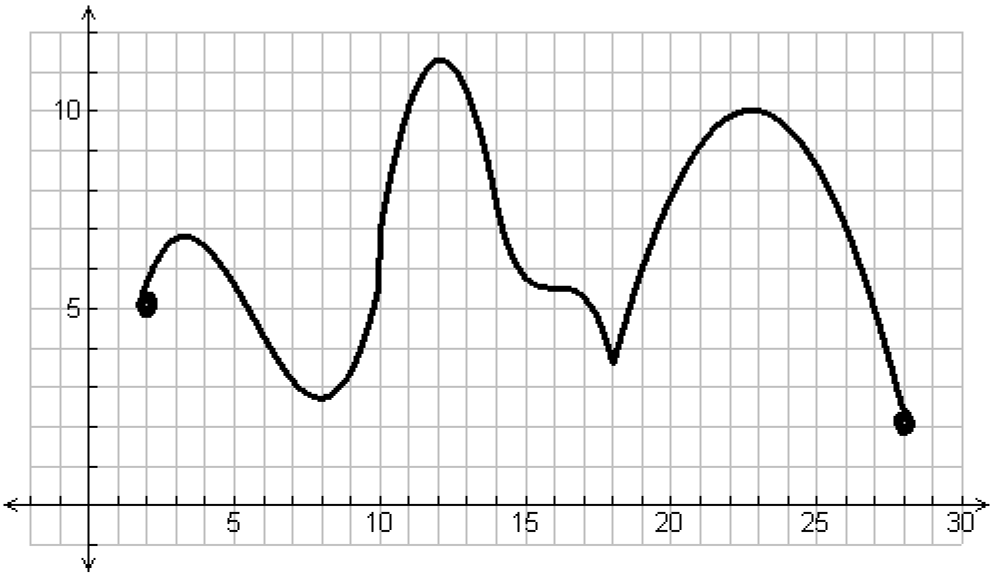
Maximum and Minimum Values

Section 4.2



> *Definitions*
> - A function has an ***absolute (or global) maximum*** at x = c if f(c) $\geq$ f(x) for all x in the domain of f. f(c) is called the maximum value of f on its domain.
> - A function has an ***absolute (or global) minimum*** at x = c if f(c) $\leq$ f(x) for all x in the domain of f. f(c) is called the minimum value of f on its domain.

The absolute maximum and minimum values are called the absolute extreme values of the function. Essentially, they are the highest and lowest y-values of the function respectively.

Refer to the function f graphed above. At what value of x does f have its absolute maximum value? What is the maximum value of the function?

At what value of x does f have its absolute minimum value? What is the minimum value of the function?

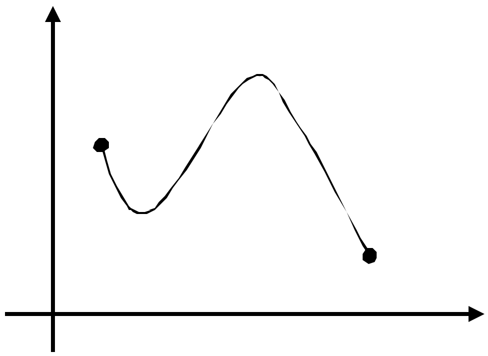
> *Definitions*
> - A function has a ***local (or relative) maximum*** at x = c if f(c) $\geq$ f(x) for all x in an open interval containing x = c.
> - A function has a ***local (or relative) minimum*** at x = c if f(c) $\leq$ f(x) for all x in an open interval containing x = c.

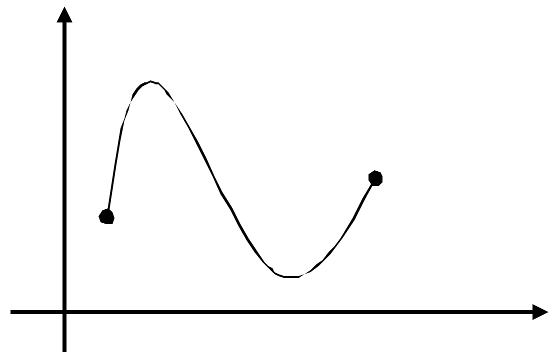
Essentially, a function has a local maximum or minimum at x = c if f has a *turning point* at (c,f(c)).

