Differentiation Techniques

Section 1.5: The Power, Sum, and Difference Rules

Given $y = f(x)$, the derivative of *f* can be represented using the following notation:

$$f'(x) = \frac{dy}{dx} = \frac{d}{dx} f(x)$$

If we let $x = a$, then the value of the derivative at $x = a$ is denoted by $f'(a) = \left.\frac{dy}{dx}\right|_{x=a}$

1. **The Power Rule**

For any real number *k*,

$$\frac{d}{dx} x^k = k \bullet x^{k-1}$$

Example ①: Find $\frac{d}{dx} x^8$

Example ②: Find $\frac{d}{dx} \sqrt[3]{x^2}$

2. The derivative of a constant times a function is the constant times the derivative of the function.

$$\frac{d}{dx}\left[c \bullet f(x)\right] = c \bullet \frac{d}{dx} f(x)$$

Example: Find $\frac{d}{dx} 3x^8$

3. **The Sum-Difference Rule**

The derivative of a sum/difference is the sum/difference of the derivatives:

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$$

Example ①: Find $\frac{d}{dx}\left[x^8 + \sqrt[3]{x^2}\right]$

Example ②: Find $\frac{d}{dx}\left[3x^8 - \frac{1}{x}\right]$ *(Hint: Write $\frac{1}{x}$ in exponential form)*

4. The derivative of a constant function is 0.

$$\frac{d}{dx}c = 0$$

Section 1.6: The Product and Quotient Rules

The derivative of a sum is the ____.

Is the derivative of a product the product of the derivatives? __________________________

Find the derivative of $x^2 \bullet x^3$ which equals x^5.

The Product Rule

If $H(x) = f(x) \bullet g(x)$, then

$$H'(x) = \frac{d}{dx}\left[f(x) \bullet g(x)\right] = f(x) \bullet \left[\frac{d}{dx} g(x)\right] + \left[\frac{d}{dx} f(x)\right] \bullet g(x)$$

That is, the derivative of a product is the first factor times the derivative of the second factor, plus the derivative of the first factor times the second factor.

Example 1: Differentiate: $f(x) = \sqrt[3]{x}(400 - 5x)$

Example 2: Differentiate: $f(x) = \left(7x^9 + 4x^3 - 50\right)\left(9 - 7x^2\right)$

Is the derivative of a quotient, the quotient of the derivatives? Try it with x^2/x^3.

The Quotient Rule

If $Q(x) = \dfrac{N(x)}{D(x)}$,

then $Q'(x) = \dfrac{D(x) \bullet N'(x) - D'(x) \bullet N(x)}{[D(x)]^2}$

That is, the derivative of a quotient is the denominator times the derivative of the numerator, minus the derivative of the denominator times the numerator, all divided by the square of the denominator.

Example 3: Differentiate: $y = \dfrac{x+2}{x-2}$

Example 4: Differentiate: $f(x) = \dfrac{x}{400-x}$

Section 1.7: The Chain Rule

1. Consider the function $y=[g(x)]^2$ Rewrite this function as $y=g(x)\cdot g(x)$, and then use the product rule to find y' in terms of $g(x)$ and $g'(x)$. Simplify your answer by combining any like terms.

2. Now let $y=[g(x)]^3$. Rewrite this function as $y=g(x)\cdot[g(x)]^2$, and then use the product rule in combination with your answer to #1 above to find y' in terms of $g(x)$ and $g'(x)$. Simplify your answer by combining any like terms.

3. Now let $y=[g(x)]^4$. Rewrite this function as $y=g(x)\cdot[g(x)]^3$, and then use the product rule in combination with your answer to #2 above to find y' in terms of $g(x)$ and $g'(x)$. Simplify your answer by combining any like terms.

4. Based on your answers to numbers 1, 2 and 3, guess a formula for y' if $y=[g(x)]^k$. This is called the Extended Power Rule. It is an application of an important rule of differentiation called the Chain Rule, which allows us to differentiate composite functions (functions of functions).

The Extended Power Rule

Suppose that $g(x)$ is a function of x. Then for any real number k,

$$\frac{d}{dx}\left[g(x)\right]^k = k\left[g(x)\right]^{k-1} \bullet \frac{d}{dx} g(x)$$

Example 1: Given $y=(1+x^2)^3$, find $\frac{dy}{dx}$.

