|^N + \epsilon\right)^2}\,|dV| = 0 \tag{333}$$

and

$$\lim_{\epsilon \to 0} \frac{N}{S_N} \int_{B_\tau} \frac{\epsilon}{\left(|x|^N + \epsilon\right)^2} |dV| = 1 \tag{334}$$

Thus

$$\begin{aligned} \min\left(\phi, B_\infty - B_\tau\right) \frac{N}{S_N} \int_{B_\infty - B_\tau} \frac{\epsilon}{\left(|x|^N + \epsilon\right)^2} |dV| \leq \\ \frac{N}{S_N} \int_{B_\infty - B_\tau} \frac{\epsilon \phi(x)}{\left(|x|^N + \epsilon\right)^2} |dV| \\ \leq \max\left(\phi, B_\infty - B_\tau\right) \frac{N}{S_N} \int_{B_\infty - B_\tau} \frac{\epsilon}{\left(|x|^N + \epsilon\right)^2} |dV| \end{aligned} \tag{335}$$

and

$$\min\left(\phi, B_\tau\right)\frac{N}{S_N}\int_{B_\tau}\frac{\epsilon}{\left(|x|^N+\epsilon\right)^2}\,|dV| \leq$$

$$\frac{N}{S_N}\int_{B_\tau}\frac{\epsilon\phi(x)}{\left(|x|^N+\epsilon\right)^2}\,|dV|$$

$$\leq \max\left(\phi, B_\tau\right)\frac{N}{S_N}\int_{B_\tau}\frac{\epsilon}{\left(|x|^N+\epsilon\right)^2}\,|dV| \qquad (336)$$

Thus

$$\lim_{\epsilon\longrightarrow 0}\frac{N}{S_N}\int_{B_\infty - B_\tau}\frac{\epsilon\phi(x)}{\left(|x|^N+\epsilon\right)^2}\,|dV| = 0 \qquad (337)$$

and

$$\min\left(\phi, B_\tau\right) \le \lim_{\epsilon\rightarrow 0}\frac{N}{S_N}\int_{B_\tau}\frac{\epsilon\phi(x)}{\left(\left|x\right|^N+\epsilon\right)^2}\left|dV\right| \le \max\left(\phi, B_\tau\right) \tag{338}$$

Finally

$$\lim_{\tau\rightarrow 0}\lim_{\epsilon\rightarrow 0}\frac{N}{S_N}\int_{B_\tau}\frac{\epsilon\phi(x)}{\left(\left|x\right|^N+\epsilon\right)^2}\left|dV\right| = \phi(0) \tag{339}$$

and we have proved equation 328 since

$$\lim_{\tau\rightarrow 0}\left(\max(\phi, B_\tau) - \min(\phi, B_\tau)\right) = 0 \tag{340}$$

Now in the fundamental theorem of geometric calculus let $\mathsf{L}(A) = GA\psi$

so that (remember that ψ communtes with dV and $\nabla_x \psi = 0$)

$$\begin{aligned} \dot{\mathsf{L}}\left(\dot{\nabla}_x dV\right) &= \dot{G}\dot{\nabla}_x dV\psi + G\dot{\nabla}_x dV\dot{\psi} \\ &= \left(\dot{G}\dot{\nabla}_x\psi + G\nabla_x\psi\right) I\,|dV| \\ &= \dot{G}\dot{\nabla}_x\psi I\,|dV| \\ &= \nabla_x G\psi I\,|dV| \end{aligned} \tag{341}$$

and

$$\begin{aligned} \oint_{\partial V} GdS\psi &= \int_V \delta(x-y)\,\psi(x)\,I\,|dV| \\ &= \psi(y)\,I \\ &= I\psi(y) \end{aligned} \tag{342}$$

or

$$\psi(y) = \frac{I^{-1}}{S_N} \oint_{\partial V} \frac{x-y}{|x-y|^N} dS \psi(x) \tag{343}$$

because ψ is a monogenic function ($\nabla\psi = 0$).

Geometric Calculus on Manifolds

The definition of a manifold that we will use is -

> *A vector manifold (generalized surface) is a set of points labeled by vectors lying in a geometric algebra of arbitrary dimension and signature. If we consider a path in the surface $x(\lambda)$, the tangent vector is defined by*
>
> $$x' \equiv \left.\frac{\partial x(\lambda)}{\partial \lambda}\right|_{\lambda_0} = \lim_{\epsilon \to 0} \frac{x(\lambda_0 + \epsilon) - x(\lambda_0)}{\epsilon} \tag{344}$$
>
> *and the path length*
>
> $$s \equiv \int_{\lambda_1}^{\lambda_2} \sqrt{|x' \cdot x'|}\, d\lambda \tag{345}$$

Pseudoscalar on Manifold -

Now introduce a set of paths in the surface all passing through the same point x. These paths define a set of tangent vectors $\{e_1, \ldots, e_n\}$. We assume these paths have been picked so that the vectors are independent and form a basis for the tangent space at point x. The outer product of the tangent vectors form the pseudoscalar, $I\left(x\right)$, for the tangent space

$$I\left(x\right) \equiv \frac{e_1 \wedge e_2 \wedge \cdots \wedge e_n}{\left|e_1 \wedge e_2 \wedge \cdots \wedge e_n\right|} \tag{346}$$

Thus

$$I^2 = \pm 1 \tag{347}$$

We require that for any point on the manifold the denominator of equation 346 is nonzero. We also assume that for all points on the manifold that $I\left(x\right)$ is continuous, differentiable, singled valued, and has the same grade everywhere.

Projection Operator -

Define the projection operator $\mathsf{P}(A)$ operating on any multivector A in the embedding multivector space as

$$\mathsf{P}(A) \equiv (A \cdot I(x))\, I^{-1}(x) \tag{348}$$

We can show that $\mathsf{P}(A)$ extracts those components of A that lie in the geometric algebra defined by $I(x)$. Since $\mathsf{P}(A)$ is linear in A if we show that if $\mathsf{P}(A_r)$ projects correctly for an r-grade multivector it will do so for a general mulitvector. If n is the dimension of the tangent space and A_r is a pure grade multivector we can write equation 348 as

$$\mathsf{P}(A_r) = \langle A_r I(x) \rangle_{|r-n|}\, I^{-1}(x) \tag{349}$$

Now consider the blades that make up the components of A_r. They will either consist only of blades formed from the tangent vectors e_i or they will contain at least one basis vector that is not a tangent vector. In the first case

$$\left\langle A_r I\left(x\right)\right\rangle_{|r-n|} = A_r I\left(x\right) \tag{350}$$

and

$$\mathsf{P}\left(A_r\right) = A_r \tag{351}$$

In the second case there is no component of $A_r I\left(x\right)$ of grade $|r-n|$ and

$$\mathsf{P}\left(A_r\right) = 0 \tag{352}$$

This is easily seen if one constructs at point x an orthogonal basis $\left(o_i \cdot o_j = \delta_{ij} o_i^2\right)$ for the tangent space $\left\{o_1, \ldots, o_n\right\}$ and an orthogonal basis for the remainder of the embedding space $\left\{o_{n+1}, \ldots, o_m\right\}$. Then any component blade of A_r is of the form

$$A_r^{i_1,i_2,\ldots,i_r} o_{i_1} o_{i_2} \ldots o_{i_r} \tag{353}$$

where $i_1 < i_2 < \cdots < i_r$. If $i_j \le n \,\forall\, 1 \le j \le r$ then

$$A_r^{i_1,i_2,\ldots,i_r} o_{i_1} o_{i_2} \ldots o_{i_r} \cdot I = A_r^{i_1,i_2,\ldots,i_r} o_{i_1} o_{i_2} \ldots o_{i_r} I \tag{354}$$

and

$$\left(A_r^{i_1,i_2,\ldots,i_r} o_{i_1} o_{i_2} \ldots o_{i_r} \cdot I\right) I^{-1} = A_r^{i_1,i_2,\ldots,i_r} o_{i_1} o_{i_2} \ldots o_{i_r} \tag{355}$$

and

$$\mathsf{P}\left(A_r^{i_1,i_2,\ldots,i_r} o_{i_1} o_{i_2} \ldots o_{i_r}\right) = A_r^{i_1,i_2,\ldots,i_r} o_{i_1} o_{i_2} \ldots o_{i_r} \tag{356}$$

If any $i_j > m$ then $A_r^{i_1,i_2,\ldots,i_r} o_{i_1} o_{i_2} \ldots o_{i_r} I$ contains no grade $|r-n|$ and

$$\mathsf{P}\left(A_r^{i_1,i_2,\ldots,i_r} o_{i_1} o_{i_2} \ldots o_{i_r}\right) = 0 \tag{357}$$

Exclusion Operator -

For an arbitrary multivector A the exclusion operator $\mathsf{P}_{\perp}(A)$ is defined by

$$\mathsf{P}_{\perp}(A) \equiv A - \mathsf{P}(A) \tag{358}$$

Intrinsic Derivative -

Given a set of tangent vectors $\{e_i\}$ spanning the tangent space the derivative intrinsic to the manifold is defined everywhere by

$$\partial \equiv e^i e_i \cdot \nabla = \mathsf{P}\left(\nabla\right) \tag{359}$$

Also note that

$$\mathsf{P}\left(\partial\right) = \partial \tag{360}$$

When we write $\mathsf{P}\left(\nabla\right)$ or $\mathsf{P}\left(\partial\right)$ the ∇ or ∂ is not differentiating the $I\left(x\right)$ in the P operator anymore than ∇ is differentiating dX in the fundamental theorem of Geometric Calculus.