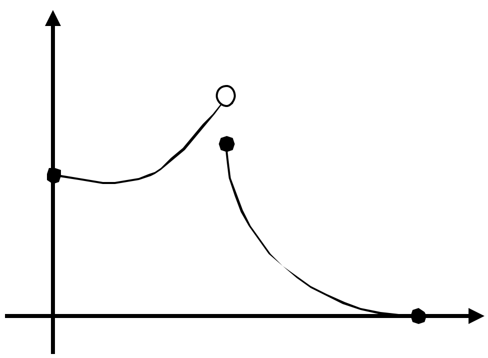
Refer again to the function in the graph above. At what value(s) of x does f have a local maximum?

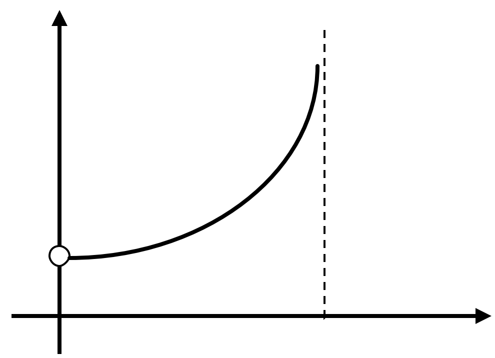
At what value(s) of x does f have a local minimum?

The Extreme Value Theorem
If *f* is continuous on a closed interval [a, b], then *f* attains an absolute maximum value *f(c)* and an absolute minimum value *f(d)* at some numbers *c* and *d* in [a, b].









Fermat's Theorem

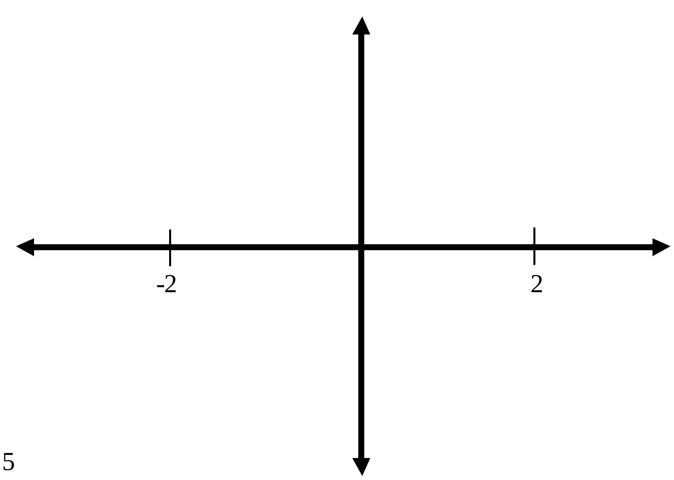
critical number

Closed Interval Method

Suppose that *f* is a continuous function over a closed interval [a, b]. To find the absolute maximum and minimum values of the function over [a, b]:

a) Find $f'(x)$
b) Determine the critical points in [a, b].
c) List the critical points of *f* and the endpoints of the interval: $a, c_1, c_2, \ldots, c_n, b$.
d) Find the function values in part c): $f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)$. The largest of these is the **absolute maximum** of *f* over [a,b]. The smallest of these is the **absolute minimum** of *f* over $[a,b]$.

Ex.: Sketch a function *f* on [-2, 2] that satisifies *all* of the following criteria.
a. *f* is continuous on [-2, 2]
b. *f* has a local and absolute maximum at $x = 0$.
c. *f* has an absolute minimum at $x = 2$.
d. *f* has no point c where $f'(c) = 0$.



Critical Numbers & Extreme Values on a Closed Interval

For each function,

(a) Find the derivative.

(b) Determine any x-values at which the derivative is zero.

(c) Determine any x-values at which the derivative is undefined.

(d) List the critical numbers of the function.

