

In **optimization** problems we are looking for the largest value or the smallest value that a function can take.

A Strategy for Solving Optimization Problems

1. Read the problem carefully. If relevant, make a drawing.
2. Label the picture with appropriate variables and constants, noting what varies and what stays fixed.
3. Assign a symbol, Q , to the quantity to be maximized/minimized.
4. Translate problem to an equation involving Q . Try to represent Q in terms of one variable by finding a relationship (in the form of equations) among the variables. Write the domain of the function.
5. Determine the maximum or minimum values using the methods described in sections 4.2 and 4.3

First Derivative Test for Absolute Extrema

Let I be the interval of all possible optimal values of $f(x)$ and further suppose that $f(x)$ is continuous on I , except possibly at the endpoints. Finally suppose that $x = c$ is a critical point of $f(x)$ and that c is in the interval I . If we restrict x to values from I (*i.e.*, we only consider possible optimal values of the function) then,

1. If $f'(x) > 0$ for all $x < c$ and if $f'(x) < 0$ for all $x > c$ then $f(c)$ will be the absolute maximum value of $f(x)$ on the interval I .
2. If $f'(x) < 0$ for all $x < c$ and if $f'(x) > 0$ for all $x > c$ then $f(c)$ will be the absolute minimum value of $f(x)$ on the interval I .

- ① **Find the rectangle with the maximum area which can be inscribed in a semicircle.**
<http://archives.math.utk.edu/visual.calculus/3/applications.1/index.html>

- ② **A box with an open top is to be constructed from a square piece of cardboard, 6 ft wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.**

MA 181 Calculus I Optimization (4.6)

- ③ A rectangular box open at the top is to be constructed so that the volume is 10 cubic meters and the base has length equal to twice its width. Material for the base of the box costs \$4 per square meter, while the material for the sides costs \$2 per square meter. Our goal is to find the dimensions of the box which will minimize the cost of the box. To help you in solving this problem, follow these steps
- (a) Draw a picture and label the dimensions of the box. Use h for the height, w for the width and write the length in terms of w .
- (b) Write an expression in terms of h and w for the cost C of the box. To do this, you need to recognize that the cost of each surface of the box is the area of that surface multiplied by the cost per square meter. Keep in mind that the box has five surfaces --- the bottom plus four sides.
- (c) Use the fact that the volume of the box is 10 m³ to relate h and w and then solve for h in terms of w .
- (d) Substitute the expression for h which you found in (c) into the expression for cost in (b).
- (e) You should now have an expression for cost in terms of w only. Use calculus to find the dimensions of the box which will give a minimum for C analytically, find the critical value(s) of C and then use either the First or Second Derivative Test to determine where the function actually has the desired minimum.
- (f) Confirm the answer which you got in part (e) by graphing the cost function on your calculator and estimating the value of w which minimizes the cost.

MA 181 Calculus I Optimization (4.6)

④ Points A and B are opposite one another on shores of a straight river 3 km wide. Point C is on the same shore as B, but 2 km down the river from B. A telephone company wishes to lay a cable from A to C. Cable laid underwater costs 5 times as much as cable laid on land.

(a) Draw and label a sketch.

(b) If it costs k \$/km to lay cable on land, how much (in terms of k) does it cost to lay cable under water?

(c) How much would it cost to lay cable directly from A to C? (Your answer will be a multiple of k .)

(d) What would be the total cost to lay cable from A to B under water, then from B to C on land?

(e) Suppose point D is located on land exactly midway between points B and C. What would be the total cost to lay the cable from A to D under water, then from D to C on land?

(f) The results so far should suggest that the cable paths described in parts (c) and (d) above are not the most cost-efficient. The path described in part (e) might be best, but perhaps there is an *optimum* location for point D, not necessarily midway between B and C. Let point D be located on a line from B to C, and let x be the distance from B to D. Find expressions for the distance from D to C, and for the distance from A to D. Write an equation for the total cost function.

(g) Use calculus to minimize the total cost function analytically.

(h) Confirm your answer to part (g) by graphing the cost function on your calculator (assume $k = 1$) and estimating the value of x (and hence the location of point D) that minimizes the cost.