We also note that if the vector a is in the tangent space that

$$a \cdot \partial = a \cdot \nabla \tag{361}$$

and that $a \cdot \partial$ is a scalar operator that gives the directional derivative in the a direction. Also since it is scalar it satisfies Leibniz's rules without using the dot notation **(remember the convention that if parenthesis are not present the operator precedence is dot product then wedge product then geometric product)**-

$$a \cdot \partial \left(AB \right) = \left(a \cdot \partial A \right) B + A \left(a \cdot \partial B \right) \tag{362}$$

Covariant Derivative -

The ∂ operator is entirely within the tangent space and if the general multivector function $A\left(x\right)$ is also entirely within the tangent space, it is still possible (even likely) that ∂A is not entirely within the tangent space. We need a covariant derivative D that will result in a multivector entirely within the tangent space. This can be done by defining

$$a \cdot DA \left(x \right) \equiv \mathsf{P} \left(a \cdot \partial A \left(x \right) \right) \tag{363}$$

so that

$$a\cdot\partial A=\mathsf{P}\left(a\cdot\partial A\right)+\mathsf{P}_{\perp}\left(a\cdot\partial A\right)=a\cdot DA+\mathsf{P}_{\perp}\left(a\cdot\partial A\right) \tag{364}$$

Again since $a\cdot D$ is a scalar operator we have

$$a\cdot D\left(AB\right)=\mathsf{P}\left(a\cdot\partial\left(AB\right)\right)=\left(a\cdot DA\right)B+A\left(a\cdot DB\right) \tag{365}$$

A component expansion of D is given in the usual way by (do not forget the summation convention)

$$D=e^{i}e_{i}\cdot D \tag{366}$$

and

$$DA_{r}=e^{i}\left(e_{i}\cdot DA_{r}\right)=\mathsf{P}\left(\partial A_{r}\right) \tag{367}$$

and

$$D\cdot A_{r}\equiv\left\langle DA_{r}\right\rangle_{r-1} \tag{368}$$

$$D \wedge A_r \equiv \left\langle DA_r \right\rangle_{r+1} \tag{369}$$

if $\alpha(x)$ is a scalar function on the manifold then

$$\partial\alpha(x) = D\alpha(x) \tag{370}$$

because in equation 370 no basis vectors are differectiated. To relate ∂ and D if the function operated on is not a scalar first construct a normalized basis $\{e_i\}$ for the tangent space at point x. Then

$$I = e_1 \wedge e_2 \wedge \ldots \wedge e_n \text{ and } I^2 = \pm 1 \tag{371}$$

and (since $a \cdot \partial$ and $a \cdot D$ are scalar operators we can move them across the wedge products without any problems)

$$\left(a\cdot\partial I\right)I^{-1}=\left(\sum_{i=1}^{n}e_1\wedge\ldots\wedge\left(a\cdot De_i+\mathsf{P}_\perp\left(a\cdot\partial e_i\right)\right)\wedge\ldots\wedge e_n\right)I^{-1} \tag{372}$$

$$=\left(a\cdot DI\right)I^{-1}+\sum_{i=1}^{n}\left(-1\right)^{i-1}\mathsf{P}_\perp\left(a\cdot\partial e_i\right)\wedge e_1\wedge\ldots\wedge\breve{e}_i\wedge\ldots\wedge e_nI^{-1} \tag{373}$$

$$=\left(a\cdot DI\right)I^{-1}+\mathsf{P}_\perp\left(a\cdot\partial e_i\right)\wedge e^i \tag{374}$$

We go from equation 373 to equation 374 by using equation 61 on page 33 in the section on reciprocal frames.

Since $\left(a\cdot D\right)I$ is a grade n multivector in the tangent space it must be proportional to I and thus commute with I so that $\left(\left(a\cdot\partial\right)I\right)I=$

$I\left(\left(a\cdot\partial\right)I\right)$. Also $I^{-1}=\pm I$ so that we have

$$\begin{aligned}\left(a\cdot DI\right)I^{-1} &= \pm\left(a\cdot DI\right)I \\ &= \pm\frac{1}{2}\left(\left(a\cdot DI\right)I+I\left(a\cdot DI\right)\right) \\ &= \pm\frac{1}{2}\left(a\cdot D\left(I^2\right)\right) \\ &= 0 \end{aligned} \tag{375}$$

Thus

$$\left(a\cdot\partial I\right)=\mathsf{P}_{\perp}\left(a\cdot\partial e_i\right)\wedge e^iI\equiv -S\left(a\right)I \tag{376}$$

Where $S\left(a\right)$ is the shape tensor associated with the manifold. Since $S\left(a\right)$ is a bivector we can write $\left(A\times B=\left(AB-BA\right)/2\right)$

$$a\cdot\partial I=I\times S\left(a\right) \tag{377}$$

since

$$S(a)\cdot I = S(a)\wedge I = 0 \tag{378}$$

and by equation 131 page 58. Where $a(x)$ and $b(x)$ are vector fields on the manifold (both are in the tangent space at point x) form (remember that for any three vector u, v, and w we have $u\cdot(v\wedge w) = (u\cdot v)\,w - (u\cdot w)\,v$)

$$\begin{aligned} b\cdot S(a) &= b\cdot\left(e^i\wedge \mathsf{P}_\perp(a\cdot\partial e_i)\right) \\ &= \left(b\cdot e^i\right)\mathsf{P}_\perp(a\cdot\partial e_i) - (b\cdot \mathsf{P}_\perp(a\cdot\partial e_i))\,e^i \\ &= \left(\left(b^j e_j\right)\cdot e^i\right)\mathsf{P}_\perp(a\cdot\partial e_i) \\ &= b^j\delta^i_j\mathsf{P}_\perp(a\cdot\partial e_i) \\ &= \mathsf{P}_\perp\left(a\cdot\dot{\partial} b^i\dot{e}_i\right) \end{aligned} \tag{379}$$

but

$$\mathsf{P}_\perp(a\cdot\partial b) = \mathsf{P}_\perp\left(a\cdot\partial\left(b^i e_i\right)\right)$$

$$
\begin{aligned}
&= \mathsf{P}_\perp\left(a\cdot\dot{\partial}\dot{b}^i e_i + a\cdot\dot{\partial}b^i\dot{e}_i\right)\\
&= \mathsf{P}_\perp\left(a\cdot\dot{\partial}b^i\dot{e}_i\right) \qquad (380)
\end{aligned}
$$

and

$$\boxed{b\cdot S(a) = \mathsf{P}_\perp(a\cdot\partial b)} \qquad (381)$$

Thus

$$a\cdot\partial b = \mathsf{P}(a\cdot\partial b) + \mathsf{P}_\perp(a\cdot\partial b) = a\cdot Db + b\cdot S(a) \qquad (382)$$

and (using the fact that the dot product of a vector and bivector are antisymmetric)

$$a\cdot Db = a\cdot\partial b + S(a)\cdot b \qquad (383)$$

Now consider the expression

$$\begin{aligned}
a \cdot D\left(b_1 \ldots b_r\right) &= \sum_{i=1}^{r} b_1 \ldots \left(a \cdot \partial b_i + S\left(a\right) \cdot b_i\right) \ldots b_r \\
&= \sum_{i=1}^{r} b_1 \ldots \left(a \cdot \partial b_i\right) \ldots b_r + \\
&\qquad \sum_{i=1}^{r} b_1 \ldots \left(S\left(a\right) \cdot b_i\right) \ldots b_r \\
&= a \cdot \partial\left(b_1 \ldots b_r\right) + \\
&\qquad \frac{1}{2} \sum_{i=1}^{r} b_1 \ldots \left(S\left(a\right) b_i - b_i S\left(a\right)\right) \ldots b_r \\
&= a \cdot \partial\left(b_1 \ldots b_r\right) + \frac{1}{2}\left(S\left(a\right)\left(b_1 \ldots b_r\right) - \left(b_1 \ldots b_r\right) S\left(a\right)\right) +
\end{aligned}$$

$$
\begin{aligned}
&\frac{1}{2}\left(\sum_{i=2}^{r-1} b_1 \ldots S\left(a\right) b_i \ldots b_r - \sum_{i=2}^{r-1} b_1 \ldots b_i S\left(a\right) \ldots b_r\right) + \\
&\frac{1}{2}\left(b_1 \ldots b_{r-1} S\left(a\right) b_r - b_1 S\left(a\right) b_2 \ldots b_r\right) \\
= a \cdot \partial\left(b_1 \ldots b_r\right) + &\frac{1}{2}\left(S\left(a\right)\left(b_1 \ldots b_r\right) - \left(b_1 \ldots b_r\right) S\left(a\right)\right) + \\
&\frac{1}{2}\left(\sum_{i=3}^{r-1} b_1 \ldots S\left(a\right) b_i \ldots b_r - \sum_{i=2}^{r-2} b_1 \ldots b_i S\left(a\right) \ldots b_r\right) + \qquad (384) \\
= a \cdot \partial\left(b_1 \ldots b_r\right) + &S\left(a\right) \times \left(b_1 \ldots b_r\right) \qquad (385)
\end{aligned}
$$

To get from equation 384 to equation 385 note that in the sums in parenthesis in equation 384 the i^{th} term in the first sum cancels the $i^{th}+1$ term in the second sum.