(e) Determine the absolute maximum and the absolute minimum of the function on the interval [-4,4]. If the function does not have an absolute maximum or an absolute minimum on [-4,4], state this and explain why this does not contradict the Extreme Value Theorem.

1. $f(x) = x^3 + 3x^2 - 9x - 2$

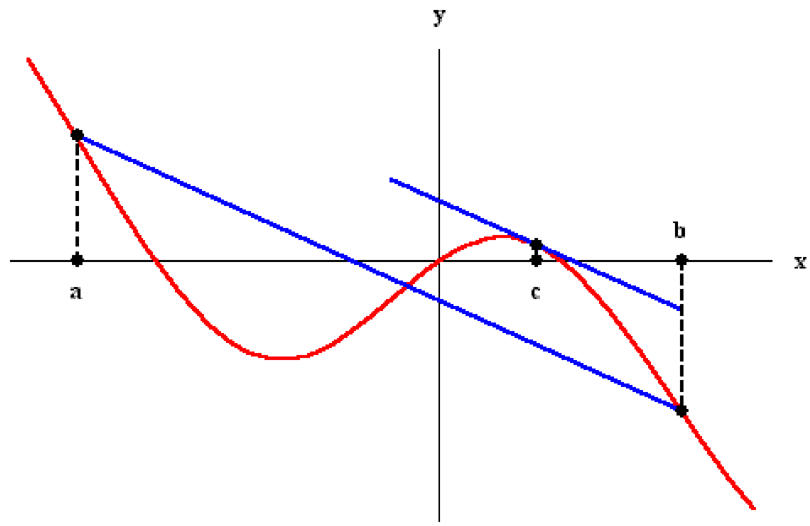
2. $m(x) = \dfrac{x}{x^2 + 9}$

3. $f(x) = (x^2 - 9)^{2/3}$

4. $h(x) = \dfrac{1}{(x^2 - 9)^{1/3}}$

Derivatives and the Shapes of Curves (4.3)

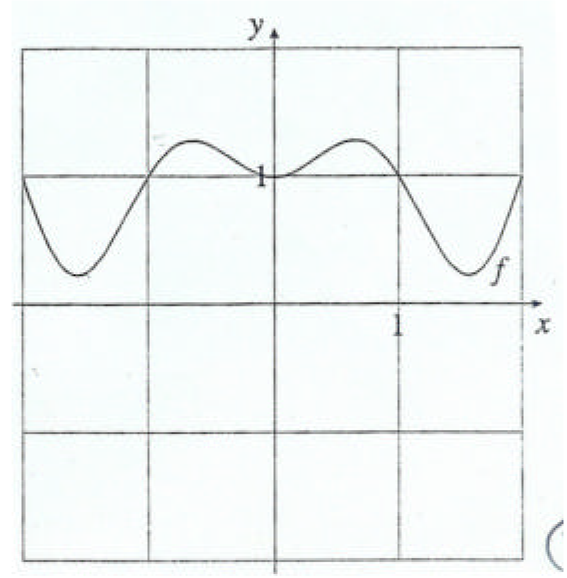
The **Mean Value Theorem** is one of the most important theoretical tools in Calculus. It states that if $f(x)$ is defined and continuous on the interval $[a,b]$ and differentiable on (a,b), then there is at least one number c in the interval (a,b) (that is $a < c < b$) such that



For each of the following functions, find the value(s) of c in the given interval which satisfies the conclusion of the Mean Value Theorem.

1. $f(x) = x^2 - 5x + 7,\ -1 \le x \le 3$ $\qquad$ $f(x) = \sin(2x) + \cos(x),\ 0 \le x \le \pi$

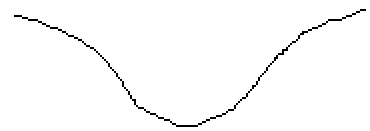
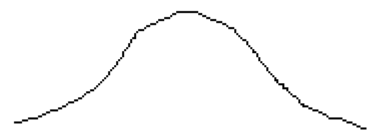
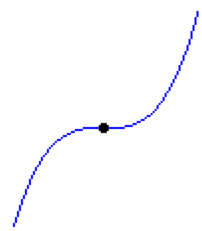
3. Use the graph of f to estimate the values(s) of c that satisfy the conclusion of the Mean Value Theorem for the interval [-2, 2].