Example 2: Differentiate: $y=\sqrt{4x-7}$

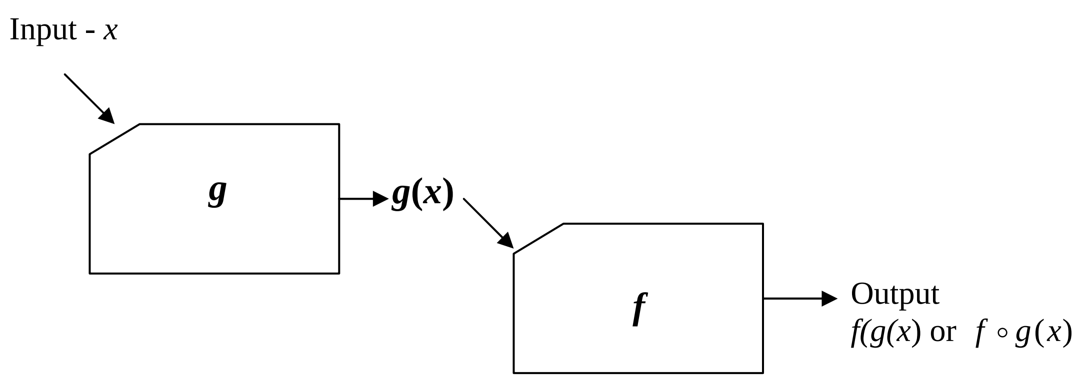
Example 3: Differentiate: $f(x)=(1+x^3)^5-(1+x^3)^4$

The Extended Power Rule is a specific case of a more general rule called the Chain Rule.

The **composed function** $f \circ g$ **,** the composition of *f* and *g*, is defined as

$$f \circ g(x) = f(g(x)) \text{ or } f \circ g(x) = f[g(x)].$$

The composition of *f* and *g* can be shown in a diagram.



Example 4: Find $f \circ g(x)$, $g \circ f(x)$, and $h \circ g(x)$, given

$$f(x) = \frac{3}{x},\ g(x) = 2x^2 + 3, \text{ and } h(x) = 2x - 1$$

The Chain Rule

The derivative of the composition of $f \circ g$ is given by

$$\frac{d}{dx}\left[f \circ g(x)\right] = \frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \bullet g'(x).$$

Recall that the Extended Power Rule is a special case of the Chain Rule. Consider the function $f(x) = x^k$. Then for any other function $g(x)$, $f \circ g(x) = [g(x)]^k$ and the derivative of the composition is

$$\frac{d}{dx}\left[g(x)\right]^k = k\left[g(x)\right]^{k-1} \bullet g'(x).$$

The Chain Rule often appears in another form. Suppose that $y = f(u)$ and $u = g(x)$. Then

$$\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}.$$

Example 5: Find $\frac{dy}{du}, \frac{du}{dx}, and\ \frac{dy}{dx}$ given, $y = \frac{15}{u^3}$ and $u = 2x + 1$.

Example 6: Find $f(x)$ and $g(x)$ such that $h(x) = f \circ g(x)$.

$$h(x) = \frac{1}{\sqrt{7x+2}}$$

Calculation Thought Experiment

The **calculation thought experiment** is a technique to determine whether to treat an algebraic expression as a product, quotient, sum, or difference. Given such an expression, consider the steps you would use in computing its value. If the last operation is multiplication, treat the expression as a product; if the last operation is division, treat the expression as a quotient, and so on.

Using the Calculation Thought Experiment (CTE) to Differentiate a Function

If the CTE says, for instance, that the expression is a sum of two smaller expressions, then apply the rule for sums as a first step. This will leave you having to differentiate simpler expressions, and you can use the CTE on these, and so on...

Examples

1. $(3x^2-4)(2x+1)$ can be computed by first calculating the expressions in parentheses and then multiplying. Since the last step is multiplication, we can treat the expression as a product.

2. $(2x-1)/x$ can be computed by first calculating the numerator and denominator, and then dividing one by the other. Since the last step is division, we can treat the expression as a quotient.

3. $x^2 + (4x-1)(x+2)$ can be computed by first calculating x^2, then calculating the product $(4x-1)(x+2)$, and finally adding the two answers. Thus, we can treat the expression as a sum.

4. $(3x^2-1)^5$ can be computed by first calculating the expression in parentheses, and then raising the answer to the fifth power. Thus, we can treat the expression as a power.