Since any multivector is a linear superposition of terms containing $b_1 \ldots b_r$ with $1 \leq r \leq n$ and a scalar we have

$$\boxed{a \cdot DA = a \cdot \partial A + S\left(a\right) \times A} \tag{386}$$

Where $a\left(x\right)$ and $b\left(x\right)$ are vector fields on the manifold write

$$a \cdot \partial b = a \cdot \partial \mathsf{P}\left(b\right) = a \cdot \dot{\partial} \dot{\mathsf{P}}\left(b\right) + \mathsf{P}\left(a \cdot \partial b\right) = a \cdot \dot{\partial} \dot{\mathsf{P}}\left(b\right) + a \cdot Db \tag{387}$$

Now substitute equation 386 into equation 387 to get

$$a \cdot \dot{\partial} \dot{\mathsf{P}}\left(b\right) = b \cdot S\left(a\right) \tag{388}$$

Case Study of a Manifold for a Model Universe

We wish to construct a spatially curved closed isotropic Minkowski space with 1, 2, and 3 spatial dimensions.

To do this consider a Minkowski space with one time dimension $\left(e_0^2 = 1\right)$ and 2, 3, or 4 spatial dimensions (unit vector squares to -1) as indicated below

$$e_0^2 = -e_1^2 = -e_2^2 = -e_3^2 = -e_4^2 = 1 \tag{389}$$

The vector manifolds will designated by

Vector Function	Spatial Dimensions	Coordinates	Components
$X^{(1)}$	1	τ, ρ	e_0, e_1, e_2
$X^{(2)}$	2	τ, ρ, θ	e_0, e_1, e_2, e_3
$X^{(3)}$	3	τ, ρ, θ, ϕ	e_0, e_1, e_2, e_3, e_4

The condition that τ does parametrize the time coordinate of the manifold is that

$$e_\tau \cdot e_\tau = \frac{\partial X^{(i)}}{\partial \tau} \cdot \frac{\partial X^{(i)}}{\partial \tau} = 1 \tag{390}$$

since then the time coordinate is given by (all spatial coordinates are fixed)

$$\int \sqrt{e_\tau \cdot e_\tau}\, d\tau = \tau \tag{391}$$

Then the manifolds in 1, 2, and 3 spatial dimensions are defined by

$$X^{(1)} = t\left(\tau\right) e_0 + r\left(\tau\right) \left(\cos\left(\frac{\rho}{r}\right) e_1 + \sin\left(\frac{\rho}{r}\right) e_2\right) \tag{392}$$

$$X^{(2)} = t\left(\tau\right) e_0 + r\left(\tau\right) \left(\cos\left(\frac{\rho}{r}\right) e_1 + \sin\left(\frac{\rho}{r}\right) \left(\cos\theta e_2 + \sin\theta e_3\right)\right) \tag{393}$$

$$X^{(3)} = t\left(\tau\right) e_0 +$$

$$r\left(\tau\right)\left(\cos\left(\frac{\rho}{r}\right)e_1+\sin\left(\frac{\rho}{r}\right)\left(\cos\theta e_2+\sin\theta\left(\cos\phi e_3+\sin\phi e_4\right)\right)\right) \tag{394}$$

Where $r\left(\tau\right)$ is the spatial radius of the manifolds. In order for equation 390 to be satisfied for all the three manifolds we must have

$$e_\tau \cdot e_\tau = \left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dr}{d\tau}\right)^2 = 1 \tag{395}$$

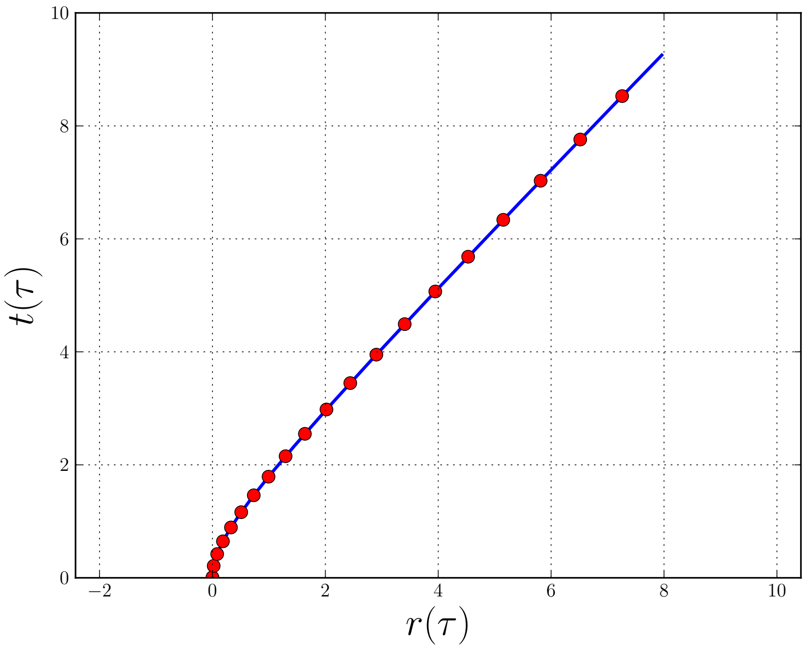
For the specific case of $r\left(\tau\right) = \frac{1}{2}\tau^2$ the derivatives and $t\left(\tau\right)$ are given by

$$\frac{dr}{d\tau} = \tau$$

$$\frac{dt}{d\tau} = \sqrt{1+\tau^2}$$

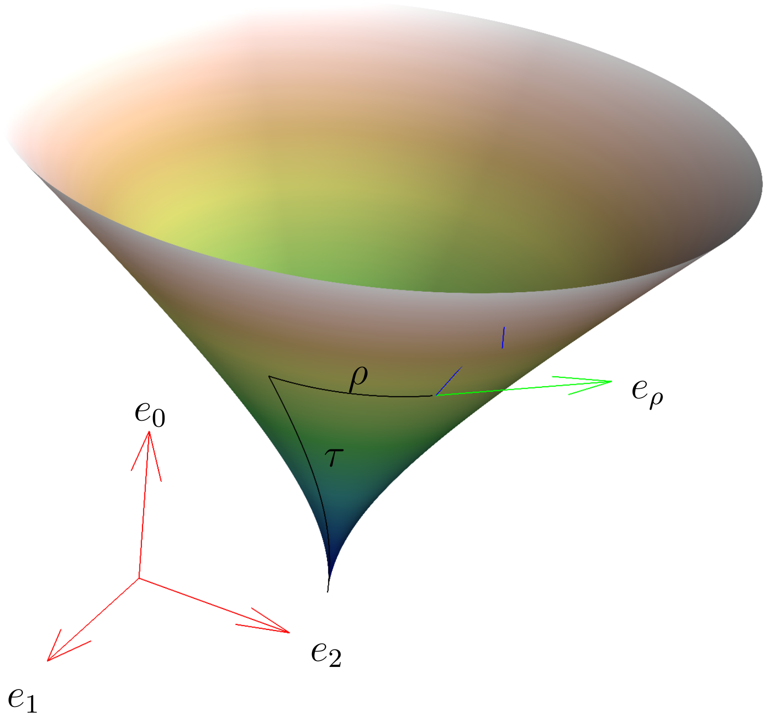
$$t\left(\tau\right) = \frac{1}{2}\left(\tau\sqrt{1+\tau^2} + \sinh^{-1}\tau\right)$$

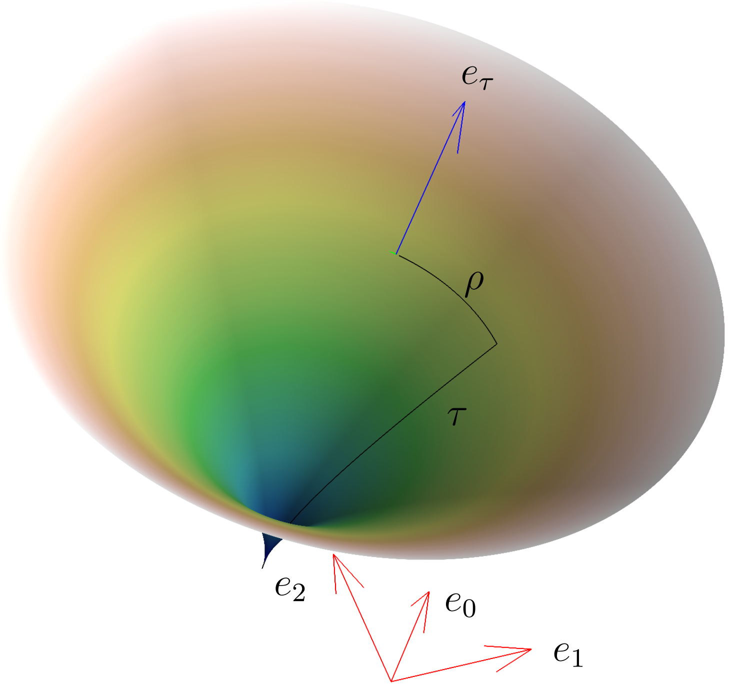
$\Delta\tau = 0.2$ for curve markers



Spatial coordinates for the model universes are given by ρ a spatial radius and the angular coordinates θ and ϕ if required. A visualization of the 1-D spatial dimensional manifold is