4. Gustavo left home at 5:00 pm and drove on a highway with a posted speed limit of 65 mph. He arrived at his destination 74 miles down the highway at 5:55 pm. Immediately upon parking a state trooper, pulled up behind Gustavo giving him a ticket for driving in excess of 80 mph. Was the state trooper justified in giving Gustavo the ticket. Justify your answer using the Mean Value Theorem. Be as specific as possible.

5. Two runners start a race at the same time and finish in a tie. Prove that at some time during the race they have the same speed. [*Hint*: Consider $f(t) = g(t) - h(t)$, where g and h are the position functions of the two runners.

Local (Relative) Extrema:

$f(c)$ *c is a critical number*	**Sign of $f'(x)$ for x in (a,c)**	**Sign of $f'(x)$ for x in (c,b)**	**Graph of f over open interval (a,b)**
Local minimum			
Local maximum			
No local maxima or minima			

p. 280

Increasing/Decreasing Test

(a) If ________________ on an interval, then f is increasing on that interval.

(b) If _________________ on an interval, then f is decreasing on that interval.

The First Derivative Test: *(Used to find local extrema)*

p. 281

The First Derivative Test.

Suppose that c is a critical number of a continuous function f.

(a) If f' changes from ________________ to negative at c, then f has a local ________________ at c.

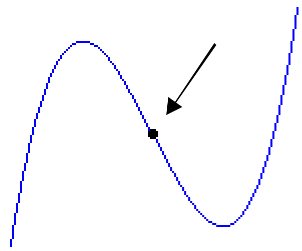
(b) If f' changes from negative to ________________ at c, then f has a local _______________ c.

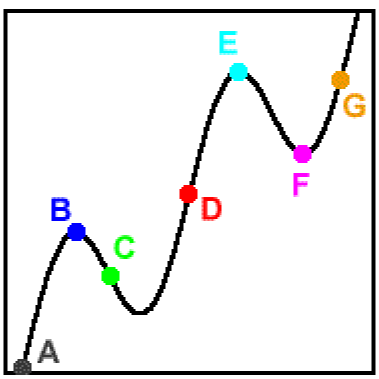
(c) If f' does not change ________________ at c (that is, f' is positive on both sides of c or negative on both sides), then f has no maximum or minimum at c.

Concavity

A function (or its graph) is called **concave upward** on an interval *I* if f' is an increasing function on *I*. It is called **concave downward** on *I* if f' is decreasing on *I*.

The **inflection point** is the point where a curve changes its direction of concavity.





	A	B	C	D	E	F	G
$f'(x) < 0$							
local minima							
f increasing							
concave up							
concave down							
$f'(x) > 0$							
local maxima							
$f'(x) = 0$							
inflection points							
f decreasing							
absolute minima							

Concavity Test: p. 282

Concavity Test

If **f**¢(**x**) > **0** for all *x* in *I*, then the graph of *f* is ______________________________ on *I*.

If _______________ for all *x* in *I*, then the graph of *f* is concave downward on *I*.

Second Derivative Test: *(Used to find local extrema)* p. 282

Suppose f'' is continuous near *c*.

(a) If $f'(c) = 0$ and _______________ then f has a _______________ minimum at *c*.

(b) If _______________ and $f''(x) < 0$, then f has a local ___________________ at c.

(c) If $f'(c) = 0$ and $f''(c) = 0$, then the Second Derivative Test fails.

