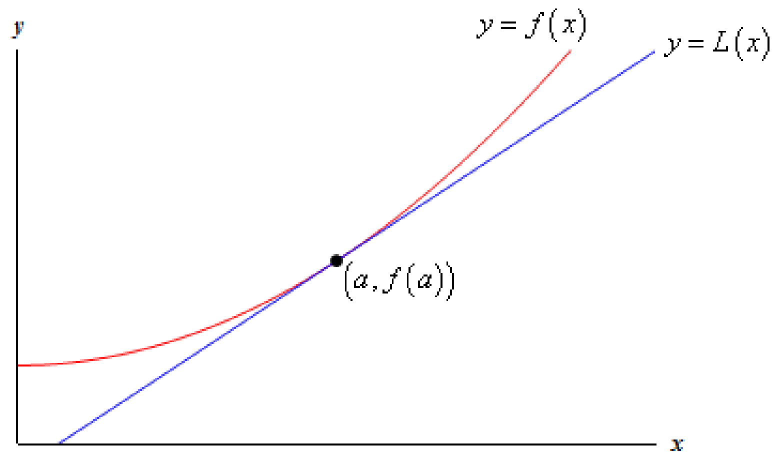
Linear Approximation & Differentials (3.9)

In this section we are going to take a look at an application of the tangent line to a function. If you zoom in to a portion of a smooth curve near a specified point, it becomes indistinguishable from the tangent line at that point. In other words:

The values of the function are close to the values of the linear function whose graph is the tangent line.



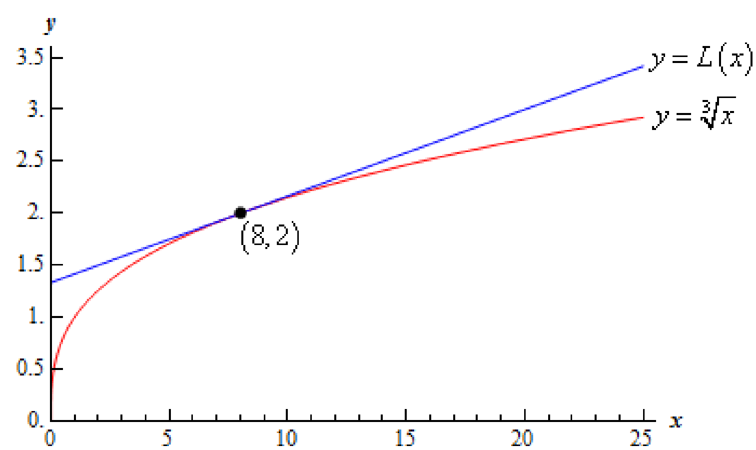
For this reason, the linear function whose graph is the tangent line to $y = f(x)$ at a specified point $(a, f(a))$ is called the *linear approximation of* $f(x)$ *near* $x = a$.

What is the formula for the linear approximation?

Examples:

① Determine the linear approximation for

$f(x) = \sqrt[3]{x}$ at $x = 8$.



① Use the linear approximation to approximate the value of $\sqrt[3]{8.05}$ and $\sqrt[3]{25}$.

Linear approximations do a very good job of approximating values $f(x)$ as long as we stay "near" $x = a$. However, the farther away from $x = a$ we get the worse the approximation is liable to be. The main problem here is that how near we need to stay to $x = a$ in order to get a good approximation will depend upon both the function we're using and the value of $x = a$ that we're using. Also, there will often be no easy way of prediction how far away from $x = a$ we can get and still have a "good" approximation.

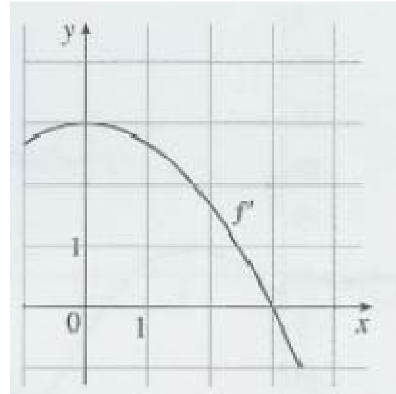
② Use linear approximation to approximate $\sqrt{4.1}$.

③ Use linear approximation to approximate $ln(1.134)$.

④ Determine the linear approximation for $\mathrm{f}(\theta) = \sin\theta$ at $\theta = 0$.

This is actually a somewhat important linear approximation. In optics this linear approximation is often used to simplify formulas. This linear approximation is also used to help describe the motion of a pendulum and vibrations in a string.

Consider the graph of $f'(x)$, the *derivative* of $f(x)$.



1. Suppose $f(2) = 4$. Use linearization to approximate $f(1.98)$ and $f(2.02)$.

2. Are your approximations overestimates or underestimates? EXPLAIN your answer.

3. Suppose you know $f(3) = 7$. Can you approximate $f(2.98)$ and $f(3.02)$? EXPLAIN your answer.

Differentials

When a physical measurement is made, there is always some uncertainty about it accuracy. For instance, if you are measuring the radius of a ball bearing, you might measure it repeatedly and obtain slightly differing results. Rather than concluding, say, that the radius of the ball bearing is exactly 1.2mm, you may instead conclude that the radius is 1.2 mm ±0.1 mm. (The actual calculation of the range ±0.1 mm is often given by a statistical formula based on the standard deviation of all the separate measurements.)

Once you have an error estimate for the radius, you might wonder how this error might effect the calculation of the *volume* of the ball bearing. In other words, if the radius is off by 0.1 mm, by how much is the volume off? To answer the question, think of the error of the radius as a change, Δr, in r, and then compute the associated change, ΔV, in the volume V. The general question is therefore:

Q If x changes by Δx, and y is a function of x, what is the associated change Δy in y?

Given a function$y = f(x)$we call dy and dx differentials and the relationship between them is given by, $dy = f'(x)dx$

Note that if we are just given $f(x)$ then the differentials are df and dx and we compute them in the same manner.

$$df = f'(x)dx$$

Examples: Compute the differential for each of the following.

(a) $y = t^3 - 4t^2 + 7t$

(b) $w = x^2 \sin(2x)$

(c) $f(z) = e^{3-z^4}$

Example: Compute dy and Δy if $y = \cos(x^2 + 1) - x$ as x changes from $x = 2$ to $x = 2.03$.

Example: A sphere was measured and its radius was found to be 45 inches with a possible error of no more that 0.01 inches. What is the maximum possible error in the volume if we use this value of the radius?