1D Universe





Where e_τ and e_ρ are

$$e_\tau = \frac{\partial X^{(1)}}{\partial \tau} \quad \text{and} \quad e_\rho = \frac{\partial X^{(1)}}{\partial \rho} \tag{396}$$

The metric tensors for the three cases are

$$g^{(i)}_{\mu\nu} = \frac{\partial X^{(i)}}{\partial \mu} \cdot \frac{\partial X^{(i)}}{\partial \nu} \tag{397}$$

where $\mu, \nu = \{\tau, \rho, \theta, \phi\}$. If we define $h\left(\tau, \rho\right)$ as

$$h\left(\tau, \rho\right) = \frac{dr}{d\tau}\frac{\rho}{r\left(\tau\right)} \tag{398}$$

Then the metric tensors for the three cases are (using sympy to do the algebra)

$$g^{(1)}_{\mu\nu} = \left(\begin{array}{cc} 1 - h^2 & h \\ h & -1 \end{array}\right) \tag{399}$$

$$g^{(2)}_{\mu\nu} = \begin{pmatrix} 1-h^2 & h & 0 \\ h & -1 & 0 \\ 0 & 0 & -\left(r\sin\left(\frac{\rho}{r}\right)\right)^2 \end{pmatrix} \tag{400}$$

$$g^{(3)}_{\mu\nu} = \begin{pmatrix} 1-h^2 & h & 0 & 0 \\ h & -1 & 0 & 0 \\ 0 & 0 & -\left(r\sin\left(\frac{\rho}{r}\right)\right)^2 & 0 \\ 0 & 0 & 0 & -\left(r\sin\left(\frac{\rho}{r}\right)\sin\theta\right)^2 \end{pmatrix}. \tag{401}$$

If we renormalize e_θ and e_ϕ to be unit vectors

$$e'_\theta = \frac{e_\theta}{\left|r\sin\left(\frac{\rho}{r}\right)\right|} \tag{402}$$

$$e'_\phi = \frac{e_\phi}{\left| r \sin\left(\frac{\rho}{r}\right) \sin\theta \right|} \tag{403}$$

The metric tensors $g_{\mu\nu}^{(2)}$ and $g_{\mu\nu}^{(3)}$ become

$$g'^{(2)}_{\mu\nu} = \begin{pmatrix} 1-h^2 & h & 0 \\ h & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{404}$$

$$g'^{(3)}_{\mu\nu} = \begin{pmatrix} 1-h^2 & h & 0 & 0 \\ h & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{405}$$

and

$$\det\left(g^{(1)}_{\mu\nu}\right) = \det\left(g'^{(2)}_{\mu\nu}\right) = \det\left(g'^{(3)}_{\mu\nu}\right) = -1 = I^2 \tag{406}$$

For the 1-Dimensional space the differential arc length is

$$(ds)^2 = \left(1 - h^2\right)(d\tau)^2 + 2h d\tau d\rho + (d\rho)^2 \tag{407}$$

so that for the light cone, $ds = 0$, we have the differential equation

$$1 - h^2 + 2h\frac{d\rho}{d\tau} + \left(\frac{d\rho}{d\tau}\right)^2 = 0 \tag{408}$$

Note that this equation also applies to the 2 and 3 dimensional case if we set $\frac{d\theta}{d\tau} = \frac{d\phi}{d\tau} = 0$. Solving for $\frac{d\rho}{d\tau}$ gives

$$\frac{d\rho}{d\tau} = h \pm 1 = \frac{1}{r}\frac{dr}{d\tau}\rho \pm 1 = \left(\frac{d}{d\tau}\ln(r)\right)\rho \pm 1 \tag{409}$$

Using the integration factor for linear first order differential equations the solution to equation 409 is $(\rho(\tau_0) = 0)$

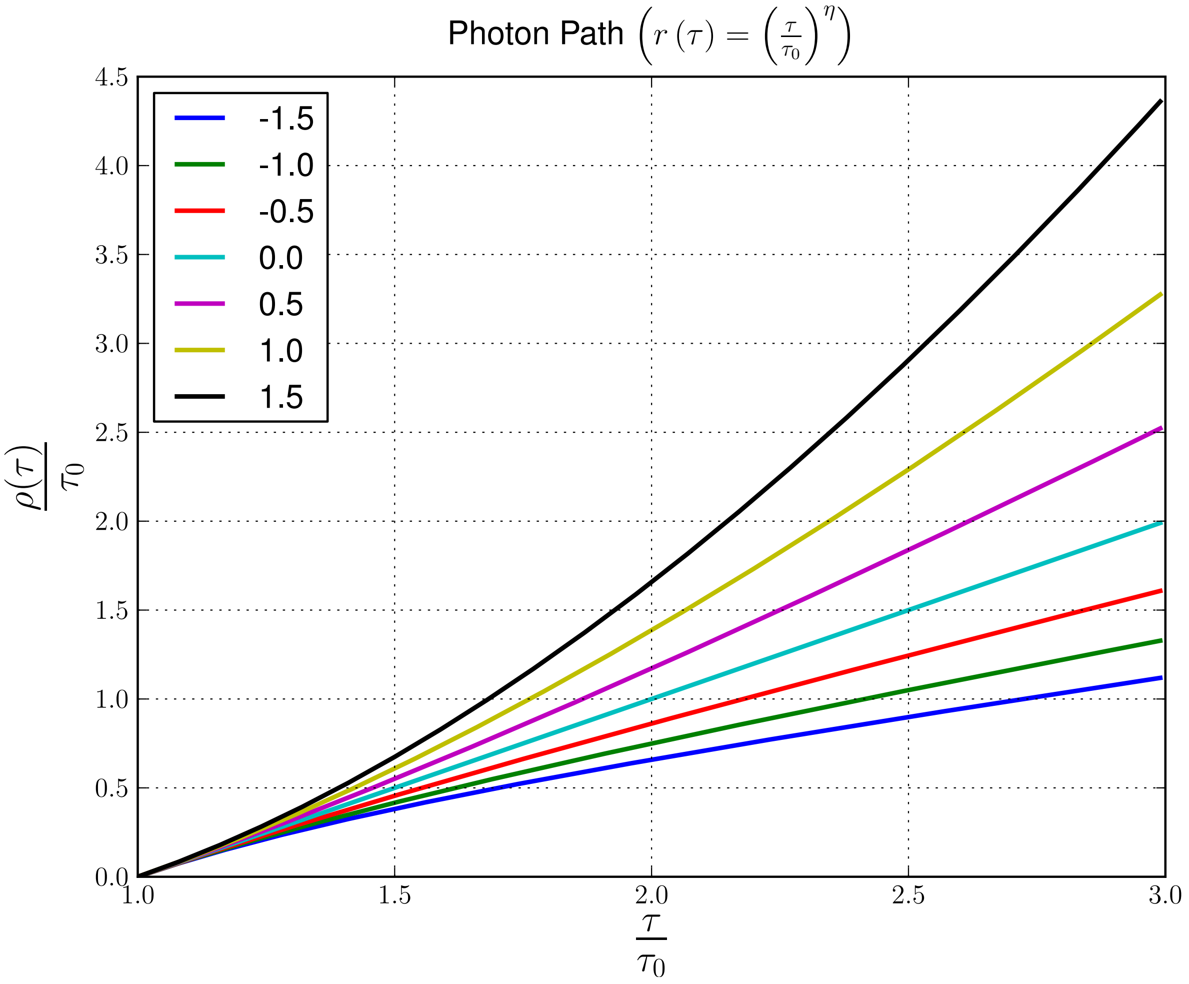
$$\rho(\tau) = \pm r(\tau) \int_{\tau_0}^{\tau} \frac{d\tau'}{r(\tau')} \tag{410}$$

Note that $r(\tau)$ and $\alpha r(\tau)$ have the same solution $\rho(\tau)$. Now consider the case that $r(\tau) = \tau^{\eta}$, then

$$\rho(\tau) = \pm \tau^{\eta} \int_{\tau_0}^{\tau} (\tau')^{-\eta} d\tau' = \left\{ \begin{array}{ll} \eta = 1, & \pm\tau \ln\left(\frac{\tau}{\tau_0}\right) \\ \eta \neq 1, & \frac{\pm 1}{1-\eta}\left(\tau - \tau_0\left(\frac{\tau}{\tau_0}\right)^{\eta}\right) \end{array} \right\} \tag{411}$$

Typical $\rho(\tau)$'s for various $-1.5 \leq \eta \leq 1.5$ are shown in the light cones plot. If $\eta > 0$ the speed of light is greater than c in flat space. If $\eta < 0$ the speed of light is less than c in flat space. Note that if the universe is curved, but not expanding or contracting the speed of light is the same as c in flat space.