Strategies for Sketching the Graph of a Function, *f*

1. ***Intercepts***. Find the *x*-intercept(s) (solve $f(x) = 0$) and the *y*-intercept (find $f(0)$ of the graph.
2. ***Asymptotes***. Find vertical, horizontal, or oblique asymptotes (if any).
3. ***Derivatives***. Find f' and f''.
4. ***Domain of f***. Find the values of *x* where the function *f* is undefined
5. Find the ***critical numbers*** of *f*. *Note: Crictical numbers must be in the domain of f.*
6. ***Increasing/Decreasing and Relative Extrema***. Use numbers found in *Step (5)* to partition the domain of *f*. Determine the intervals where *f* is increasing/decreasing by finding the interval(s) where $f'(x) > 0$ or $f'(x) < 0$. Use this information or the second derivative to determine the relative extrema.
7. ***Hypercritical numbers***: Determine candidates of inflection points by finding values of *x* where $f''(x) = 0$ or $f''(x)$ does not exist. *Note: Hypercritical numbers must be in the domain of f.*
8. ***Concavity***. Use the *x*-values from *Step (7)* to partition the domain of *f*. Determine the concavity by checking to see where $f''(x) > 0$ or $f''(x) < 0$
9. ***Sketch the graph of f***. Use the information from *Steps (1-8)* to sketch the graph of *f*, plotting extra points [using your calculator] as needed.

Example: Given $f(x) = \dfrac{x^2 - 1}{x^3}$

a) Determine domain.

b) Find asymptotes.

c) Find intervals of increase/decrease.

d) Find local extrema.

Round outputs to the nearest hundredth.

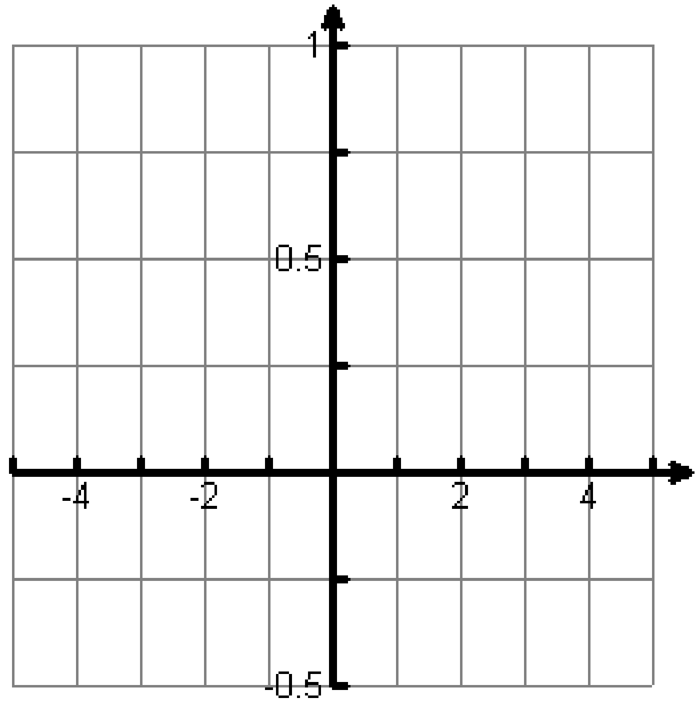
e) Find interval(s) of concavity

f) Find inflection points.

Round outputs to the nearest hundredth.

g) Find intercepts.

h) Sketch the graph of *f*. Label all extrema and inflection points.



a)
b) vertical:
horizontal:
c) increasing:
decreasing:
d) local maxima:
local minima:
e) concave up:
concave down:
f) inflection points:
g)

Section 4.3

For each function below:

a) Find all critical numbers (possible extrema)
b) Find the interval(s) of decrease and increase.
c) Find the local maximum and minimum values
d) Find all hypercritical numbers (possible points of inflection)
e) Find the interval(s) of concavity
f) Find the inflection points.
g) Use the information to sketch the graph of the function.