Light Cones

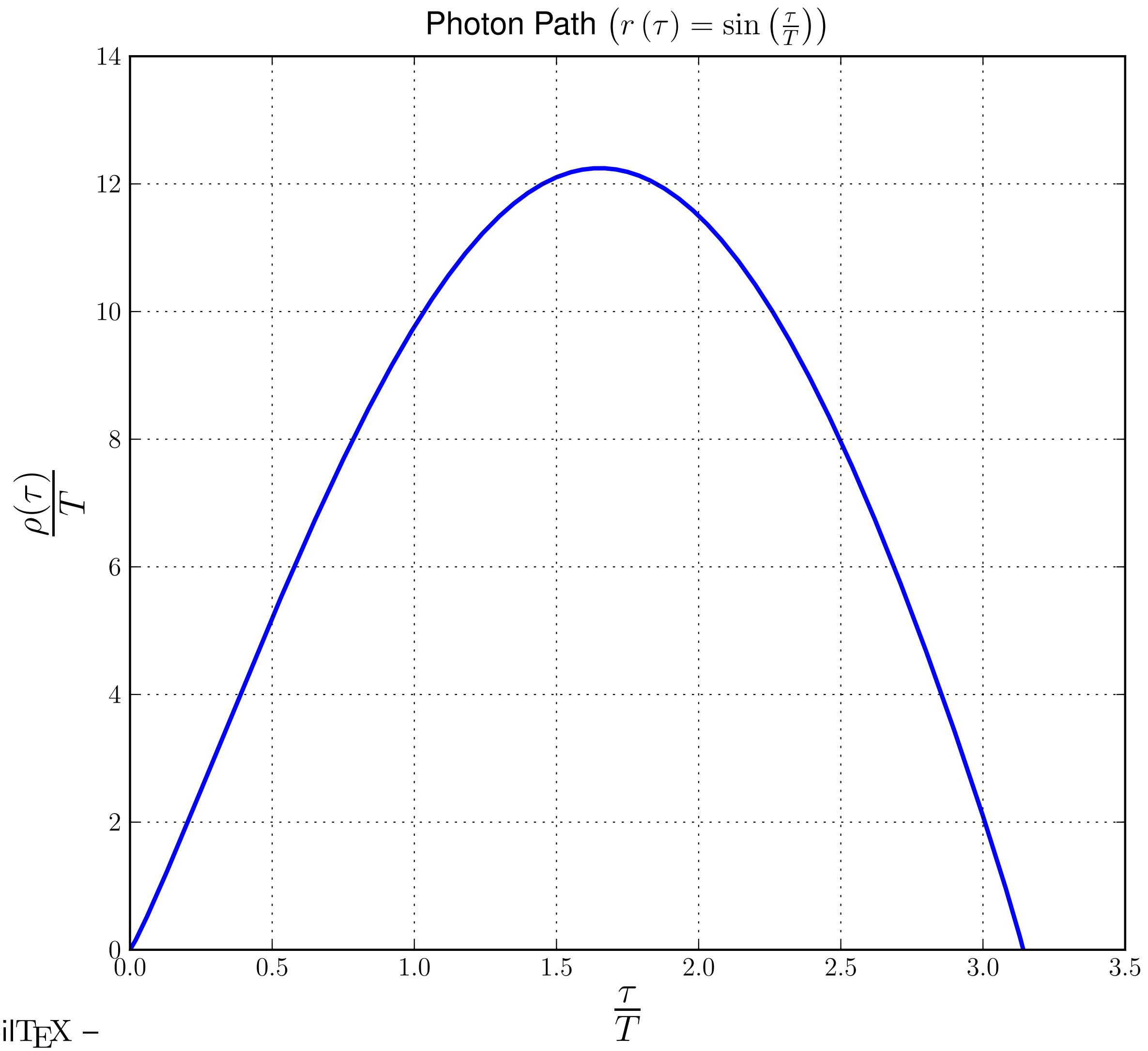

Photon Path $\left(r\left(\tau\right)=\left(\frac{\tau}{\tau_0}\right)^{\eta}\right)$
$\frac{\rho(\tau)}{\tau_0}$
$\frac{\tau}{\tau_0}$
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
1.0
1.5
2.0
2.5
3.0
-1.5
-1.0
-0.5
0.0
0.5
1.0
1.5

For a time periodic universe $r\left(\tau\right)=\sin\left(\frac{\tau}{T}\right)$ where T is twice the period of the universe and $\rho\left(\tau_0\right)=0$, then

$$\rho\left(\tau\right)=T\sin\left(\frac{\tau}{T}\right)\ln\left|\frac{\csc\left(\frac{\tau_0}{T}\right)+\cot\left(\frac{\tau_0}{T}\right)}{\csc\left(\frac{\tau}{T}\right)+\cot\left(\frac{\tau}{T}\right)}\right| \qquad (412)$$

The plot of equation 412 is shown below

Periodic Light Cone



The Edge of Known Space

Another kinematic question to answer is under what conditions light cannot access parts of the universe. The critical quantity is

$$\lambda\left(\tau\right)=\frac{\rho\left(\tau\right)}{\vartheta r\left(\tau\right)} \tag{413}$$

where ϑ is the measure of how far from the observer you are. Since the universe is spatially periodic the maximum value of ϑ is π. If $\lambda\left(\tau\right)\geq 1$ for some finite τ you can access the distance defined by ϑ. Substituting equation 410 into equation 413 gives

$$\lambda\left(\tau\right)=\frac{1}{\vartheta}\int_{\tau_0}^{\tau}\frac{d\tau'}{r\left(\tau'\right)} \tag{414}$$

so that the connection condition is

$$\int_{\tau_0}^{\tau}\frac{d\tau'}{r\left(\tau'\right)}\geq\vartheta. \tag{415}$$

First consider a linear expansion model of the form

$$r\left(\tau\right)=r_0\left(1+\alpha\left(\frac{\tau}{\tau_0}-1\right)\right) \tag{416}$$

where $r\left(\tau_0\right) = r_0$. Then

$$\frac{\tau}{\tau_0} \geq \frac{1}{\alpha} e^{\alpha\vartheta\left(\frac{r_0}{\tau_0}\right)} - 1 \tag{417}$$

In a linearly expanding universe the photon time of flight increases exponentially with distance. Now consider super-linear expansion of the form $(\eta > 1)$

$$r\left(\tau\right) = r_0 \left(1 + \alpha \left(\frac{\tau}{\tau_0} - 1\right)^{\eta}\right) \tag{418}$$

Then

$$\frac{\tau_0}{r_0}\alpha^{-\frac{1}{\eta}}\int_0^{\alpha^{\frac{1}{\eta}}\left(\frac{\tau}{\tau_0}-1\right)}\frac{d\mu}{1+\mu^\eta}\geq\vartheta, \tag{419}$$

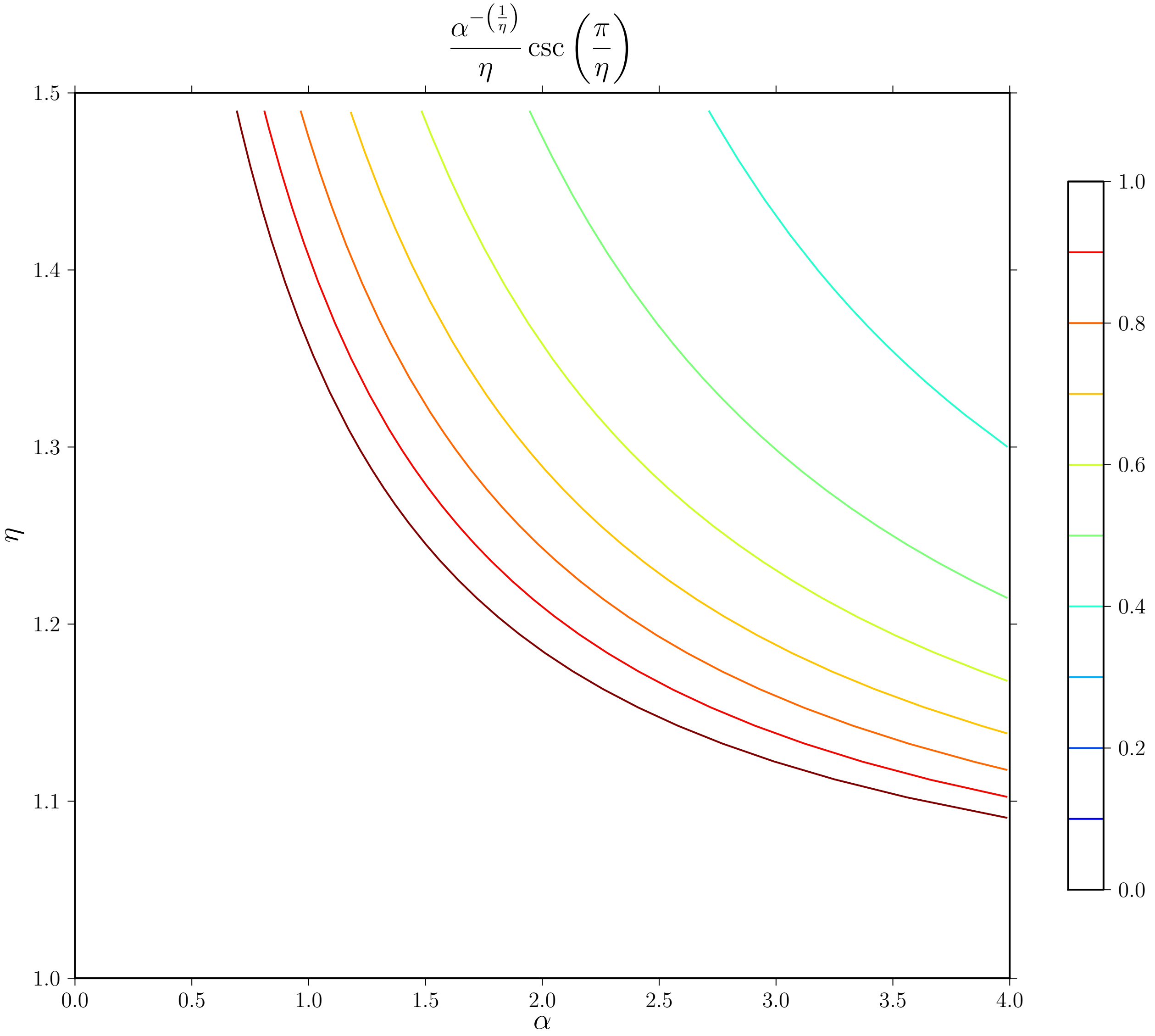
but the intergral in equation 419 does not have a closed form solution unless we let $\tau\rightarrow\infty$. In that case[7] we can write

$$\frac{1}{\eta}\alpha^{-\frac{1}{\eta}}\csc\left(\frac{\pi}{\eta}\right)\geq\left(\frac{\vartheta}{\pi}\right)\left(\frac{r_0}{\tau_0}\right) \tag{420}$$

A contour plot of the left side of equation 420 is shown below

---

[7] $\int_0^\infty \frac{dx}{1+x^\eta}$ is 3.241-2 in "Gradshteyn and Ryzhik"


$\frac{\alpha^{-\left(\frac{1}{\eta}\right)}}{\eta}\csc\left(\frac{\pi}{\eta}\right)$
1.5
1.4
1.3
1.2
1.1
1.0
η
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
α
1.0
0.8
0.6
0.4
0.2
0.0

The right side of equation 420, $\left(\frac{\vartheta}{\pi}\right)\left(\frac{r_0}{\tau_0}\right)$, is interpeted as follows -

1. $\frac{\vartheta}{\pi}$ is the fractional distance around the closed spatially periodic universe. $\frac{\vartheta}{\pi} = 1$ is as far as one can go before the distance from the observer starts to decrease.
2. $\frac{r_0}{\tau_0}$ is a measure of inflation. Immediately after an inflationary epoch $\frac{r_0}{\tau_0} >> 1$.

Thus equation 420 determined the maximum distance $\frac{\vartheta}{\pi}$ that a photon can propagate in a finite ammout of time.

Another question to consider is under what conditions $\frac{r_0}{\tau_0}$ will increase as τ_0 increases or when will the following be true

$$\frac{r(\tau)}{\tau} \geq \frac{r(\tau_0)}{\tau_0}. \tag{421}$$

Equation 421 is equivalent to

$$\alpha \left(\frac{\tau}{\tau_0} - 1 \right)^{\eta - 1} \geq 1 \tag{422}$$

so that if $\eta > 1$ then $\frac{r(\tau_0)}{\tau_0}$ will eventually grow as τ_0 increases.

Coordinates and Derivatives

In a region of the manifold we introduce local coordinates x^i and define the frame vectors as

$$e_i = \frac{\partial x}{\partial x^i} \tag{423}$$

From the definition of ∂ is follows that $e^i = \partial x^i$. The $\{e_i\}$ are referred to as tangent vectors and the reciprocal frame $\left\{e^i\right\}$ as the cotangent vector (or 1-forms). The covarient derivative along a coordinate vector, $e_i \cdot D$, satisfies

$$e_i \cdot DA = D_i A = e_i \partial A + S\left(e_i\right) \times A \equiv \partial_i A + S_i \times A \tag{424}$$

This defines both D_i and S_i.

The tangent frame vectors satisfy

$$\partial_i e_j - \partial_j e_i = \left(\partial_i \partial_j - \partial_j \partial_i\right) x = 0 \tag{425}$$

Using the P operator on equation 425 gives

$$D_i e_j - D_j e_i = 0 \tag{426}$$

while using $\mathsf{P}_\perp$ gives

$$e_i \cdot S_j = e_j \cdot S_i \tag{427}$$

For arbitrary vectors a and b in the tangent space equation 427 becomes

$$a \cdot S(b) = b \cdot S(a) \tag{428}$$

In terms of the coordinate vectors the shape tensor becomes

$$S(a) = e^k \wedge \mathsf{P}_\perp (a \cdot \partial e_k) \tag{429}$$

and

$$S_i = e^k \wedge \mathsf{P}_\perp (e_i \cdot \partial e_k) = e^k \wedge \mathsf{P}_\perp (e_k \cdot \partial e_i) \tag{430}$$

Then

$$\partial \wedge e_i = e^k \wedge \partial_k e_i = e^k \wedge (\mathsf{P}(\partial_k e_i) + \mathsf{P}_\perp (\partial_k e_i)) = D \wedge e_i + S_i \tag{431}$$

Letting $a = a^i e_i$ be a constant vector in the tangent space gives the general result

$$\partial \wedge a = D \wedge a + S(a) \tag{432}$$

Additionally

$$\begin{aligned} \partial \wedge a &= \partial \wedge (\mathsf{P}(a)) \\ &= \dot{\partial} \wedge \dot{\mathsf{P}}(a) + \mathsf{P}(\partial \wedge a) \\ &= D \wedge a + \dot{\partial} \wedge \dot{\mathsf{P}}(a) \end{aligned} \tag{433}$$

Thus

$$\dot{\partial} \wedge \dot{\mathsf{P}}(a) = S(a) \tag{434}$$

Note that if a and b are any two vectors in the embedding space then $\mathsf{P}(a \wedge b) = \mathsf{P}(a) \wedge \mathsf{P}(b)$ and if $\phi(x)$ is a scalar function on the manifold we have

$$\begin{aligned}
\partial \wedge \partial\phi &= \partial \wedge \mathsf{P}(\nabla\phi) \\
&= \dot{\partial} \wedge \mathsf{P}\left(\dot{\nabla}\phi\right) + \dot{\partial} \wedge \dot{\mathsf{P}}(\nabla\phi) \\
&= \mathsf{P}\left(\dot{\nabla}\right) \wedge \mathsf{P}\left(\dot{\nabla}\phi\right) + \dot{\partial} \wedge \dot{\mathsf{P}}(\nabla\phi) \\
&= \mathsf{P}(\nabla \wedge \nabla\phi) + \dot{\partial} \wedge \dot{\mathsf{P}}(\nabla\phi) \qquad (435)
\end{aligned}$$

but $\nabla \wedge \nabla = 0$ so

$$\partial \wedge \partial\phi = S(\nabla\phi) \qquad (436)$$

Since $S(a)$ for any vector a lies outside the manifold we have

$$D \wedge (D\phi) = 0 \tag{437}$$

Letting $\phi(x) = x^i(x)$, then

$$D \wedge \left(Dx^i\right) = D \wedge e^i = 0 \tag{438}$$

so that for a general vector $a = a_i(x)\, e^i$ we have

$$D \wedge a = D \wedge \left(a_j e^j\right) = e^i \wedge e^j \left(\partial_i a_j\right) = \frac{1}{2} e^i \wedge e^j \left(\partial_i a_j - \partial_j a_i\right) \tag{439}$$

Equation 439 is isomophic to the definition of the *exterior derivative* of differential geometry.