1. $f(x) = 2 + x - x^2 - x^3$
2. $f(x) = (x+3)^{2/3} - 5$
3. $f(x) = (x+1)^{1/3}$
4. $f(x) = x^2 \ln x$
5. $f(x) = \dfrac{x^2}{x^2+1}$

Group Work 3, Section 4.3
Graphing with the Derivative (Form A)

This exercise is designed to illustrate how numerical information from a function and its derivatives can be used to get a very good sense of how the function looks. While it is a good idea to use your graphing calculator to check your final answers, it would be missing the point to use it earlier.
Consider the function

$$f(x) = \frac{2x}{x^2 - 1}$$

1. Where are the zeros (roots) of this function?

2. On what intervals is this function increasing? On what intervals is it decreasing?

3. Where are the local maxima and minima?

4. It is a fact that $f''(x)$ simplifies to $4x\frac{x^2+3}{(x^2-1)^3}$. Where is f concave up? Where is f concave down?

5. Where are the inflection points?

6. Does this graph have any vertical asymptotes? If so, what are they? If not, why not?

7. What appears to happen to $f(x)$ when x gets very large? What appears to happen when x gets very large and negative?

8. Using this information, sketch a graph of this function on a separate piece of paper.

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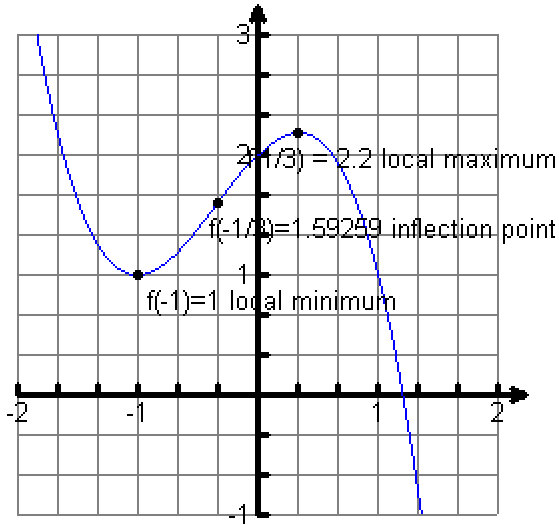
Section 4.3

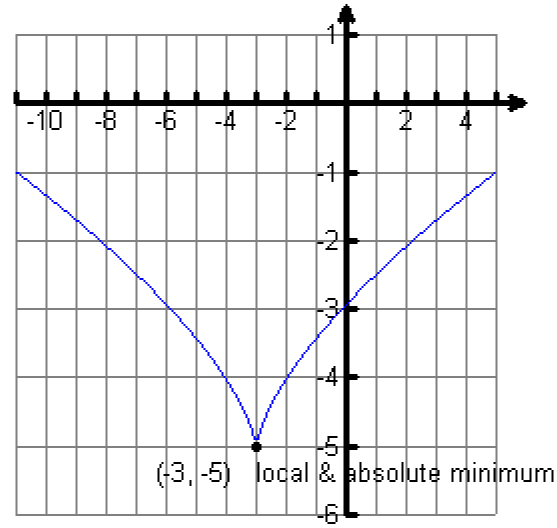
Derivatives and the Shape of Curves - Answers

	$f(x)=2+x-x^2-x^3$	$f(x)=(x+3)^{2/3}-5$	$f(x)=(x+1)^{1/3}$	$f(x)=x^2\ln x$	$f(x)=\frac{x^2}{x^2+1}$
$f'(x)$	$1-2x-3x^2$	$\frac{2}{3}(x+3)^{-1/3}$	$\frac{1}{3}(x+1)^{-2/3}$	$x+2x\ln x$	$\frac{2x}{(x^2+1)^2}$
$f''(x)$	$-2-6x$	$-\frac{2}{9}(x+3)^{-4/3}$	$\frac{-2}{9(x+1)^{5/3}}$	$3+2\ln x$	$\frac{2(1-3x^2)}{(x^2+1)^3}$
Critical #s	1/3,-1	-3	-1	$e^{-1/2}\approx .6$	0
Increasing	(-1,1/3)	(-3,∞)	(-∞,-1) and (-1,∞)	$(e^{-1/2},\infty)$	(0,∞)
Decreasing	(-∞,-1) and (1/3,∞)	(-∞,-3)	Nowhere	$(0,e^{-1/2})$	(-∞,0)
Local max	(1/3, 2.2)	None	None	None	None
Local min	(-1,1)	(-3,-5)	None	$(e^{-1/2},-.18)$	(0,0)
Hypercritical #s	-1/3	-3	-1	$e^{-3/2}$	$\pm\sqrt{3}/3\approx\pm.6$
Concave up	(-∞, -1/3)	Nowhere	(-∞,-1)	$(e^{-3/2},\infty)$	$(-\sqrt{3}/3,\sqrt{3}/3)$
Concave down	(-1/3, ∞)	(-∞,-3) and (-3, ∞)	(-1, ∞)	$(0,e^{-3/2})$	$(-\infty,-\sqrt{3}/3)$ and $(\sqrt{3}/3,\infty)$
Inflection points	(-1/3, 1.6)	none	(-1,0)	$(e^{-3/2},-.07)$	$(\pm\sqrt{3}/3,1/4)$