Riemannian Geometry

We shall now relate the shape tensor to the metric tensor and Christoffle connection. The metric tensor is defined by

$$g_{ij} \equiv e_i \cdot e_j \tag{440}$$

and the Christoffle connection by

$$\Gamma^i_{jk} \equiv (D_j e_k) \cdot e^i \tag{441}$$

so that the components of the covariant derivative are given by

$$(a \cdot Db) \cdot e^i = a^j \left(D_j \left(b^k e_k\right)\right) \cdot e^i$$

$$= a^j \left(\partial_j b^i + \Gamma^i_{jk} b^k \right) \tag{442}$$

The Γ^i_{jk} can be expressed in terms of the g_{ij} by considering the following relations. First, the Γ^i_{jk} are symmetric in the j and k indices.

$$\Gamma^i_{jk} - \Gamma^i_{kj} = (D_j e_k - D_k e_j) \cdot e^i \tag{443}$$

Second, the curl of the basis vectors is given by

$$D \wedge e_i = D \wedge \left(g_{ij} e^j \right) = (D g_{ij}) \wedge e^j \tag{444}$$

By equation 443 we can write

$$\Gamma^i_{jk} = \frac{1}{2} e^i \cdot (D_j e_k + D_k e_j)$$

$$= \frac{1}{2} e^{i} \cdot \left(\left(e_j \cdot D \right) e_k + \left(e_k \cdot D \right) e_j \right) \tag{445}$$

Now apply equation 473 (Appendix A) to each term in equation 445 to get

$$\begin{aligned} \left(e_j \cdot D \right) e_k + \left(e_k \cdot D \right) e_j &= e_j \cdot \left(D \wedge e_k \right) + e_k \cdot \left(D \wedge e_j \right) + \\ &\quad \left(e_j \cdot \dot{e}_k \right) \dot{D} + \left(e_k \cdot \dot{e}_j \right) \dot{D} \\ &= e_j \cdot \left(D \wedge e_k \right) + e_k \cdot \left(D \wedge e_j \right) + D \left(g_{jk} \right) \\ &= e_j \cdot \left(D g_{kl} \wedge e^{l} \right) + e_k \cdot \left(D g_{jl} \wedge e^{j} \right) + \\ &\quad D \left(g_{jk} \right) \end{aligned} \tag{446}$$

Now apply equatioin 470 (Appendix A) to $e_j \cdot \left(D g_{kl} \wedge e^{l} \right)$ and

$e_k \cdot \left(Dg_{jl} \wedge e^j\right)$ giving in the first case

$$\begin{aligned} e_j \cdot \left(Dg_{kl} \wedge e^l\right) &= \left(e_j \cdot \left(Dg_{kl}\right)\right) e^l - \left(e_j \cdot e^l\right) Dg_{kl} \\ &= \left(\left(e_j \cdot D\right) g_{kl}\right) e^l - \delta^l_j Dg_{kl} \\ &= \left(D_j g_{kl}\right) e^l - Dg_{kj} \\ &= \left(\partial_j g_{kl}\right) e^l - \partial g_{kj} \end{aligned} \tag{447}$$

so that equation 445 becomes

$$\begin{aligned} \Gamma^i_{jk} &= \frac{1}{2} e^i \cdot \left(\left(\partial_j g_{kl}\right) e^l + \left(\partial_k g_{jl}\right) e^l - \partial g_{kj}\right) \\ &= \frac{1}{2} e^i \cdot \left(\left(\partial_j g_{kl}\right) e^l + \left(\partial_k g_{jl}\right) e^l - e^l \partial_l g_{kj}\right) \end{aligned}$$

$$= \frac{1}{2} g^{il} \left(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{kj} \right) \tag{448}$$

which is the standard formula for the $\Gamma^{i}_{\;jk}$.

Now define the commutator bracket $[A, B]$ of the multivectors A and B by (note there is no $\frac{1}{2}$ factor)

$$[A, B] \equiv AB - BA \tag{449}$$

Now form equation 470 and use the Jacobi identity to reduce the double commutator products on the r.h.s. of the equation

$$[D_i, D_j]\, A = \partial_i \left(\partial_j A + S_j \times A \right) + S_i \times \left(\partial_j A + S_j \times A \right)$$

$$
\begin{aligned}
&-\partial_j\left(\partial_i A+S_i\times A\right)-S_j\times\left(\partial_i A+S_i\times A\right)\\
&=\left(\partial_i S_j-\partial_j S_i\right)\times A+\left(S_i\times S_j\right)\times A
\end{aligned}
\tag{450}
$$

However

$$
\begin{aligned}
\left(\partial_i S_j-\partial_j S_i\right)&=-\partial_i\left(\left(\partial_j I\right)I^{-1}\right)+\partial_j\left(\left(\partial_i I\right)I^{-1}\right)\\
&=-S_j I S_i I^{-1}+S_i I S_j I^{-1}\\
&=-2S_i\times S_j
\end{aligned}
\tag{451}
$$

where we have used that $S\left(a\right)I=S\left(a\right)\times I$ so that $S\left(a\right)$ and I anticommute to reduce the second line of equation 451

$$
\left[D_i,D_j\right]A=-\left(S_i\times S_j\right)\times A \tag{452}
$$

The commutator of the covariant derivatives defines the Riemann tensor

$$\mathsf{R}\left(a,b\right)\equiv\mathsf{P}\left(S\left(b\right)\times S\left(b\right)\right)\tag{453}$$

Since $\mathsf{R}\left(a,b\right)$ is a bilinear antisymmetric fuction of a and b we may write

$$\mathsf{R}\left(a\wedge b\right)=\mathsf{P}\left(S\left(b\right)\times S\left(a\right)\right)\tag{454}$$

or

$$\mathsf{R}\left(e_i\wedge e_j\right)=\mathsf{P}\left(S\left(e_j\right)\times S\left(e_i\right)\right)\tag{455}$$

Since both $S\left(a\right)$ and $S\left(b\right)$ are bivectors we can use equation 478 (Appendix A) to reduce $S(b) \times S(a)$

$$\begin{aligned}
S(b) \times S(a) &= \left(e^k \wedge \mathsf{P}_\perp(b \cdot \partial e_k)\right) \times \left(e^l \wedge \mathsf{P}_\perp(a \cdot \partial e_l)\right) \\
&= \left(e^k \cdot \mathsf{P}_\perp(a \cdot \partial e_l)\right) \mathsf{P}_\perp(b \cdot \partial e_k) \wedge e^l \\
&- \left(e^k \cdot e^l\right) \mathsf{P}_\perp(b \cdot \partial e_k) \wedge \mathsf{P}_\perp(a \cdot \partial e_l) \\
&+ \left(\mathsf{P}_\perp(b \cdot \partial e_k) \cdot e^l\right) e^k \wedge \mathsf{P}_\perp(a \cdot \partial e_l) \\
&- \left(\mathsf{P}_\perp(b \cdot \partial e_k) \cdot \mathsf{P}_\perp(a \cdot \partial e_l)\right) e^k \wedge e^l \qquad (456)
\end{aligned}$$

In equation 456 the first and third terms are zero. The second term is entirely outside the tangent space and the fourth term is entirely inside the tangent space. Also note that since the second term consists of bivectors that are entirely outside the tangent space that term commutes with all multivectors A in the tangent space so that the commutator of the second term with A is zero. Thus the Riemann tensor reduces to

$$\mathsf{R}\left(a\wedge b\right)=-\left(\mathsf{P}_{\perp}\left(b\cdot\partial e_u\right)\cdot\mathsf{P}_{\perp}\left(a\cdot\partial e_v\right)\right)e^u\wedge e^v \tag{457}$$

or

$$\mathsf{R}\left(e_i\wedge e_j\right)=-\left(\mathsf{P}_{\perp}\left(\partial_j e_u\right)\cdot\mathsf{P}_{\perp}\left(\partial_i e_v\right)\right)e^u\wedge e^v \tag{458}$$

To calculate the Riemann tensor in terms of the Christoffel symbols note that

$$\begin{aligned}\mathsf{R}\left(e_i\wedge e_j\right)\cdot e_k &= \left[D_i,D_j\right]e_k\\ &= D_i\left(\Gamma^a_{jk}e_a\right)-D_j\left(\Gamma^a_{ik}e_a\right)\end{aligned}$$

$$\begin{aligned}
&= \left(\partial_i \Gamma^a_{jk}\right) e_a + \Gamma^a_{jk} D_i e_a \\
&- \left(\partial_j \Gamma^a_{ik}\right) e_a - \Gamma^a_{ik} D_j e_a \\
&= \left(\partial_i \Gamma^a_{jk} + \Gamma^b_{jk} \Gamma^a_{ib}\right) e_a \\
&- \left(\partial_j \Gamma^a_{ik} - \Gamma^b_{ik} \Gamma^a_{jb}\right) e_a
\end{aligned} \tag{459}$$

so

$$\begin{aligned}
\left(\mathsf{R}\left(e_i \wedge e_j\right) \cdot e_k\right) \cdot e^l &= \left(\left(\partial_i \Gamma^a_{jk}\right) + \Gamma^b_{jk} \Gamma^a_{ib}\right) \delta^l_a \\
&- \left(\left(\partial_j \Gamma^a_{ik}\right) + \Gamma^b_{ik} \Gamma^a_{jb}\right) \delta^l_a \\
&= \partial_i \Gamma^l_{jk} + \Gamma^b_{jk} \Gamma^l_{ib} \\
&- \partial_j \Gamma^l_{ik} - \Gamma^b_{ik} \Gamma^l_{jb}
\end{aligned} \tag{460}$$

Using equation 458 we have

$$(\mathsf{R}\left(e_i\wedge e_j\right)\cdot e_k)\cdot e^l = -\left(\mathsf{P}_\perp\left(\partial_j e_u\right)\cdot\mathsf{P}_\perp\left(\partial_i e_v\right)\right)\left(\left(e^u\wedge e^v\right)\cdot e_k\right)\cdot e^l \tag{461}$$

Using equation 477 (Appendix A) to reduce $\left(\left(e^u\wedge e^v\right)\cdot e_k\right)\cdot e^l$ gives

$$\left(\left(e^u\wedge e^v\right)\cdot e_k\right)\cdot e^l = g^{ul}\delta^v_k - g^{vl}\delta^u_k \tag{462}$$

Substituting equation 462 into equation 461 gives

$$\begin{aligned}(\mathsf{R}\left(e_i\wedge e_j\right)\cdot e_k)\cdot e^l &= \mathsf{P}_\perp\left(\partial_j e_k\right)\cdot\mathsf{P}_\perp\left(\partial_i e_v\right)g^{vl}\\ &\quad-\mathsf{P}_\perp\left(\partial_j e_u\right)\cdot\mathsf{P}_\perp\left(\partial_i e_k\right)g^{ul}\\ &=\mathsf{P}_\perp\left(\partial_j e_k\right)\cdot\mathsf{P}_\perp\left(\partial_i e^l\right)\\ &\quad-\mathsf{P}_\perp\left(\partial_j e^l\right)\cdot\mathsf{P}_\perp\left(\partial_i e_k\right)\end{aligned} \tag{463}$$

because

$$\begin{aligned} \mathsf{P}_\perp\left(\partial_i e_v\right) g^{vl} &= \mathsf{P}_\perp\left(g^{vl}\partial_i e_v\right) \\ &= \mathsf{P}_\perp\left(\partial_i\left(g^{vl}e_v\right) - \left(\partial_i g^{vl}\right)e_v\right) \\ &= \mathsf{P}_\perp\left(\partial_i\left(g^{vl}e_v\right)\right) \\ &= \mathsf{P}_\perp\left(\partial_i e^l\right) \end{aligned} \tag{464}$$

Finally

$$\begin{aligned} {R_{ijk}}^l &= \mathsf{P}_\perp\left(\partial_j e_k\right)\cdot\mathsf{P}_\perp\left(\partial_i e^l\right) - \mathsf{P}_\perp\left(\partial_j e^l\right)\cdot\mathsf{P}_\perp\left(\partial_i e_k\right) \\ &= \partial_i\Gamma^l_{jk} + \Gamma^b_{jk}\Gamma^l_{ib} - \partial_j\Gamma^l_{ik} - \Gamma^b_{ik}\Gamma^l_{jb} \end{aligned} \tag{465}$$

Which is the standard form of the Riemann tensor in terms of the Christoffel symbols. Note that

$$\begin{aligned} R_{ijkl} &= R_{ijk}{}^{v} g_{vl} \\ &= \mathsf{P}_{\perp}\left(\partial_j e_k\right) \cdot \mathsf{P}_{\perp}\left(\partial_i e_l\right) - \mathsf{P}_{\perp}\left(\partial_j e_l\right) \cdot \mathsf{P}_{\perp}\left(\partial_i e_k\right) \quad (466) \end{aligned}$$

From equation 466 and equation 425 $(\partial_i e_j = \partial_j e_i)$ we can see that the symmetries of the covariant Riemann tensor are

$$R_{ijkl} = -R_{jikl},\ \ R_{ijkl} = -R_{ijlk},\ \text{and}\ R_{ijkl} = R_{klij}$$

To prove the first Bianchi identity form $\mathsf{R}\left(e_i \wedge e_j\right) \cdot e_k$ and use equation 426 $(D_j e_k = D_k e_j)$ to get

$$\mathsf{R}\left(e_i \wedge e_j\right) \cdot e_k = D_i D_j e_k - D_j D_i e_k$$

$$\begin{aligned} &= D_i D_k e_j - D_j D_k e_i \\ &= \left[D_i, D_k\right] e_j - \left[D_j, D_k\right] e_i + D_k \left(D_i e_j - D_j e_i\right) \\ &= \mathsf{R}\left(e_i \wedge e_k\right) \cdot e_j - \mathsf{R}\left(e_j \wedge e_k\right) \cdot e_i \end{aligned} \tag{467}$$

or

$$F\left(a, b, c\right) = a \cdot \mathsf{R}\left(b \wedge c\right) + c \cdot \mathsf{R}\left(a \wedge b\right) + b \cdot \mathsf{R}\left(c \wedge a\right) = 0 \tag{468}$$

However $F\left(a, b, c\right)$ is a linear function of a, b, and c. Also $F\left(b, a, c\right) = -F\left(a, b, c\right)$ and $F\left(a, c, b\right) = -F\left(a, b, c\right)$ so since F is antisymmetric in all arguments we may write

$$F\left(a, b, c\right) = F\left(a \wedge b \wedge c\right) \tag{469}$$

Thus equation 468 contains $n\binom{n}{3} = \dfrac{n^2\left(n-1\right)\left(n-2\right)}{6}$ scalar coefficients. Since the Riemann tensor is a bivector valued function of a bivector the degrees of freedom of the tensor is no more than $\left(\dfrac{n\left(n-1\right)}{2}\right)^2$ and equation 468 reduces the degrees of freedom by $n\binom{n}{3}$ so that the total degrees of freedom or the Riemann tensor is

$$\left(\frac{n\left(n-1\right)}{2}\right)^2 - n\binom{n}{3} = \frac{1}{12}n^2\left(n^2-1\right).$$

Appendix A - BAC-CAB Formulas

Using the geometric algebra module in *sympy*[8] several formulas containing the dot and wedge products can be reduced. Let a, b, c, d, and e be vectors, then we have

$$a \cdot (b \wedge c) = (a \cdot b)\, c - (a \cdot c)\, b \tag{470}$$

$$\begin{aligned} a \cdot (b \wedge c \wedge d) = {} & (a \cdot d)\,(b \wedge c) - (a \cdot c)\,(b \wedge d) \\ & + (a \cdot b)\,(c \wedge d) \end{aligned} \tag{471}$$

$$a \cdot (b \wedge c \wedge d \wedge e) = -\,(a \cdot e)\,(b \wedge c \wedge d) + (a \cdot d)\,(b \wedge c \wedge e)$$

---

[8] http://docs.sympy.org/

$$- (a \cdot c)\,(b \wedge d \wedge e) + (a \cdot b)\,(c \wedge d \wedge e) \tag{472}$$

If in equation 470 the vector b is replace by a vector differential operator such as ∇, ∂, or D (we will use D as an example) it can be rewritten as

$$\begin{aligned} (a \cdot D)\,c &= a \cdot (D \wedge c) + (a \cdot \dot{c})\,\dot{D} \\ &= a \cdot (D \wedge c) + \dot{D}\,(a \cdot \dot{c}) \\ &= a \cdot (D \wedge c) + \dot{D}\,(\dot{c} \cdot a) \end{aligned} \tag{473}$$

Cyclic reduction formulas are

$$a \cdot (b \wedge c) + c \cdot (a \wedge b) + b \cdot (c \wedge a) = 0 \tag{474}$$

$$a\,(b \wedge c) - b\,(a \wedge c) + c\,(a \wedge b) = 3a \wedge b \wedge c \tag{475}$$

$$a\,(b \wedge c \wedge d) - b\,(a \wedge c \wedge d) + c\,(a \wedge b \wedge d) - d\,(a \wedge b \wedge c) = 4a \wedge b \wedge c \wedge d \tag{476}$$

Basis blade reduction formula

$$\begin{aligned}(a \wedge b) \cdot (c \wedge d) &= ((a \wedge b) \cdot c) \cdot d \\ &= (a \cdot d)\,(b \cdot c) - (a \cdot c)\,(b \cdot d)\end{aligned} \tag{477}$$

Finally one formula for reducing the commutator product of two bivectors

$$(a \wedge b) \times (c \wedge d) = (a \cdot d)\,b \wedge c - (a \cdot c)\,b \wedge d$$

$$+ (b \cdot c)\, a \wedge d - (b \cdot d)\, a \wedge c \qquad (478)$$

Appendix B - Blade Orientation Theorem

A blade only depends on the relative orientation of the vectors in the plane defined by the blade. Since any blade can be defined by the geometric product of two orthogonal vectors let them be $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$. Then any two vectors in the plane can be define by:

$$a = a_x\boldsymbol{e}_x + a_y\boldsymbol{e}_y \tag{479}$$

$$b = b_x\boldsymbol{e}_x + b_y\boldsymbol{e}_y \tag{480}$$

and any rotor in the plane by

$$R = ab = (a \cdot b) + (a_xb_y - a_yb_x)\,\boldsymbol{e}_x\boldsymbol{e}_y \tag{481}$$

as long as

$$RR \wedge \dagger = 1 \tag{482}$$

but

$$R\boldsymbol{e}_x\boldsymbol{e}_y = \boldsymbol{e}_x\boldsymbol{e}_y R \tag{483}$$

and

$$R\boldsymbol{e}_x R^\dagger R\boldsymbol{e}_y R^\dagger = R\boldsymbol{e}_x\boldsymbol{e}_y R^\dagger = \boldsymbol{e}_x\boldsymbol{e}_y RR^\dagger = \boldsymbol{e}_x\boldsymbol{e}_y \tag{484}$$

and absolute orientations of $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ does not matter for $\boldsymbol{e}_x\boldsymbol{e}_y$.

References

1. Cambridge site: www.mrao.cam.ac.uk/∼clifford
2. Arizona State site: modelingnts.la.asu.edu
3. Chris Doran and Anthony Lasenby, "Geometric Algebra for Physicists," Cambridge University Press, 2003.
4. David Hestenes and Garret Sobczyk, "Clifford Algebra to Geometric Calculus," Kluwer Academic Publishers, 1984.
5. David Hestenes, "New Foundations for Classical Mechanics (2^{nd} edition)," Kluwer Academic Publishers, 1999.