1.

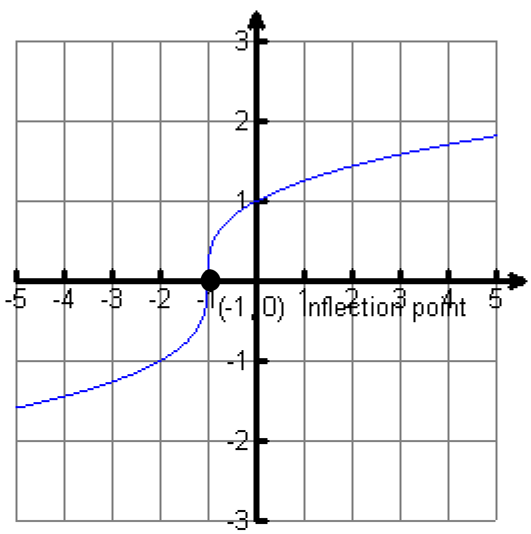

(1/3) = 2.2 local maximum
f(-1/3)=1.59259 inflection point
f(-1)=1 local minimum

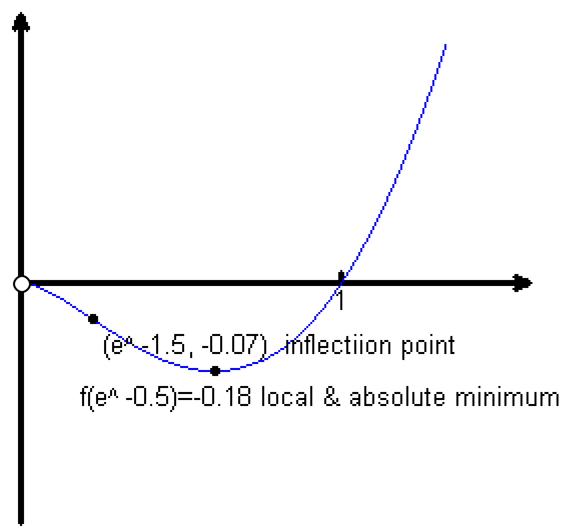
2.


(-3, -5) local & absolute minimum

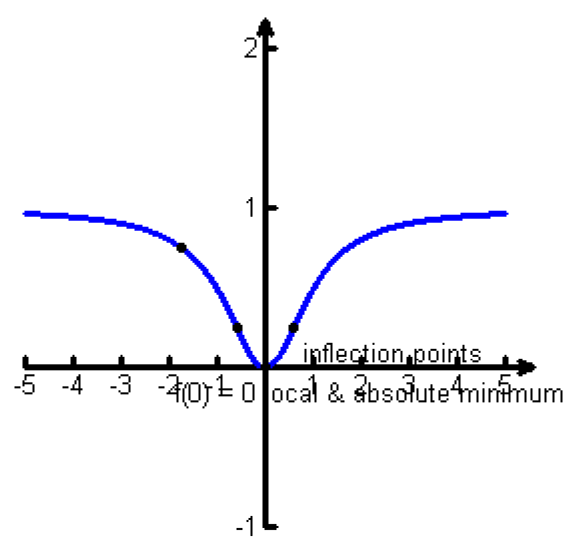
3

4.


(-1, 0) Inflection point


(e^-1.5, -0.07) inflectiion point
f(e^ -0.5)=-0.18 local & absolute minimum

5.


inflection points
f(0) = 0 local & absolute minimum